Satellite Attitude Dynamics and Control
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Lecture – 08
Rotation (Contd.)

Welcome to the 8th lecture. So, will continue with the last 7th lecture what we have been discussing about the Egorov’s theorem. So, we were working out how to get a tilde and pi from the C matrix which is the rotation matrix. So, continuing with that.

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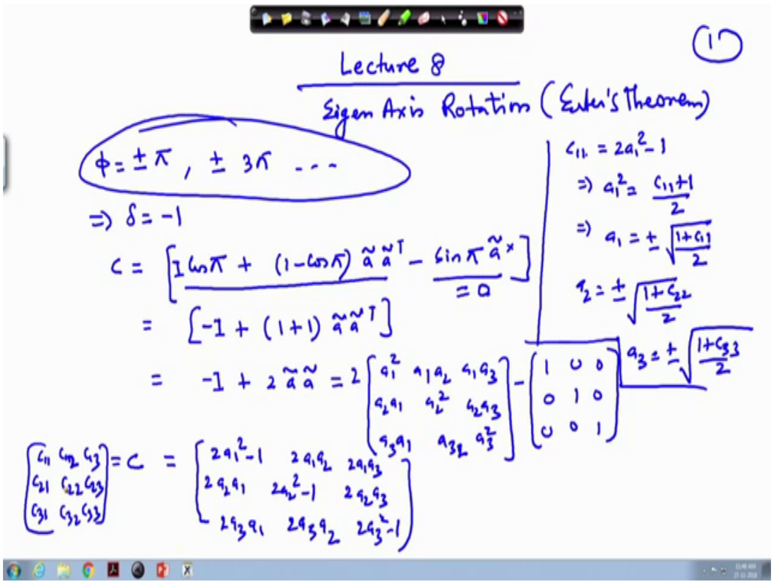

So, what we required to discuss that what happens when pi equal to plus minus pi plus minus 3 pi and so on. So, for this case we know that once phi equal to plus minus pi, so the corresponding value of delta is this implies delta equal to minus 1 this already we have observed. And therefore, the C can be written as cos cos minus pi. So, here we will replace this as say if we are using pi, so cos pi plus 1 and 1 multiplied by I, 1 minus cos pi and multiplied by a tilde a tilde transpose minus sin pi times a tilde. So, what we observed from this place that these term vanishes, this is, this is pi. So, this term is 0 and we are left only with this term. So, this is minus I plus 1 cos pi equal to minus 1, so this is plus 1 terms a tilde a tilde transpose, so minus I plus 2 times a tilde a tilde transpose.

And then purpose we need to expand it, so if we expand it this will look something like this a 1 square, this particular term a 1 a 2, a 1 a 3, a 2 a 1, a 2 a square, a 2 a 3, 3 a 2 and a 3 a square and minus I which is 1 1 1 0 0 ok. Therefore, this C can be written as 2 a 1 a square minus 1 and of course, this will be 2 a 1 a 2, 2 a 1 a 3. Similarly 2 a 2 a 1, 2 a 2 a square minus 1, 2 a 2 a 3 and here 2 a 3 a 1, 2 a 3 a 2 and this side a 2 a 3 a square minus 1.

Now, of course, you know that C is c 11, c 12, c 13, c 21. So, compare this terms, ok. So, if we compare this terms and writing it in this portion. So, we considered c 11 is equal to 2 a 1 a square minus 1 and this implies a 1 a square is equal to c 11 plus 1 divided by 2, and this implies a 1 is equal to plus minus 1 plus c 11 divided by 2 under root.

Now, the problem here is that plus and minus sin is appearing, so which sign to choose, ok. So, in early if we look for the other one of a 2 will be plus minus 1 plus c 22 divided by 2 under root and a 3 will be plus minus 1 plus c 33 divided by 2 under root. So, all of them having plus plus minus sin, so this is the ambiguity, so with this value we get this ambiguous case which will need to resolve but, this so happens that this can be resolved by using the hog diagonal terms. So, if we look into the hog diagonal terms we go on the next phase.

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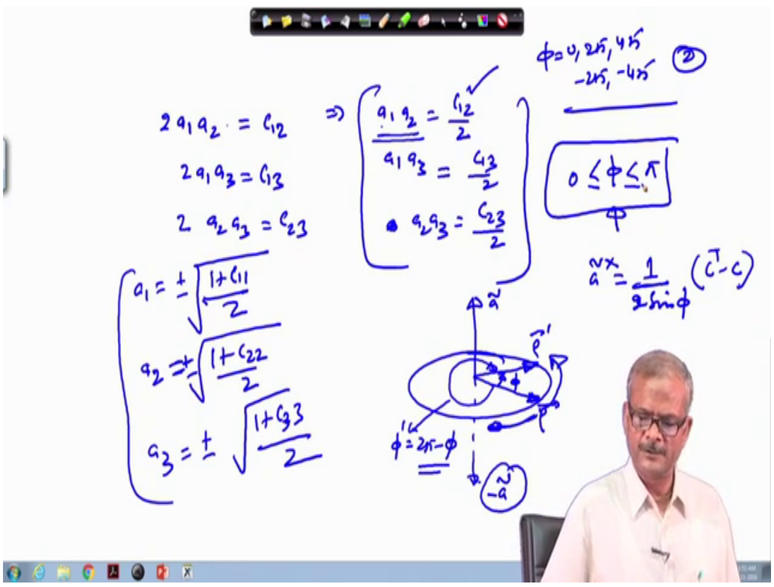

So, for the hog diagonal terms we can have 2 a 11, a 1 a 2 minus 0 a, 0 we already removed. So, we removed here also this get reduced to c 12. And this implies a 1 a 2 equal to c 12 divided by 2; similarly we will have a 1 a 3 from the other terms.

So, here a 2 a 3 equal to c 23 divided by 2 and this gives us c 13 divided by 2. So, now a 1 and a 2 this c is fixed is not it because this is given C matrix is given. So, this right and side is fixed and left hand side in the previously we are getting a 1 equal to plus minus 1 plus c 11 divided by 2 under root. So, here this can be (Refer Time: 06:13) there and let us say the a 2 also the same way 1 plus c 22 divided by 2 under root with plus minus sin, ok. However, if you look into this multiplication whatever the sin of this c 12 this value suppose this is 1 for minus phi whatever say accordingly a 1 and a 2 must be chosen. So, using this will be able to resolve this ambiguity at least here in this case.

And therefore it is not as problematic as the case with phi equal to 0 2 phi 4 phi or minus 2 phi, minus 4 phi etcetera. This case is ambiguous, but this ambiguity is resolved and therefore, a 1 a 2 a 3 and the pi can be decided how much it is. Now, some other issue also we have to look into the issue with the involved with the pi angle. So, there can be an ambiguity with the pi angle itself because, if we look here say for if this is my a vector and from here I give rotation from this place to this place by angle this pi, ok. So, in the same thing can also be achieved by this is your initial position and here we are currently. So, if we go doing like this first we have rotated here from this side to this side, this is rotation like this.

And if we rotate this way, so, along the minus a tilde axis ok; so if we rotate along this axis by say phi prime from here to here phi prime equal to 2 point minus phi. So, we get the same vector. So, here the problem arises because, the same position can be achieved either by through rotation phi here this way or either through rotation given in the opposite way which is another rotation is given this way what about the minus a tilde have both are going to the same position. So, this will create problem.

So, to restrict this to resolve this problem it can be done that this is restricted to phi is restricted to 0; to phi is restricted to 0 to phi. So, whenever this problem arises we need to resolve it carefully to work out this particular solution. And if you do this then ours whole ambiguity will be removed and we able to the problem correctly and this part this is also obvious from 1 by 2 find phi times c transferred minus c.

So, what we can see that if phi we give sorry we get certain a cross if we give 2 pi minus phi here in this place. So, we will get a minus sin. So, that rotation is just opposite one, and both will lead to the same position. That is if this is say (Refer Time: 09:46) root vector. So, here it will be the root line vector. So, either you go this way or either you go this way both are possible and to resolve this ambiguity therefore, we are keeping this phi values whenever possible of course, whenever possible to restricted to 0 to pi. Otherwise, on the case to case bases we need to discuss resolved this particular problem.

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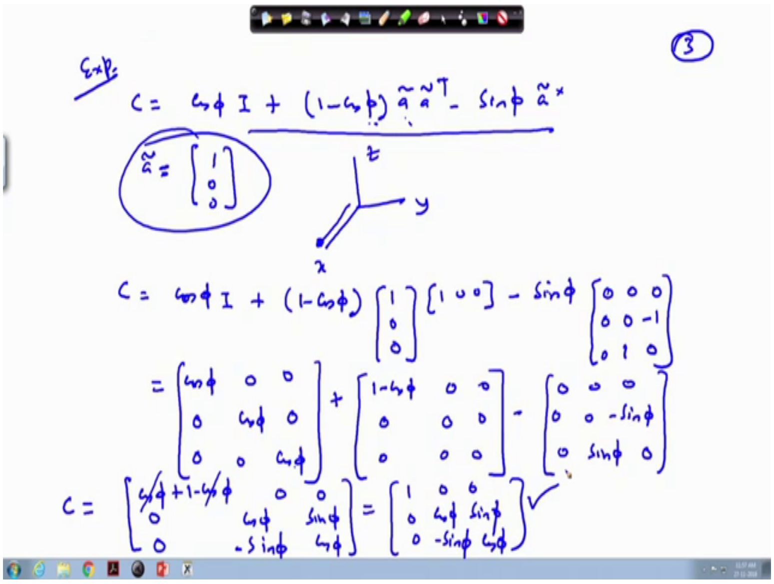

And as an example lastly we consider this matrix, consider that a tilde is given to be 1 0 0 ok. So, that means, we it is a just like we have the x, y and z axis. So, this is indicating vector along this direction, ok. So, a is a unit vector this is laying along this direction.

Now, we resolved to this so, this becomes cos phi I plus 1 0 0 a cos. So, a cos the diagonal element will be 0 and here minus a 3 which is 0 a 2, which is also 0 and a 1 with minus sin here this is minus 1 and here we are getting that reason one all other terms are 0. And if we add all of them, so this is you are cross phi 0 0 0, cos phi and this will give you only one term here, 1 minus cos phi which will and rest other terms will be 0 and in the same way here 0 0 minus sin phi and 0 sin phi 0.

So, if you add all of them. So, this will yield cos phi plus 1 minus cos phi and then this term is 0. So, that term is removed rest others they are 0 0. So, this is 0 0, similarly here 0 0 0 so, this terms remain 0. The second term cos phi 0 here it is 0, second term is cos phi and third term here 0 0 and this is minus sin phi and this minus minus that gets plus. So, this is sin phi and again this term will be 0 0, this is 0 0 and this is sin phi. So, sin phi comes here and lastly this cos phi 0 and 0 so, this is cos phi. So, here this comes with minus sin so, we are putting minus sin. This, this term it cancels out and that lift it with 1 0 0 0 cos phi sin phi, 0 minus sin phi and cos phi.

So, we considered this C matrix if a is the vector given x axis. So, this get reduce to this format and we are we are from earlier discussion this is nothing but rotation about the x axis y angle phi, so this is what this theorem states. So, this is the rotation about the vector a, by angle phi and exactly what we are getting here in this place.

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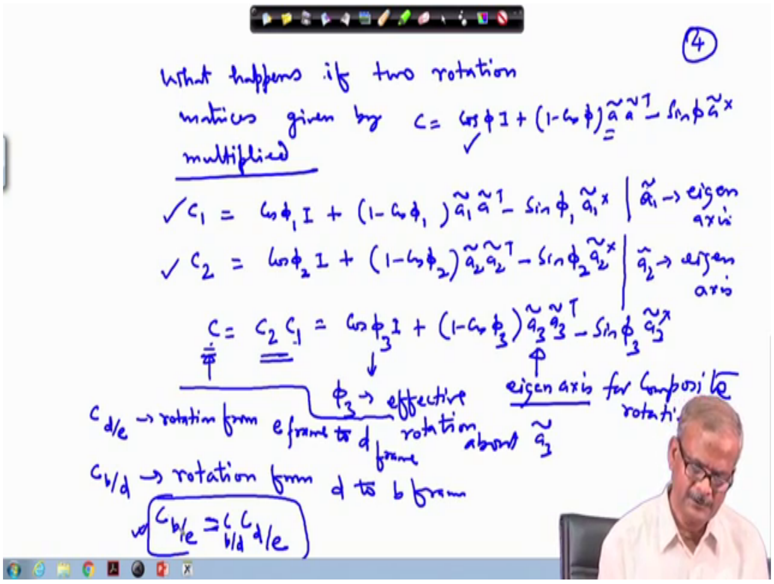

Now, the question arises what happens if two rotation matrices given by C equal to cos phi I plus 1 minus cos phi sin phi a tilde the transpose plus sin phi a tilde cos. So, these are multiplied, means we have composite rotation.

So, if this rotation indicates rotation about the a tilde axis y phi and we have another rotation. So, let us say that c 1 is one rotation which is defining we defined as cos phi 1, a 1 tilde a 1 tilde transpose minus sin phi 1 a 1 tilde plus. And another rotation matrix is been defined as so, if the give this rotation first, that is rotation about the a 1 Eigen axis; this is rotation about a 1 tilde Eigen axis.

Next, we give rotation about the a 2 tilde for the same body, a 2 tilde Eigen axis. So, obvious this one is composed of 3 rotations in general, this also it is a composed of 3 rotations. So, if we operate like this. So, this is composing of 3 rotation this is composing of 3 rotation, so total it is a composite rotation. One rotation must total it is equivalent to single rotation. So, from here also if we multiply them, so it should term out that this is a value into single rotation which can be written in the format, we should be able to write in a format as a cos phi 3 I plus 1 cos phi 3 a tilde 3 transpose minus sin minus 3 it a tilde 3 cos.

So, where which the tilde 3 this is the null Eigen axis for the composite rotation. And phi 3 this is the effective or the resulted rotation this operating by c 1 and c 2 rotation matrix. This is the effective rotation about a 3 tilde. However, proving this it is very strenuous and mathematics is very long, so rather will put is has the tutorial problem and here I will give you the final result.

So, obviously, if this is the case data I will representing this by this operation of 2 matrixes c 2 operating on the c 1, then it this also phi all the properties of the rotation matrix, ok. This needs to be proved. So, it be tried to do that, so this we have written as let us say the fist rotation we are writing as c d e which is the rotation from e frame to d frame. This is the rotation from e frame to d frame and then from d frame to d frame. So, d is the intermediate frame d to b frame. Means c b e we can write as c b d and c d and this must satisfy all the properties that we have proved earlier for this to be a rotation matrix.

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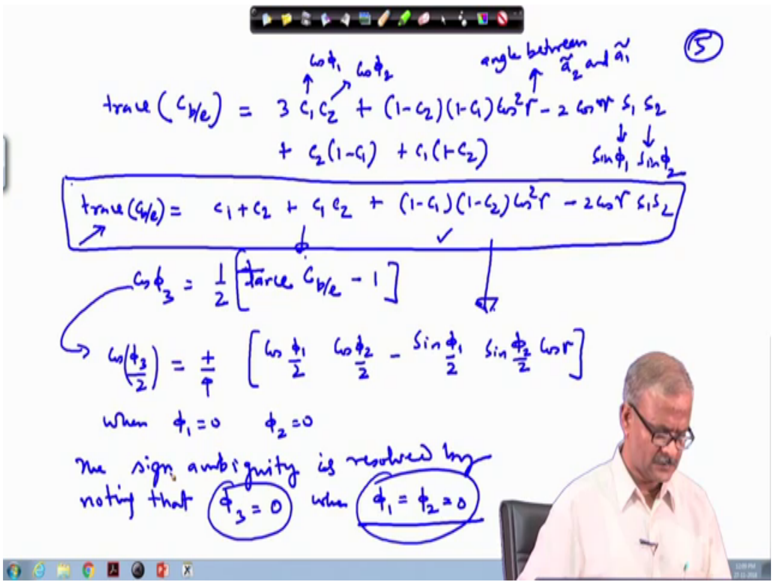

So, for the composite rotation this c b slash c it can be proved that will set as a tutorial problem this 3 times c 1, c 2, where c has the usual rotation of cos. So, this is indicating cos phi 1, this is indicating cos phi 2, where phi 1 and phi 2 are the rotation about the a 1 vector and the a 2 vector respectively gamma this is the angle between a 2 tilde and a 1 tilde. So, in the early decision they getting sin phi 1 and this is phi 2 ok. And lastly we have we have 2 terms plus c 2 times 1 minus c 1, c 1 times c minus c 2 and this can be further simplified.

So, if we try to simplify this, this can look this format c 1, c 2. So, this is trace c b slash e. We have a number of indent cities need to be used to prove this final form and it is a various viewers. So, we are a wording this and will take up this as a tutorial problem rather than doing it in the as lecture problem ok. So, from this place then you will get cos phi 3 as usual 1 by 2 this trace of c b slash e minus 1 ok. It is possible to put this in little more simplified format and this can be done using some simple rules, but for the time viewing we are not going to fonder over all those things.

So, if we try to work out this where you remember that c 1 is cos phi 1, c 2 is cos phi 2 and so on and here s 1, s 2 they are appearing as sin phi 1 and sin phi 2. So, it can be proved that cos phi 3 divided by 2 this equal to plus minus cos phi 2 divided by 2 minus sin phi 1 sin phi 1 divided by 2. So, this can be reduced into this format. Remember that of this particular one. Using this, so insert this into this and then finally, this can be written in this format. So, from this place it is easy to work out this phi 3 and 2.

Now, if this plus minus sin what is appearing here this must be resolved ok. What we see that when phi 1 equal to 0 and phi 2 equal to 0. So, this part will get reduced to 0 and this part will get reduced to plus 1 ok. And therefore, so the sin ambiguity whatever he is a dishing here this can be resolved, because we are if you are not getting any rotation consider if we are not getting a rotation phi 1 and phi 2 they are 0, therefore, the total the composite rotation must be 0 there is a no rotation. So, phi 3 must also be 0. So, left hand side this trance out to be plus 1. So, right hand side also must be plus 1 and therefore, this sin gets resolved here. So, we pick up the plus sin from that place so, using this technique we will be able to resolved the sin problem.

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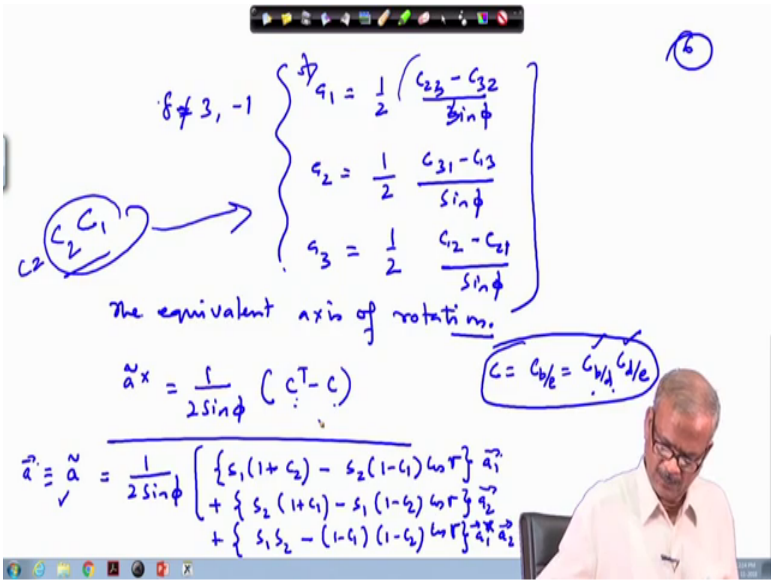

And in the same way then it you can define if delta as of here written delta not equal to 3 and minus 1. So, in that case you are defining this a 1 as 1 by 2, c 2 3 minus c 3 2 by sin phi and so on 1 by 2. So, whatever discuss, already we have discuss for single C matrix. Once we multiplied once the c 1 is multiplied with c 2. So, for that case also the same thing is applicable. So, here in this case this is referring to this case where you are C matrix, the final C matrix you are writing. So, components will be deterring in this way and a 1 is defined like this and vice versa the same way you can do exactly ok.

So, this process will be more it will be more consolidated once we look into the problem set the tutorial set once we try to work out, in fact do the mathematics so it will be much more the concept will be become much more a stronger.

For finding out this is the equation we have been using ok. So, for this here c is nothing but c b slash e which we written as b slash d, c b slash e. So, we need to find out and then need to take the transpose of this put here in this place and then workout with this to solve the problem. So, is very very very lengthy and various viewers ok.

This portion we are going to skip and we are reserving it for our this particular tutorial problems. I will write the final result here, it can be written as 2 sin phi, remember gamma is angle between vector a 1 and a 2 ok. So, this is what you get once you write it this format on the right hand side, this is consequence of; here we need to remove this put particular part ok. So, as a final result this is what we get, a tilde if you write in vector rotation, so simply you write as a. So, this quantity will be given here in this format.

Obviously, this is quite long a still, but seeing the whole thing the original if you try to prove these putting these C matrix from this place C transpose then C and trying to work out with stages over a number of phases. But it so happen that this can also be little bit more simplified and put into a simpler format.

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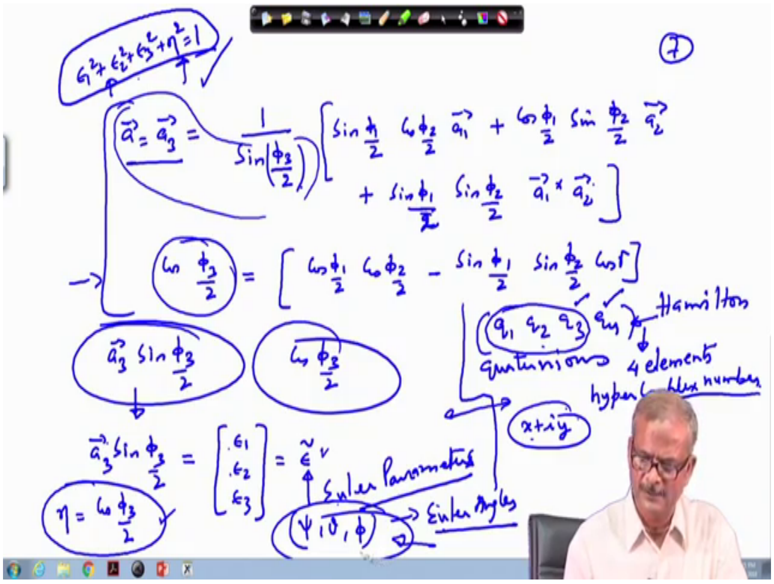

So, this a 3 matrix or the maybe a 3 vector or the a vector which is the composite axis Eigen value rotation, or composite Eigen axis. So, this can be written as 1 by the previous equation what we written it, it can be reduced in this format.

It is a little more simplified format here cross phi 3 divided by 2 this is return as ok. So, if once we know this, so we can observe something very interesting from this place that this suggest that a 3 sin phi 3 by 2. This is of special importance and similarly cos phi 3 by 2 this is our also special importance. And this week end generalizes a tilde sin a tilde 3 sin phi 3 divided by 2 this we can write in terms of epsilon 1, epsilon 2, epsilon 3, epsilon tilde. Actually here in this place were the quaternion rotation is start rather than telling that it is a quaternion these are call the Eigen parameters, because the quaternion, we use the rotation q 1, q 2, q 3, q 4. So, this is what quaternion and here it was developed by Hamilton ok.

And this is specially deals with the 4 elements; 4 elements hyper complex number. So, till 12th we done just like x plus i y, this we consider as a complex number ok. But here what we see that this is 2 element complex number ok, but this one is defined as 4 element complex number where this is the real part and this 3 happen to be the complex part. So, if so happens that whatever the equations developed using this they can also be derived using this, but this is basically the complex rotation and this we are working with the matrix rotation.

So, rather than calling this as the quaternion we call as the Euler parameters. However, if you of doing the mathematics, so if it equivalently you can say that we are working with the quaternion, but mistake since this is Euler parameter this set is which calls the quaternion which was obviously, developed by Hamiltonian ok.

So, if we defined epsilon tilde like this and then we defined eta as cos phi 3 divided by 2 which is appearing here in this place. So, this particular part and this part this is of special significance and what we see that epsilon 1, epsilon 2, epsilon 3 and eta these are the 4 parameters in world, but in rotation we have only 3 rotation angles in world which is psi, theta, phi ok. These are the only 3 Euler symbol involved, these are the Euler angle.

So, here one extra parameter we are getting but it so happens that as we see later that epsilon 1 is square plus epsilon 2 a square, epsilon 3 is a square plus eta a square this turns out to be 1 ok. And therefore, one of them can we expressed in terms of the other 3. So, only 3 independent parameters are there not 4. As here in the case Euler reprehensive, some 3 rote independent rotation can given, similarly so corresponding to that the 3 parameters should appear here. However, here it is 4 but 1 is depended on the other 3.

Now, advantages of using particular rotation is that it is free from trigonometric rotation. So, to get of the trigonometric rotation we can use the Euler parameter or the quaternion whatever we say equivalent, but not in the restriction. So, of our trigonometric process it is gets reduce very simplified as a big process. And therefore, in many problems where the large angles are involved rather than working in terms of psi eta phi we rather work in terms of epsilon 1, epsilon 2, epsilon 3 and eta which are the Euler parameters.

So, we will continue with this in the next lecture where we wind up these quaternion and whatever the mathematics we not develop, so if mathematics will go as part of the tutorial where we will be able to work out those. If you face any problem at that time, so will interact and then resolve all those problem.

Thank you very much for the listening.