Satellite Attitude Dynamics and Control
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Lecture – 05
Kinematics of Rotation (Contd.)

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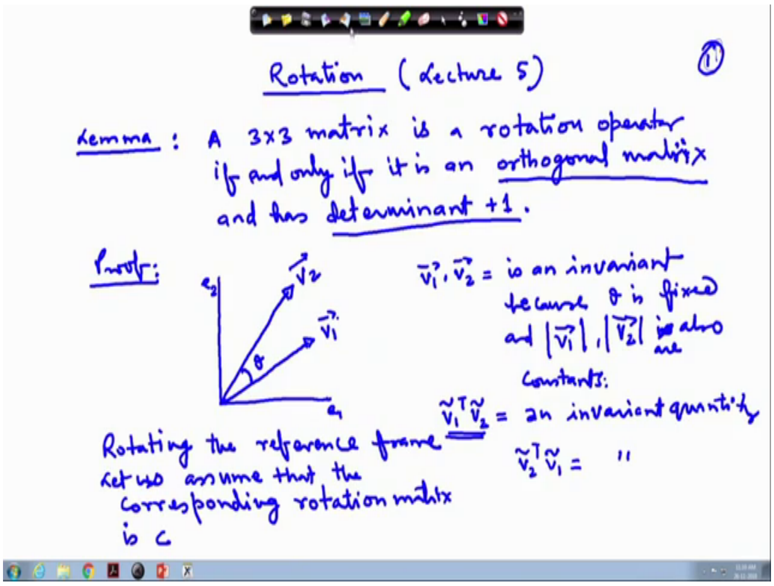

Good morning and welcome to the 5th lecture on Rotation. So, the last time we have been discussing about one lemma, so we will continue with that lemma, continue with that lemma. For 3 into 3 matrix is a rotation operator if it is an orthogonal matrix, we have already discussed about what the orthogonal matrix is and has determinant plus 1.

We have a short proof for this. So, let us consider the frame in which we have two vector arbitrary vectors V 1 and V 2, angle between them is theta. So, obviously we can observe that if we rotate the reference frame, say this is e 1, e 2. So, if we rotate the reference frame, the angle between these two vectors will not change. So, this implies that V 1 dot V 2 is an invariant, because theta is fixed and V 1 magnitude V 2 magnitude this is also, these are constants means they are not changing.

So, in the vector rotation as we have written, this implies that V 1 tilde transpose V 2 tilde, this is an invariant quantity. Now, if we give rotation to this frame e 1 and e 2, so rotating frame rotating the reference frame, let us say the corresponding rotation matrix is let us assume that matrix is C.

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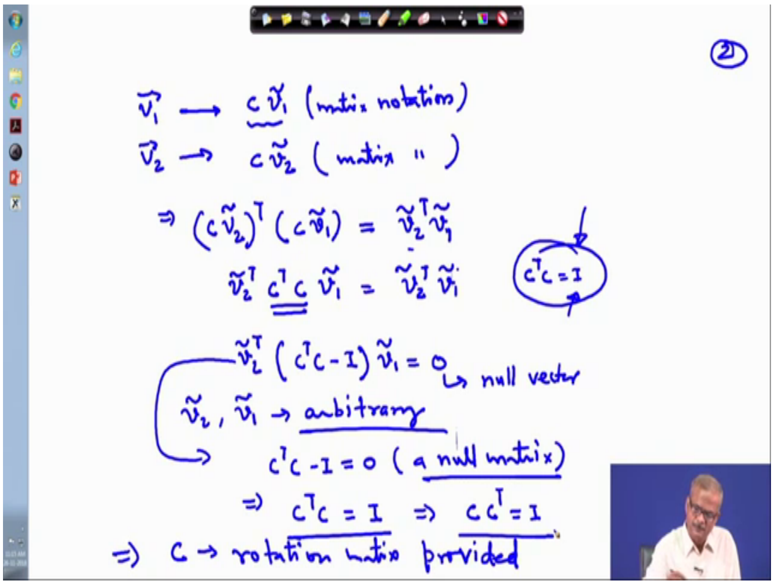

Therefore in the new frame the vector we have taken this inner product, we could have also written this as V 2 tilde transpose V 1 tilde, this is an invariant quantity, so it is a equivalent. Now, if we rotate the frame, so p by rotating the frame, let us say that the V 1 vector in the rotated frame is represented by C times v 1 tilde ok, where we this side we are writing in matrix notation. And similarly the V 2 vector, it will be denoted by in the new frame ok.

And therefore, this implies v 2 tilde C times, v 1 tilde or v 2 tilde whatever the way we write C times v 2 tilde transpose times C times v 1 tilde, this must be equal to, this must be equal to v 1 tilde v 2 tilde transpose v 1 tilde. So, if this quantity will be equal to this quantity, because the vector the angle between these two vectors, it is not changing and the vector length itself it is not changing ok. And their only the frame is rotated, therefore this must be equal to this quantity.

And we can write this as v 2 tilde transpose, and we know that C transpose C will be equal to I, if it is a rotation matrix ok, so this we are so this is what we are looking for, so we can rearrange this thing. So, we have taken it on this side, and taken out v 2 tilde transpose, v 1 tilde transpose, and the right hand side this is a null vector ok.

Now, this already we have mentioned that v 2 tilde and v 1 tilde, they are arbitrary. And therefore, to hold this it is required that C transpose C minus I, because the right hand is null null vector. And these are not null vectors ok, these two are arbitrary, and therefore this quantity must satisfy (Refer Time: 07:58) that is a null matrix ok. And this implies C transpose C equal to I, and this in turn implies C C transpose equal to I, as we have proved earlier also ok. And therefore, this implies that has it was required that C transpose C should be I in this lemma. So, therefore c this is a rotation matrix provided C determinant this equal to plus 1 ok.

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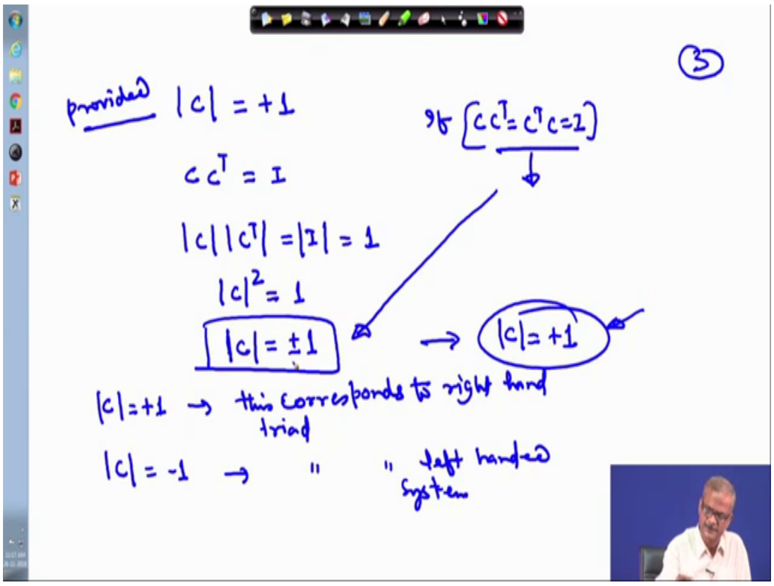

So, we have C times C transpose, this equal to I. And if we take the determinant of this ok and determinant of the transpose is the same as a determinant of the original matrix, therefore this quantity equal to 1. And C equal to plus minus 1 so, given this we have started with the assuming that if C C transpose equal to C transpose C equal to I proved in this lemma, and then once we have got this, then we are getting this quantity.

Now, out of this the plus 1 C determinants this equal to plus 1, this corresponds to so right hand triad. While C equal to minus 1, this corresponds to left handed system. Therefore, we reject C equal to minus 1, and we written only C equal to plus 1. And therefore, whatever the we are stated in the lemma, so this is proved that this C C transpose is equal to or C transpose C equal to I, and C determinant is also equal to plus 1. And why this minus one has been rejected, we will see on the next page.

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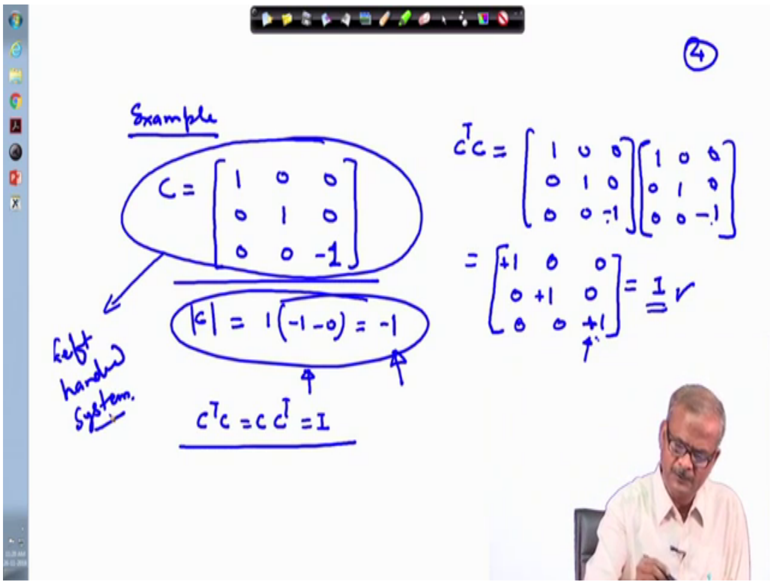

Let us consider a example consider an example, 0 0 minus 1. So, this matrix if we do C transpose C or C transpose C equal to 1 0 0 0 1 0 0 0 minus 1, it will remain the same. And once we take this product ok, so this will be 1, this we considered rest of the others will be 0 ok, in this column the rows the same way, and here also this is taking product of this and this turns out be 0, this and here this also turns out to be 0, and this is minus 1.

So, here this minus 1 and looking into this determinant, this is plus 1 ok. So, looking into this determinant, so this is (Refer Time: 12:06) your I ok, but so all of them are plus 1. What about the determinant of this? So determinant of this is 1 times minus 1, minus 0, this is the only one existing here, so this is minus 1. But, this does not satisfy the other property that is C transpose C equal to C C transpose, this is equal to I ok, which is very much clear.

As we take the last one, which is essentially we have to look into 0 times 0 is 0, 0 times 0 is 0, minus 1 times minus 1 is plus 1. So, therefore C C transpose is I, the identity matrix. But, the determinant is not plus 1, but rather it is a minus 1, and this matrix this belongs to the left hand triad, left hand system left handed system.

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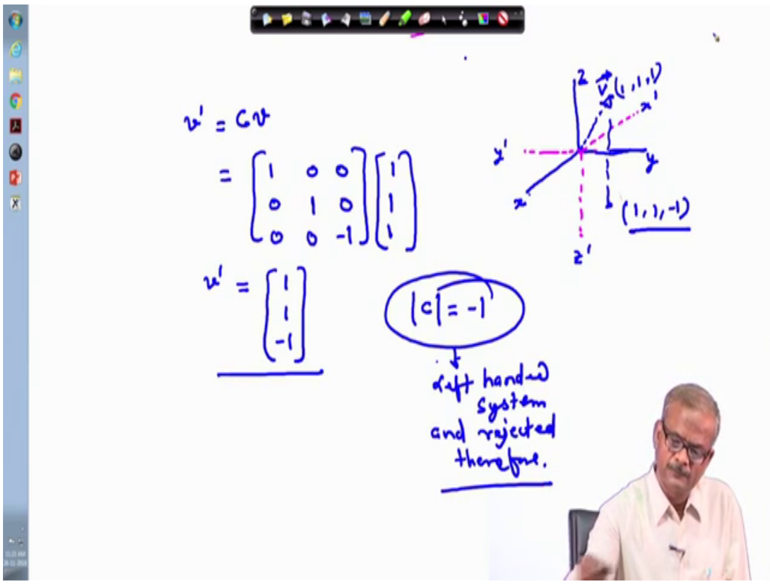

As we can observe from the operation on C by any vector say, let us assume that this is operating on some vector nu, where we will assume this to be 0 1 0 0 0 minus 1, and here we assume this new vector to be 1 1 1 means, it is belonging to so let us say this is a x, y and z. And this we will write at z times, this is a x prime, and this is y prime. So, we have many point here, which is 1, 1, 1, and on this then you are operating y, so that simply implies that there is a vector from this point to this point, this is your vector V ok.

And you are operating by this the rotation matrix, which is given by 1 1 minus 1 here the diagonal elements, off diagonal elements are 0. So, this implies nu prime will become this is 1, this is 1, and this gets reduced to minus 1 ok. So, this point gets map to the point and which is reflected in the x y plane, and this is the point here 1 1 minus 1. So, we can see that the x, y and z prime, it does not form a right hand triad ok, this forms left hand triad.

So, therefore if your C determinant of the C matrix this equal to minus 1, so that is means it does not satisfy our requirement of the lemma. So, this is belonging to the left hand left handed system, and we reject it therefore and rejected ok.

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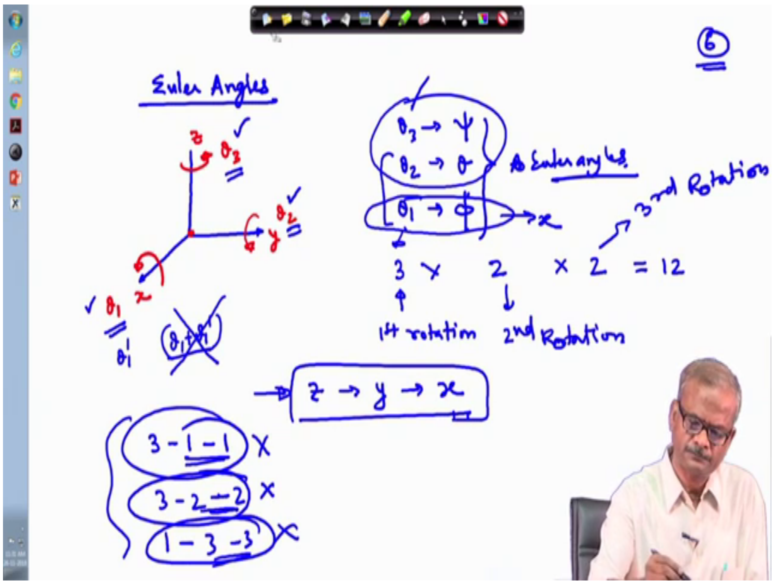

So, we have now proved the lemma, and we will look into the some basic of the angular representation Euler angles. So, let us say this is a x, y, z frame, and this is the origin here. And this is a right handed triad, so rotation about this x-axis will represented by theta 1, rotation about this axis will represent by theta 2, rotation about the z-axis represented by theta 3.

So, it is a common that theta 3 is a quite often represented by psi ok, theta 2 it is represented by theta, and theta 1 is represented by phi, so these are called the Euler angles. But, here we stick to this general representation theta 1, theta 2, and theta 3, because it is of a quite often we use these notations psi, phi for some other angles, then that a that will be a confusion. So, we will always keep theta 1, theta 2, and theta 3 as the angular displacements. So, you can see that we have three degree of a freedom in the system in the rotational freedom, and that we can represent by theta 1, theta 2, and theta 3.

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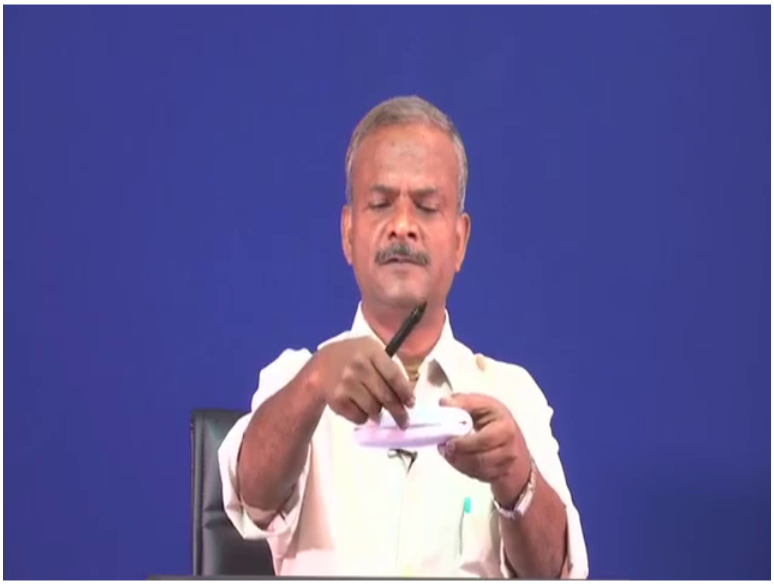

Now, how do we represent a rotate it is a matter of concern like a say if I have this mobile or say if this is a box ok, so this is a box and outside here it is a written some word ok, so I keep it like this. So, let us say the x-axis is along this direction, y-axis along this direction, z-axis along this direction, so this forms a right handed triad, as we can see by the right hand rule.

Now, I want to push it into a configuration, which is represented like this. So, how we can do it? I have to bring it into this way ok. So, here I can rotate it like this, so this rotation is about the z-axis ok. So, first rotation we are giving the z-axis, and this axis we have taken as a x-axis. So, next rotation we will a depending on what we are looking for so from here we want to go into this directions, so first like about the z, and the second rotation about like this about the x-axis, so that gives me this orientation.

Now, is it possible to get into the same orientation using some other way? So, let us look into this one, first rotation about the x-axis ok. And this is your y-axis, now yeah this is the y-axis, so next rotation about the y-axis. So, we get to the same orientation by these two different rotations. So, here what do I want to tell you that we can choose these rotations in different sequences, and by giving different magnitude to them, we can get to the same orientation. So, this Euler angle representation of any orientation, it is not unique ok. This is a very basic concept, which we must understand that this Euler angle representation, it is not unique. We can get to the same orientation by giving different amount of values to psi, theta, and phi, it may be positive, it may be negative.

Now, also we have already learned that about say this is the x-axis. So, about the x-axis, we cannot give two sequential rotation means first write theta 1 and then write theta 1 prime, so this will simply implies theta 1 plus theta 1 prime, this already we have proved. So, this kind of rotation is ruled out ok. So, the first rotation out of this 3, I can choose in 3 ways ok. So, this is your 1st rotation. The next rotation, I cannot choose again say the about the theta 3 ok. If I choose this means it is like the same thing, is the same rotation of higher magnitude ok, so that is ruled out. So, next I have to choose from these 2 from these 2, I can choose in 2 ways. So, this is your 2nd rotation ok.

Now, once we have done this suppose that we have chosen this. Then we are left with these two, because first choosing about the 3rd, and then choosing about the 1st, and then I choose about any of these two if either the z or the y, so it is not the rotation about the same axis means this is about the x-axis, so this will be a along the y and the z-axis.

And therefore, the next rotation is along the two. If we are not taking any sequence rotation sequentially about the same axis means, there are total 12 ways of choosing this orientation representation means using the Euler angle, we can achieve to the same orientation in 12 different ways. But, we have to definitely give different magnitude 3 of them just like here in this case for this box, we considered from this orientation to this orientation, it was a simple one that first we rotated about this is the x-axis rotation, then the y-axis rotation ok, z-axis rotation is 0.

The other way we did here z-axis rotation, and then the x-axis rotation, y-axis rotation here in this case is 0. So, this way you can compose positive, negative of course the other way this is a much simpler case, other cases will be very difficult to look into look into physically. So, this is your 3rd rotation.

But, this remains a critical fact that, we can get to any orientation in different ways. And how do we represent that orientation, it is not unique. But, however if you choose one of the rotation procedure say the first rotation about the z-axis, then the next orientation rotation about the y-axis, and next rotation about the x-axis. Then is a and I strict to this throughout your computation, then all scores you get whatever the result you get that will be unique, there will be no confusion anywhere.

So, if we are ruling here ruling out the configuration like 3 1 1, so this is ruled out. Similarly, 3 2 2 this kind of rotation, this will be ruled out or 1 3 3, these are ruled out, because these two are about the same axis, and therefore this these two are not allowed ok.

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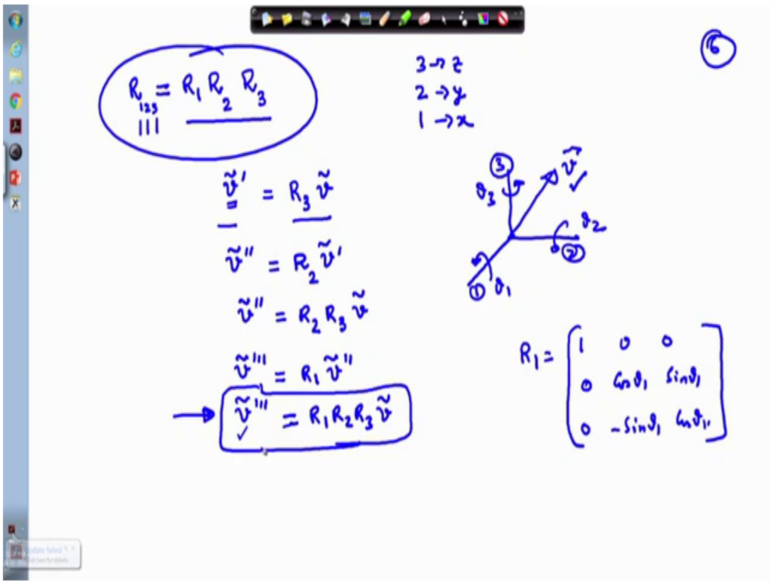

And how do we represent the rotation, we have already looked into so if a say the rotation if I represent it like this that first rotation I give about the third axis, which here in this case 3 will represent the z-axis, 2 will represent the y-axis, and 1 will represent the x-axis, so we will write it like this.

So, we are giving sequential rotations like this hence 1st rotation about the 3rd axis, 2nd rotation about the 2nd axis, and 3rd about the 1st axis. So, if let us say that I have any vector nu ok, and I am operating on this vector by R 3 means rotation about the 3rd axis, and this gives me nu tilde prime ok.

Now, I will operate on this vector ok. So, what we have done that I have a reference frame, this is a vector here v, and this is 1, 2, and these are the 3 axis, and I am rotating this axis. So, first rotation I am giving about the 3rd one ok. And as a result in the new coordinate system for the new different system, we have new is represented a nu prime. Then the next rotation, I give about the y-axis, which here presented by 2. So, this will operate on nu prime, and from here we will get as nu double prime.

So, nu prime is nothing but from this place R 3 times nu tilde ok, so this is nu tilde double prime. Similarly, we give the rotation about the first axis, so we will operate on nu tilde double prime, and this gives me nu tide triple prime. So, and then putting nu tilde double prime from this place, so this becomes R 1, R 2, R 3 ok. So, once you have given these three rotation general rotation ok, after that this your original vector it appears like this, its components in the rotated reference frame, it will change. So, here theta 1, theta 2, and theta 3.

And we are well aware that the R 1, which is a rotation about the first axis, it will be given by 1 0 0 0 0 already we have discussed this, cos theta 1 sin theta 1 minus sin theta 1 times cos theta 1 ok. Similarly, the R 2 and R 3 this has to be written. And once we insert into this, we get the transform vector or the coordinates of the vector nu in the rotator reference frame.

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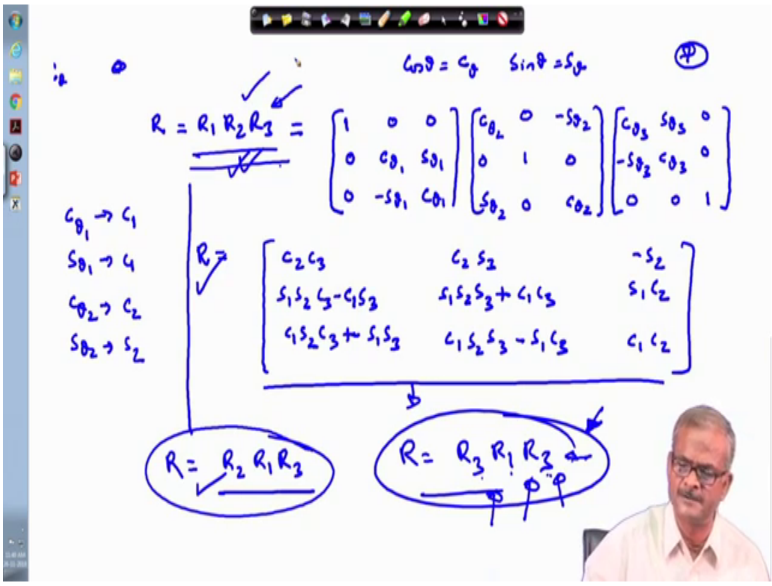

Therefore this R that becomes R 1, R 2, R 3 1 0 0, and we will use a shorter notation for representing the cos theta, we will write as C theta, and we will tag as theta 1, theta 2 whatever it is and sin theta as s theta. So, this is the shorter notation, so this becomes C theta for this 1, this is C theta 1, S theta 1 0 minus s theta 1 C theta 1. So, to remember that we are operating from this side, this is first operating on the new vector, then this will operate, then this will operate.

Thereafter, for the y here in this case this is C theta 2 C theta 2, and this just reduced to S theta 2, and here minus S theta 2 has already we have previously discussed. For the z one, and we have then C theta 3 C theta 3. And do the matrix multiplication, once we carry out this, so I will write here the final result without taking much time.

So, this will can be indicated as let us say theta 1, we further reduced this notation to C will represent C theta 1 will use this representation C theta 1 has C 1, S theta 1 has S 1, similarly other C theta 2 is equal to C 2 and S theta 2 equal to S 2, so if writing becomes here simplified. So, this will be indicated by C 2 C 3 S 1 S 2 C 3 minus C 1 S 3, you can check it verify it ok. So, taking doing this matrix multiplication, it tells us, this R, this is the rotation matrix ok.

Now, if you take a different situation sequence, let us say R 2 R 1 R 3, so this sequence will give you another matrix, which will be different from this ok. But, as per our discussion that whatever be the way of rotation ok, if we give proper magnitude to these values ok, somewhere one may be if a theta 1 may be positive in one case, another case it may be negative and so on.

So, by giving that we can come to the same orientation, because this orientation representation using this Euler angles, they are not unique. And therefore, it can be represented in many ways out of this, this is one of the way. This can be another way of doing this. Another way we can write it like as R 3 R 1 R 3 and so on ok. So, no sequential rotation about the same axis here 3rd axis, 1st axis and then the 3rd axis. But, here this one creates problem, when the rotation involves are very small ok. So, these kind of cases quite often if the angles involved are large, so you can use this, otherwise you should restrict to this particular one ok.

In the satellite control, quite often if you have to orient the satellite from the initial position to a final position, so if a you can do only using these two rotations that is about the 3rd axis, and the one at 1st axis. It means if you have two controllers along these two axis, then you may be able to work it out. But, you have to look further because it is a complicated, if you rotate about the 3rd axis, it may happen that the satellite is rotating about the 2nd axis also because of the universal coupling. So, those things are very complex and must be resolved. But, however this is preferable only if the angle involved are large, while here in this case even if the angles are the small, you can work with this.

So, thank you very much. We stop for this lecture here, and we will continue with, in the next lecture.