Satellite Attitude Dynamics and Control
Prof. Manoranjan Sinha
Department of Aerospace Engineering
Indian Institute of Technology, Kharagpur

Lecture – 07
Rotation

Welcome to the 7th lectures. So, we have been discussing about the aerospace theorems last time. So, we will continue with that.

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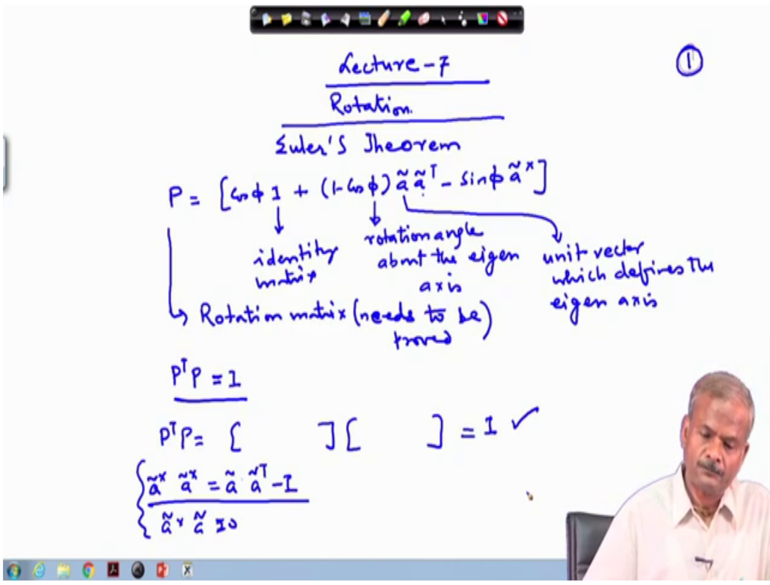

So, what we saw that if we assume this matrix P to be of the form cos phi I plus 1 minus cos phi times a tilde a tilde transpose minus sin phi times a tilde cross. So, this is identical to same matrix P we are been trying to prove. Here phi is the rotation about the eigen axis; rotation angle about the eigen axis. And a tilde this is the unit vector which defines the eigen axis ok. And I this is the identity matrix. So, for P to prove this that this is a rotation matrix this needs to be proved.

So, what we need to show that P transpose P this equal to I this is one of the properties of the rotation matrix, then also we have need to check that indeed phi is the rotation about the a axis. So, what we suggested last time that once we have start writing this P transpose P, so enter this so this will turn out to be indeed as I under certain characteristic that we have to assume a cross we have to write this a cross a is equal to a tilde transpose times a tilde times a tilde transpose minus I and a tilde cross a tilde this equal to 0. So, I have given you the equation last time. So, in that expanded equation, if you use these quantities. So ultimately it will get reduced to I which I am not going to prove here, because we will we are going to take this in the as a tutorial problem ok.

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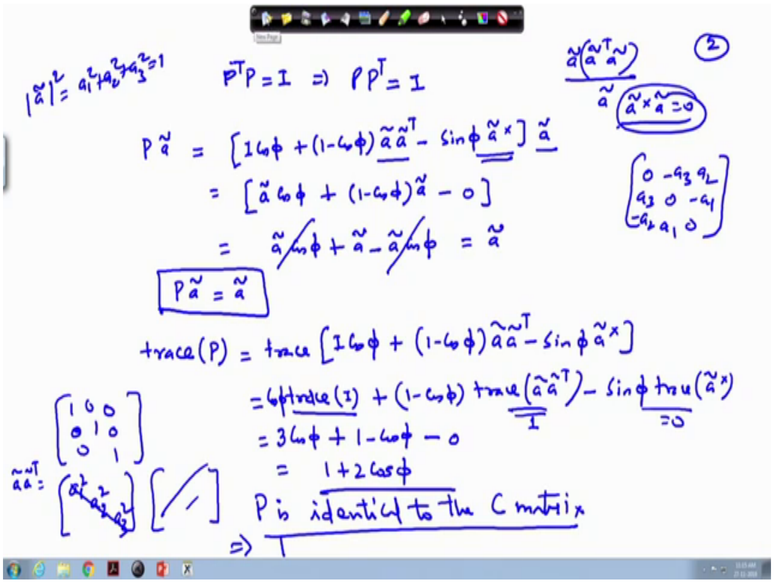

So, if P transpose P, this equal to y. So, this will also imply as per our earlier derivation, this will also imply that P times P transpose equal` to I ok. Let us check that P is in this entered the rotation matrix about the vector a. So, if P operate on a tilde by this P matrix, so we have I cos phi plus 1 minus cos phi tilde transpose minus n. And this we are operating way on the vector a.

So, here this will get reduced to a tilde cause phi plus this is the identity matrix multiplied by this vector a that will give you the vector a itself. This is a scalar. So, it remains as it is. And here we have 1 minus cos phi and will use our property that in the transpose a tilde. So, you can look at like this, this second part, here, and this and this. So, if we look into this. So, this is a tilde transpose a tilde because this is a unit vector, and therefore this gets reduced to a tilde. So, this is a tilde. And already we have discussed that a tilde cross a tilde, this equal to 0. And therefore, these terms gets reduced to 0. So, we have a tilda cos phi plus a tilda minus a tilde cos pi. And this term, this term, drops out giving us a tilde. So, this is P times a tilde this equal to a tilde. So, this is indeed rotation about the a axis, because a axis remains unaltered.

Therefore, now the last thing we need to find out that that trace of this P matrix is indeed the trace of the C matrix that we have used earlier. So, we will write this as. So, we have written two properties and the last one that trace P as I cos phi plus phi a tilde cross we can break it. So, this will be trace I and here cos phi plus 1 minus cos phi trace a tilde a tilde transpose minus sin phi trace a tilde cross ok. So, this is cos phi and trace I, this is I is the identity matrix, all diagonal terms are 0 here ok.

So, some of the diagonal elements, this is 3. So, we will put 3 here ok, and this trace a times a transpose. So, if you write here a times this is a tilde times a tilde transpose. So, this will turn out to be a symmetric matrix whose diagonal terms will be a 1 square, a 2 square and a 3 square ok. And, already we know that because a tilde magnitude this equal to a square equal to a 1 square plus a 2 square plus a 3 a square this equal to 1. And therefore, this gets reduced to 1 and we have 1 minus cos phi here in this place and minus sin phi times trace of a cross.

So, a cross here 0 0 0, and here minus a 3 a 2 minus a 1 a 1 minus a 2 and a 3; so the diagonal sum of diagonal elements here, it is 0 and therefore, this term drops out this is 0, this quantity is 1. So, here this becomes 0. And what ultimately we get 1 plus 2 cos pi. So, the trace of P is exactly the same of the same format as the trace of the C matrix that is the rotation matrix. So, in all the ways that we see that P is identical to identical to these C matrix.

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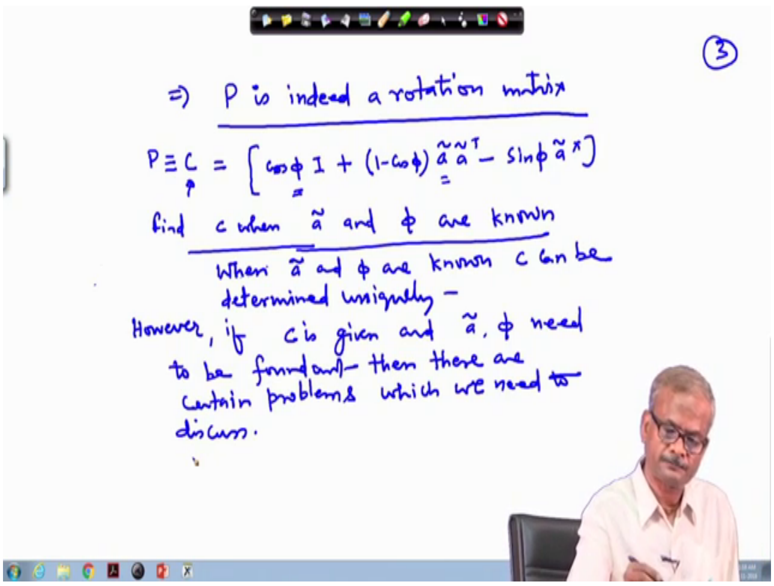

And therefore, this implies this implies that P is indeed a rotation matrix. So, this means that P is identical to C. And therefore the C we can write as cos phi I plus 1 minus cos phi a tilde times a tilde transpose minus sin phi a tilde cross. Now, the questions will arrives that either we are given phi and a tilde. So, we can find out C or either C is given, then vice versa we have to find out phi and a tilde. So, it so happens that finding C when a tilde and phi are known. And this is a straight forward process, and gives C.

So find C, when a tilde, phi are known, then a tilde and phi are known, C can be determined uniquely. However, if C is given, and a tilde, phi need to be found out, then there are certain problems which we need to discuss.

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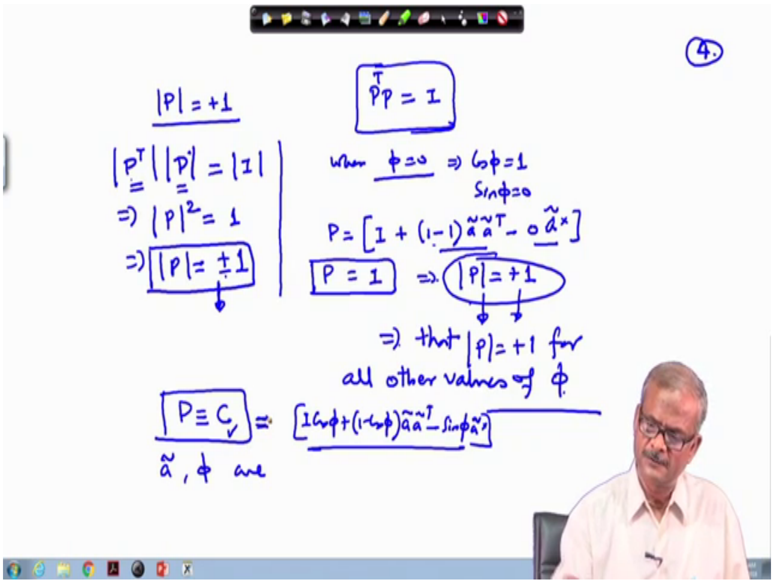

Besides we have forgotten one more thing that we need to prove that this P tilde determinant this equal to plus 1. So, already we know that we have proved that P tilde transpose P, this equal to I. And therefore, this P tilde square will be equal to ok. So, this part we are aware of ok. And this determinant, we need to prove that this is equal to 1. If we do that only then our job is done.

So, we are start with this P tilde transpose, we will take the determinant of this. So, this will be P tilde. And this will be determinant of phi. Then this implies, this determinant, this determinant they are the same transpose taking transpose does not change the determinants. And on the right hand side, we will have this has one. So, this implies P tilde equal to plus minus 1.

Now, out of this plus and minus, we need to stick with plus 1. It will be clear when you will look into that when phi equal to 0 ok. So, this implies cos phi equal to 1, and sin phi equal to 0. And therefore, if you put these values in the P matrix, so this will get reduced I plus 1 minus 1 times a tilde times a tilde transpose minus sin phi which is 0 times a tilde cross. So, here this part is 0, this part is 0, and P equal to I. And this implies P determinant this equal to plus 1 ok. In this case, it is a very simple, because there is no a square involved here P determinant equal to plus 1.

And because this is a rotation matrix and its determinant is plus 1. So, it must happen that it is a determined, if we give different values to the phi this pi. So, still we should get a determinant value of plus 1. And therefore, we need to neglect here the minus sign. So therefore, so this implies that P equal to plus 1 for all other values of. So, this way we have proved all the requirements for P 2 v o rotation matrix. And already we have written that therefore, P is identical to C. And then we have written in the format I cos phi plus 1 minus cos phi a tilde a tilde transpose minus sin phi times a tilde cross. So, this is our rotation matrix. Now, given this quantity that you are a tilde and phi are known. So, if we expand this ok, so we can find out the elements of this C matrix.

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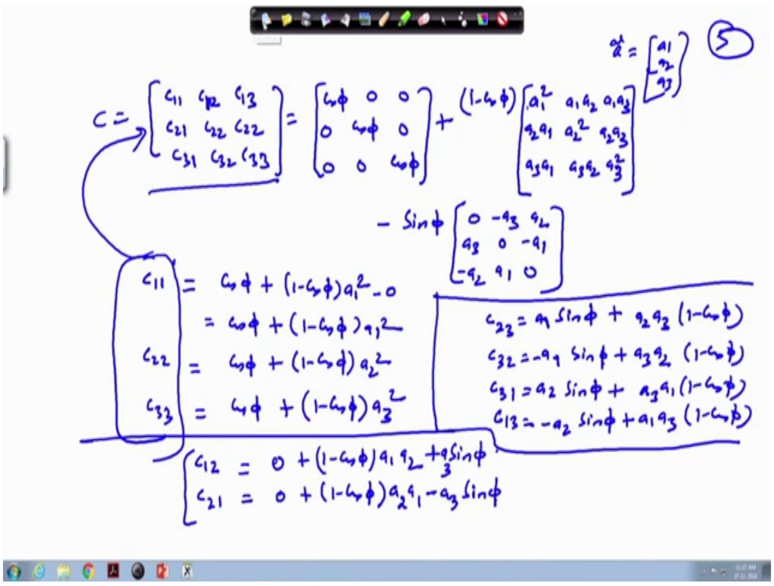

So, on the left hand side if we write C s C 11 C 12 C 13, and on the right hand side then we will have here cos phi, cos phi, cos phi, we are expanding all the terms this the first term and then plus 1 minus cos phi times a tilde a tilde transpose. So, that quantity is a 1 a square and here a 2 a 2 a 1 a 3, similarly a a 2 a 1 a 2 square a 2 a 2 and then a 3 a 1 a 3 a 2 and a 3 square. So, this is your the second term. And the third term of course, this is minus phi sin phi a cross which is a q symmetric matrix minus a 3 a 2 a 1 a 1 a 3 and this is minus a 2.

So, if we add the right hand side and compare this terms, so we can get the corresponding values let us say C 11. So C 11, we need to add all the first term. So, this will be cos phi plus 1 minus cos phi times a 1 square and here this first term is 0, so that put it as 0. So, this is your first term which is cos phi plus 1 minus cos phi a 1 square. In the same way if you write, so C 22 this will be cos phi. Again in the same way plus 1 minus cos phi times a 3 square and C 33 equal to cos phi plus 1 minus cos phi a 3 square. So, your diagonal elements of this matrix C is known this way which remains to find out the all diagonal terms.

So, go in the same way and find out the terms corresponding to let us say this C 12. C 12 here this first term this term is corresponding here 0. So, this is 0 plus 1 minus cos phi times a 1 a 2. And here this term is plus sin phi times a 3. So, a 3 will write in the ahead. So, this is yours C 12 ok. And in the same way C 21 if you look into this, so C 21 is this quantity here we have to add all this quantities. So, C 21 will be 0 plus 1 minus cos phi from this place, then a 1, a 2 or a 2 a 1 whatever we want to write. So, let us write that way only because these are the scholars it does not matter. So, what sequence we write.

And lastly the last term we have to pick up. So, here C 21 this term we have to pick up. So, this comes with minus a 3 sin pi. So, this is minus minus a 3 sign pi. So, this way if you continue, we can complete this cycle. So, we will have C 23 equal to a 1 sin phi plus a 2 a 3 1 minus cos phi, C 32 equal to minus a 1, and C 31 equal to a 2 sin phi plus cos phi and C 13 2 minus a 2 sin phi plus a 1 a 3 1 minus cos pi. So, this way all the elements of this C matrix its known. So, given phi the rotation angle about the eigen axis and the this victor a itself which is constituted of a 1, a 2 and a 3 ok. So, we will be able to find out this the rotation matrix. And also we need to look in the opposite way in the vice versa.

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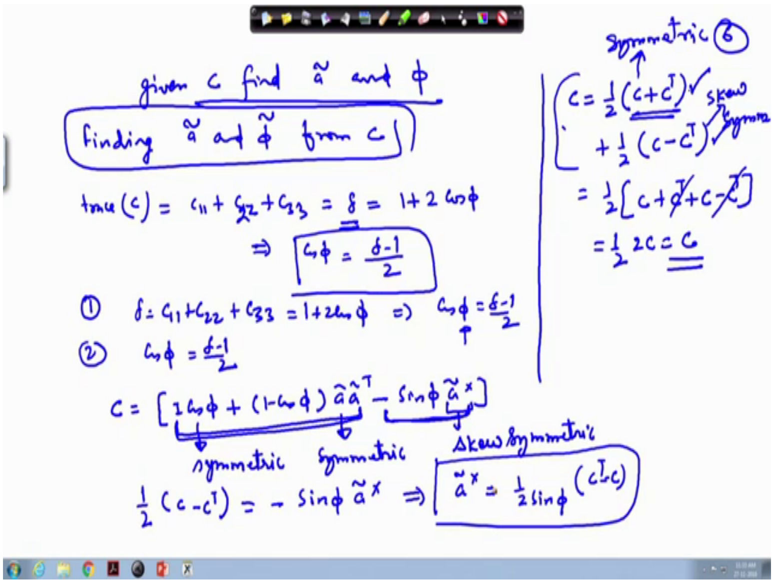

Also we need to look into just the opposite way that is given C find a tilde and pi. So, this process it creates certain problem, and care must be taken while working with the problem ok. So, before we do this let us look into or maybe we can continue with whatever we have been doing until return back to this problem.

So, the first we have the problem here. Finding a tilde and phi from C, this is (Refer Time: 22:16) my problem, which we need to sort out. So, already we have seen that if C is given, so the trace of C trace C equal to C 11 plus C 12 C 22 plus C 33, and this we can write as delta. And this quantity have we have written as 1 plus 2 times cos phi, so C is given.

So, from here what we have if C is given, so this quantity is known ok. So, from this place then cos phi, this becomes delta minus 1 divided by 2 ok. The other part we have to also look into so 1 is the first step is our delta write, this as C 11 plus C 22 plus C 33 equal to 1 plus 2 cos phi, and from there you write cos phi equal to delta minus 1 divided by 2. So, the 2nd step is what we get cos phi equal to delta minus 1 divided by 2.

Now, what we are left with that we have to find out a vector ok. There are certain problems involve, so we have to resolve all these problems even with the angles, which will come across shortly. So, we have just C matrix the rotation matrix, which we are writing as I cos phi plus 1 minus cos phi a tilde a tilde transpose minus sin phi a tilde cross ok.

So, if you look into this, this is a symmetric term, this is also a symmetric term, because a tilde times a tilde transpose this is a symmetric matrix, similarly I is a symmetric matrix. And therefore, this together it is a symmetric term while here on this side, this is skew symmetric term this is skew symmetric matrix.

And you must be aware of in from your elementary matrix algebra that C can be written as 1 by 2 times C plus C transpose plus 1 by 2 times C minus C transpose; if we break it and add it out, so if we can check whether we have done the right thing or not. So, 1 by 2 this cancels out, times 2 C equal to C. So, this simply says that a any matrix can be broken into two terms. One is symmetric one, and another one is the skew symmetric one. So, this portion this is symmetric ok, so this is symmetric while this one is skew symmetric.

So therefore, we get certain advantage from this place that 1 by 2 times C minus C transpose, here we will looked that this term is symmetric term, and this term is turning out to be this skew symmetric. So, 1 by 2 times C minus C transpose must be equal to sin phi times a tilde cross, and this implies that a tilde cross we can write as 1 by 2 sin phi times C transpose minus C. So, this constitutes our venue for finding out this particular expression a 1, a 2, and a 3, but there are certain problems involve, which we need to sort out.

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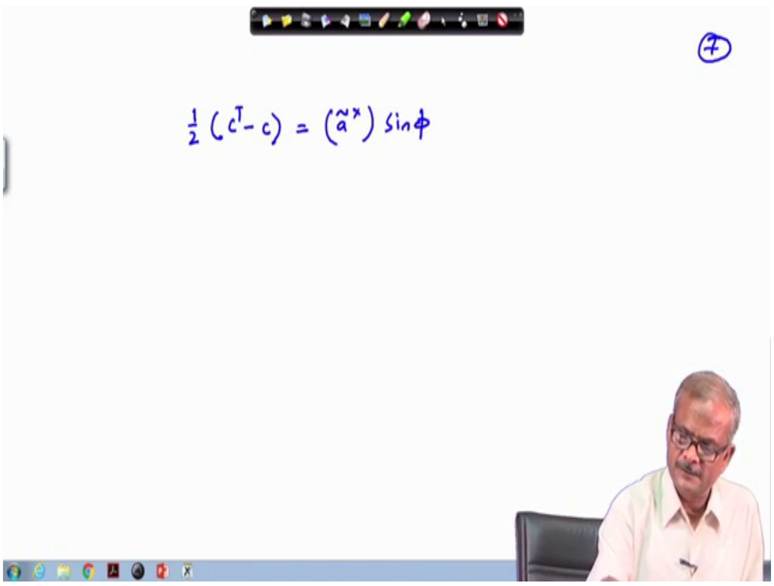

So, we have here 1 by 2 C transpose minus C, this we are writing as a tilde cross sin phi or either we can write sin phi here in this place.

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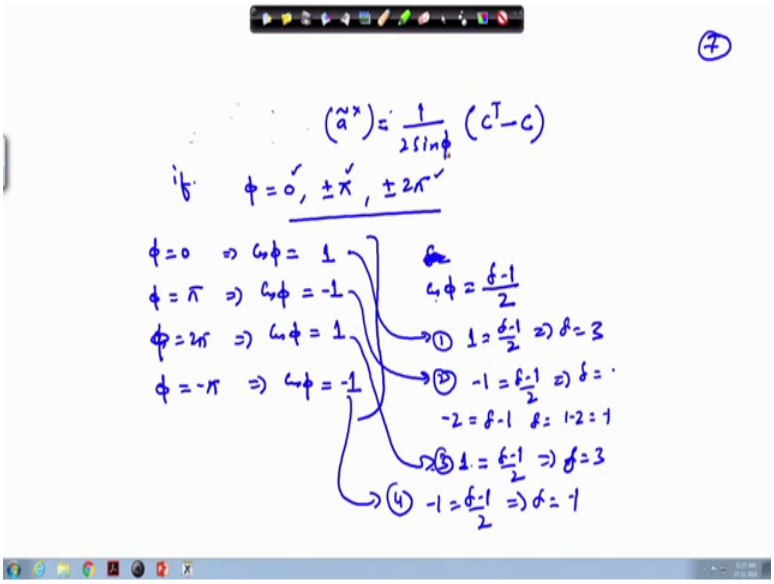

So, a tilde cross, we can rub it rub this side: a tilde cross this equal to 1 by 2 sin phi C transpose minus C. Now, if phi equal to 0 or plus minus pi or plus minus 2 pi, so we can say that sin phi reduces to 0, and therefore a cross is not defined ok. So, this is a problem that for this values, we are we face certain problem.

So, so let us explore it. Once the corresponding cos phi, and phi equal to 0, so this implies cos phi, this equal to 1, and phi equal to pi, cos phi equal to minus 1, and when phi equal to 2 pi ok. And lastly when phi equal to minus pi, phi equal to minus 1. So, the corresponding delta value is will look into, so delta equal to delta we have cos phi we have written as delta minus 1 divided by 2.

So, what happens if here in this case that if we choose the first one phi equal to 0, so delta this gets reduced to 1. If we chose this one this is the 2nd one, so phi equal to minus 1 cos phi equal to minus 1 and then delta minus 1 divided by 2, so this implies delta equal to minus 2 plus 1 equal to this should turn out to be minus 2 minus 2 equal to delta minus 1 1 minus 2 equals to minus 1 ok, and phi equal to this one we need to correct here we have written it wrongly. So, phi equal to 1 if we put cos phi equal to 1 here, so this quantity this is 1 equal to delta minus 1 divided by 2, and this implies delta equal to 3.

So, similarly the 3rd one 1 equal to delta minus 1 by 2, this implies delta equal to 3. And for the 4th one again we get this is the 3rd one, and the 4th one is minus 1 delta equal to minus 1 ok. So, we are getting either the value of delta equal to minus 1 or either 3 for all these values, where sin phi is vanishing ok.

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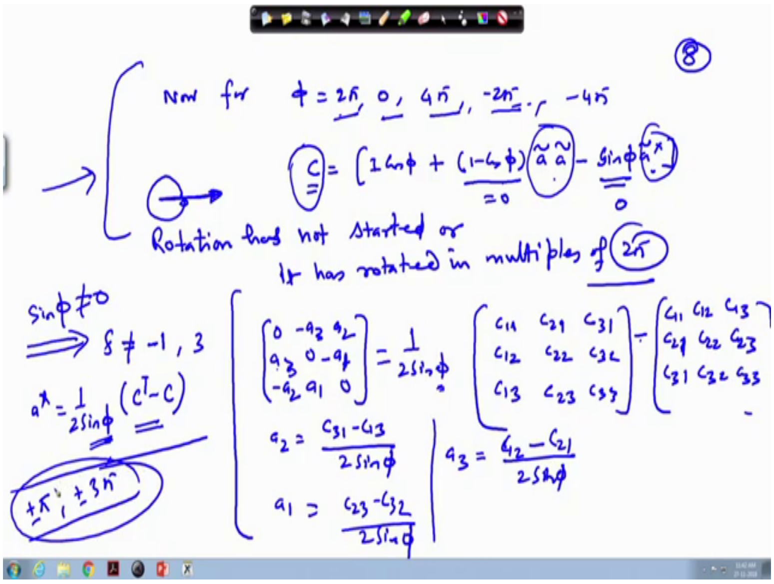

Now, for phi equal to 2 pi, 0 or 4 pi or minus 2 pi and so on. So, it simply implies that suppose this is a vector, and this has been rotated by 2 pi. So, this vector will again come to the same position is not it or either it has not been rotated, so it remains in the same position.

And if you look into the similarly for the other 4 pi minus 2 pi etcetera, so if you look into the C, so here I times cos phi plus 1 minus cos phi a tilde a tilde minus sin phi a tilde cross ok. So, if you put all this quantities, where sin phi is vanishing, so this quantity is 0 and for all these 0 2 pi, this quantity will vanish this will also be 0. So, this a term is vanishing, is not it? Here also this term involving a is vanishing.

So, therefore from given C matrix, it is not possible to find out a. So, for this case, so this indicates either the motion has not yet started, this indicates that rotation has not started or it has rotated in multiples of 2 pi ok. So, for this particular case, where 2 pi, 0, 4 pi, minus 2 pi and so on, minus 4 pi. So, we would not be able to find out a vector ok, which is very much visible from this place ok.

For all other cases, when delta is not equal to minus 1 or 3, so for these cases we can solve it ok. So, for those cases it is a very easy to write, because already we have written a cross matrix equal to 1 by 2 sin phi C transpose minus C. So, we just need to insert C transpose C, and phi f of course we need to we have already defined this file. So, in the case, where this file is non-zero, this is sin phi is non-zero. So, this quantity is known. And the right hand side, this is known, and therefore a cross can be defined uniquely ok.

So, certain problems are involved. So, one problem was here that we have resolved; for this particular case, where sin phi is non-zero ok. So, for that particular case, we are able to uniquely define it, and that you can do by writing a tilde cross. So here 0 0 0, minus a 3, a 2, then here minus a 1 a 1, minus a 2, a 3; ok; and on the right hand side expand this, by writing 1 by 2 sin phi and C transpose minus C, so that will turn out to be you have the elements like C 1, here it will be transpose of this C 12, C 13, then C 21, C 22, and C 23, this is transpose of the C matrix C 31, C 32, C 33, and then subtract from here the C matrix, which is C 11, C 12, C 13, C 21, C 22, C 23, C 31, C 32, C 33. And the compare the corresponding terms. So, if we do so will be able to get the quantity is here a 1, a 2, a 3 etcetera.

So, what we see that this a 2 from this place a 2 is this minus this, so C 31 minus C 13 divided by 2 sin phi a 2 ok. Similarly, a 1 we can check from this place this is C 23 minus C 32 divided by 2 sin phi ok. Similarly, we will have a 3 we can check from this place C 12 minus C 21 divided by 2 sin phi. So, this gives you uniquely a 1, a 2, a 3. So, your phi is defined, and also a 1, a 2, a 3 is defined.

And there are certain other problems, we will of course discuss during course of time; what is remaining here that we need to discuss the case, where air for 2 pi, 0, 4 pi, we have already done. We need to discuss about the plus minus pi, plus minus 3 pi discuss is remaining. So, we will discuss it in the next lecture.

Thank you for listening. We continue this in the next lecture.