Space Flight Mechanics
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Lecture – 01
Conic Section

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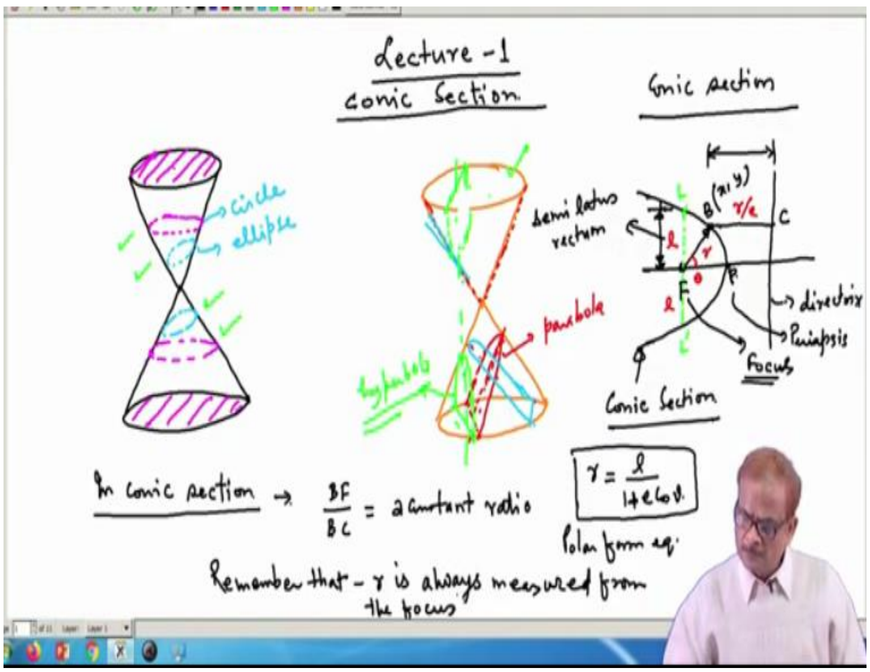

Welcome to the lecture number 1 on this space flight mechanics. So today, we will be discussing about the conic section. So, in this very first lecture, we will discuss in very brief about the conic section and more materials, you will find numerous materials, basically on internet you can look for or either many books are there and as such a very good book is by SL Loney on Coordinate geometry, there also you will find about the conic section.

So, today just for the introduction purpose, we are going to this conic section and there after whenever required I will discuss further about this conic section, to start with so, let us say that we have a cone here like this and another cone; so if we take a section of this cone like this okay, so if we pass a horizontal plane which is parallel to the base, this is the base of the conic section. Basically, these are 2 cones and base of the cones.

So, if we draw a plane parallel to this base of the cone either here or here in this place, so we get a circle and which is very obvious because your base is a circle and therefore, this will also be a circle. If we incline this little bit that is to say like if we make it like this, so this generates an ellipse, so this is your circle and this is ellipse now, if we take any section whose plane is parallel to this generator, this is the generator of the cone, okay.

This line; the inclined line is called the generator of the cone, so if you take any plane, which is parallel to this generator say, I pass a plane like this, okay so this will cut in this place and this place like this, so this is; this one and this one they are parallel to each other, so I can have a line like this, so you can see that the conic section which is get generated here, this we call as the parabola.

Remember that this is parallel to the; this plane is parallel to the generator, one of the generator, so either it can be this way or either it can you can draw it from this side, okay, so either this way or another colour I can show it like this, it can be parallel to this generator, so if it is parallel to this generator it will go like this. On the other hand, if we draw any section; if we take any section of these two cones, so we get hyperbola.

So, hyperbola say if I take a plane like this and then we; this plane is cutting like this and here also it is cutting like this, so this set whatever we are getting here, this is your hyperbola. So, other than and what the plane which is not parallel to the generator, it will generate hyperbola and you can see the case of the circle and ellipse, here this is confined only to one of the cone, okay here either it is confined here or either it is confined here.

While in the case of the parabola, we are taking 1 plane, so we are getting this or either we can get it on this cone but in the case of the hyperbola you are getting here and here also in both the places by the same plane, you will get here in this place and this place also. So, with this brief, so let us go into how do we define the conic section, this figure is little distorted and so now, we go into the definition of the conic section.

In the conic section, if we take any point what we have shown here this curve, this is your conic section, this can be ellipse, this can be part of an ellipse, this can be parabola, or this can be hyperbola, so this is conic section. So, in the conic section, if we take any point on this curve, so let us say this B is coordinated x, y, so this distance; the distance from this place to this place; B to C and BF, they bear a constant ratio.

$$\frac{BF}{BC} = \text{Constant ratio}$$

So, the conic section definition will write this as conic section BF by BC, this will be in a constant ratio and this line we call as a directrix, here this is called the focus, this point which will write as P, this is called as the periapsis and of course here, your *r* is the in magnitude wise, this is the distance from the focus to the point on the conic section and this l we call as the semi latus rectum, so this is the distance from this point to this point.

Later on what we will learn that this conic section equation can be written as

$$r = \frac{l}{1 + \mathrm{e}\cos\theta}$$

so this is a general equation in polar form and remember that r is always measure from the focus which is shown here, F is the focus; always measure from the focus, okay.

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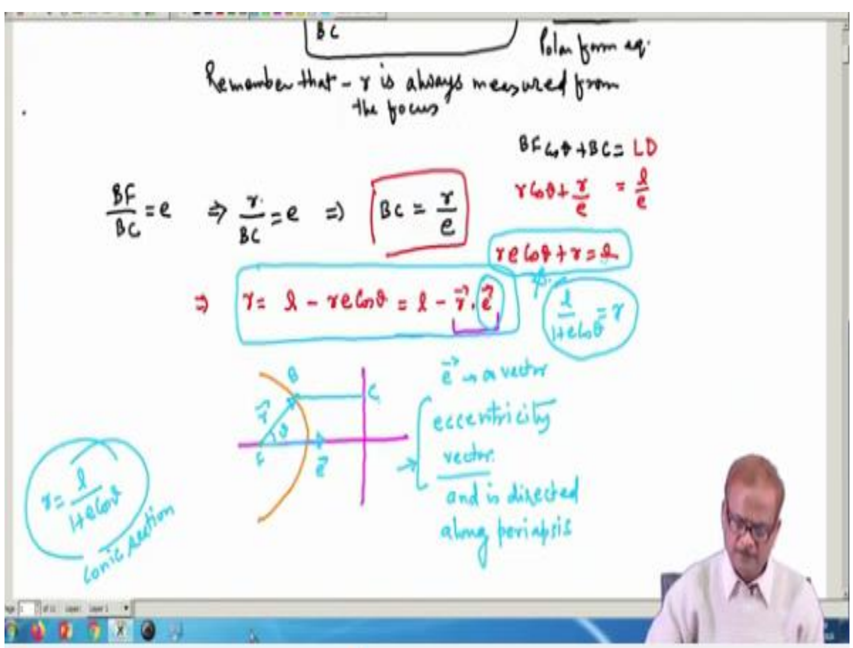

So, we have BF by BC as we have shown here in this place; BF/BC is a constant ratio and this ratio we write as e,

$$\frac{BF}{BC} = \mathrm{e}$$

so your BF in the figure; this is r, if you look here this is r. So, this implies,

$$\frac{r}{BC} = \mathrm{e}$$

and this implies,

$$BC = \frac{\mathrm{r}}{e}$$

Now, look here in this figure, the total distance from this place F to C, from F to C means the distance taken from this place to this place.

This, I am not talking about this inclined distance, I am talking about this distance, so this distance is your; you can write this as BF cos theta and plus BC and this nothing but your we can write this as, here let us make another line here, so you can take it from this place and this will be your l by e, why? Because if you look this point is also on the conic section, so by the definition of this conic section, BF instead of your BF now, this will be your LF.

So, LF is nothing but your l, semi latus rectum already we have written and this point let us write this as the D, so LF/LD, okay, so here

$$\frac{LF}{LD} = \mathrm{e}$$

and therefore

$$LD = \frac{1}{\mathrm{e}}$$

so we are using this information here, so this equal to LD here in this place, okay and this LD is nothing but l/e and BF is nothing but your r, so r cosθ and BC this is r/e, already we have written, this is r/e, so here we write as r/e.

We will simplify it little bit, so we can write it as

$$re\cos\theta + r = l$$

and this implies

$$r = l - re \cos\theta$$

and this can also be written as a dot product, this is $re \cos\theta$. So, what we observe from this place, this particular quantity, $\vec{r}.\vec{e}$; so r is a vector, so let us draw another figure. If this is focus, this is your $\vec{r}$, this point you have taken as F and this we have taken as B and here we have taken this as the point C, so this is your $\vec{r}$.

And θ angle we have indicated like this, so that implies that the $\vec{e}$ is lying along this periapsis, so here $\vec{e}$ is directed along this line, okay and later on, we are going to prove this fact that e is a vector, so a vector, so eccentricity is called the eccentricity vector. So, once we advance, so we will prove this fact that e indeed is a vector which is lying along the periapsis and falls along or directed along the periapsis.

So, this is one of a very simple representation of the conic section and this we have got from this place, so what we can do that we can rewrite this, if we write it like this;

$$\frac{l}{1 + e\,cos\theta} = r$$

we take r as common in this particular equation and write it like this, so we also get it in this form, so

$$r = l/(1 + e\,cos\theta)$$

this is also a representation for the conic section and this is valid for both for ellipse, parabola as well as hyperbola.

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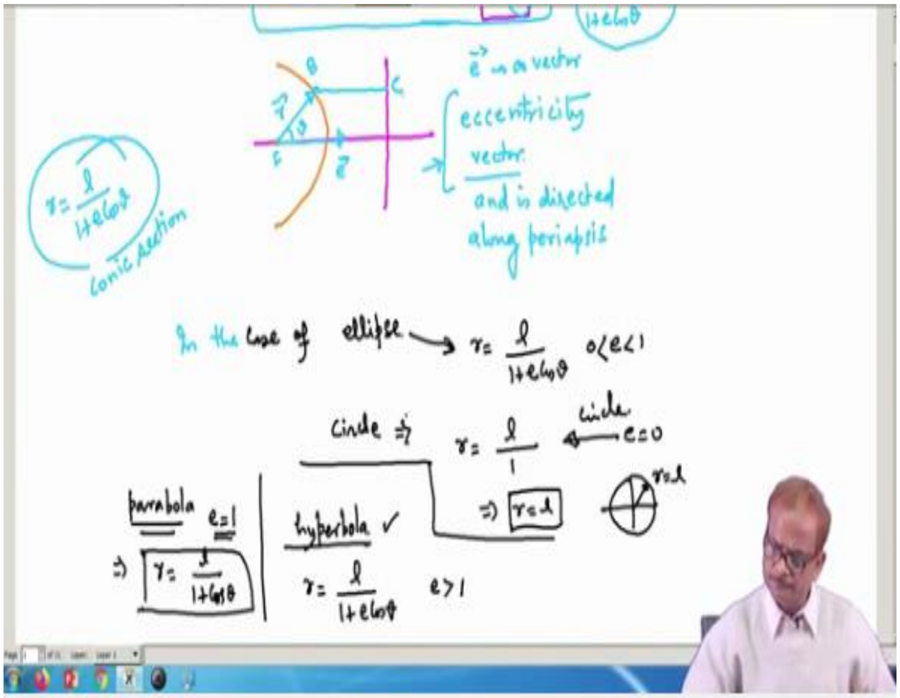

So, in the case of ellipse, we have

$$r = \frac{l}{1 + e\,cos\theta}$$

so where *e* is < 1 and > 0,

if we have the case of the circle, so in that case your *e* will be equal to 0, so for the circle

$$r = l/1$$

, so this implies $r = l$, so that happens in the case of the circle that your radius of the circle this is constant, $r = l$. On the other hand, for parabola $e = 1$ and so this implies that

$$r = \frac{l}{1 + \ cos\theta}$$

In the case of hyperbola, *r* will be given by the same equation but here, *e* will be greater than 1, so we can see that this case is this is for the ellipse, this is the case of the circle, here this is the case of the parabola, where eccentricity is equal to 1 and this is the case of the hyperbola, where eccentricity is greater than 1.

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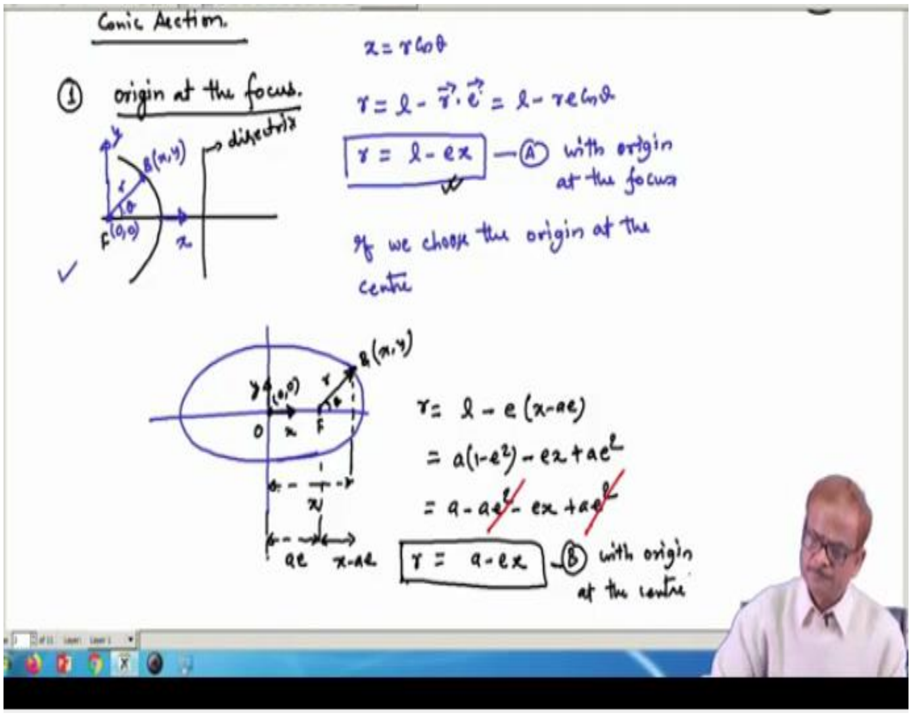

So, this is our basic definition for the conic section now, we can advance little bit in this area but I am not going to discuss in great details because in mathematics you will find many books, where this conic sections have been discussed in great details, so for our purpose, we require some introductory knowledge about this and later on as and when required, so I will introduce you the further concepts.

Now, for the conic section we can take origin at the as we have shown earlier, origin at the focus, so if we take origin at the focus that means this is your directrix, here the focus is located, this is *r*, this angle is θ, so taking focus at this point and let us say this point we have earlier chosen as B, so we will write this as B xy, this xy and this point we can show as 0, 0, so this is in a plane, here we have shown this figure in a plane.

And therefore, the only in terms of x, y coordinates otherwise if you have 3 dimensional, so in that case you will not have ellipse but it will be have called ellipsoid, if you have parabola in the 2 dimensional plane, we will call and say dimensional plane as a paraboloid and so on, just like in 3 dimension we have a sphere and 2 dimension we have the circle but here we use this term like the ellipsoid and the paraboloid and so on.

So, if we measure distance from this focus F, so any distance towards the right we take this as the positive and this is your x axis and vertically here in this direction this is your y axis. So, here in this case, x as it is apparent this will be

$$x = r \cos \theta$$

and therefore the equation of the conic section that we have written

$$l - \vec{r}.\vec{e}$$

means

$$l - re \cos\theta$$

so this can be written as

$$l - ex$$

with origin at the focus, okay. Instead, if we choose the origin at the centre, so we have this case say, we have the case of an ellipse, this is the focus, this is r, this is theta, okay, instead of choosing the origin here in this point at F, we choose the origin at the centre.

So, let us say this point is O, so we measure the distances along x here and y along this, so therefore the coordinate of this point with respect to this origin which will be marked 0,0. this will change, so now the distance to this point B is still the coordinates will be x, y, okay, so this distance is x but this distance will be a times e, this is still I have not shown you, so this distance is a times e and therefore, this distance will be $x - ae$.

And then we use this equation, so we can replace this as $l - e$, now x will be replaced by $x - ae$, also we will be proving that

$$l = a(1 - e^2)$$

which we have not done still, we expand and write it like this, so this becomes

$$a - ae^2 - ex + ae^2$$

these two terms they cancel out and what we get here $a - ex$, so this is the description with origin at the centre.

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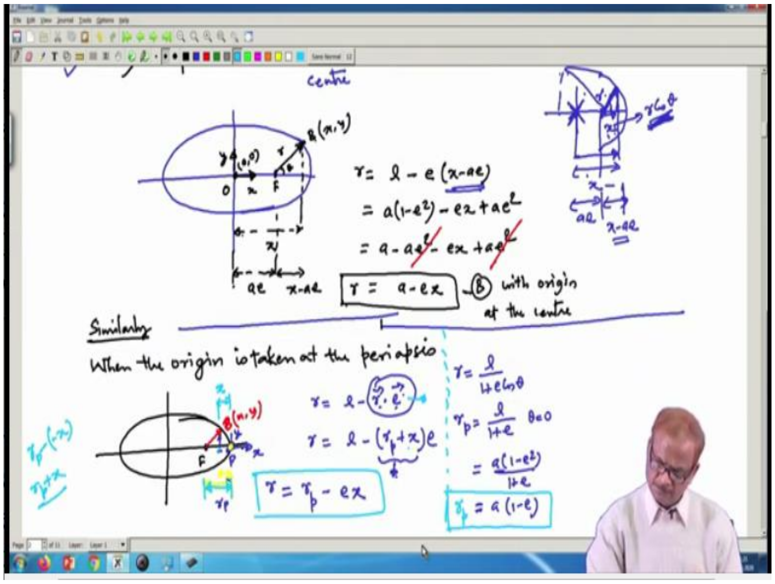

So, similarly when the origin is taken at the periapsis in this figure, this is your focus, *r* is the distance to the point B, x, y always the coordinate will be written in terms of x and y but this time the distances are measured from this place, this is your perigee position, so along this we measure x and along this, we measure y, so we can see that we have to measure distances from here and then go up toward to this point but the equation for; this basic equation remains the same,

$$l - \vec{r}.\vec{e}$$

there is no change in this.

So, for this particular case this can be written as now, this r can be replaced by *r* perigee; *r* perigee is the distance from this point to this point and plus x; $r\,perigee + x$, also we can see that this from the equation for the r, this is the equation for the conic section in this case, we have represented here ellipse so, r perigee this will be equal by $1 + e$, because here theta equal to 0 for this point, okay here we have written t and l already we have written as a times $1 - e^2$, so this becomes $a(1 - e)$

Also, so in various ways we can express this but here the better way to write this is as $r_p - ex$ and of course here, *e* we are missing, so what we have done that this is now representing the; see the; because we are shifting the centre origin, so initially what we did that we have this has the distance x but later on once we shifted to this place, so we are measuring this place from this place, so this does not change, here what we have written as $\vec{r}.\vec{e}$, so this $\vec{r}.\vec{e}$represents this is your r and this is θ.

So, what we have represented this x here, so this is representing

$$x = r\,cos\theta$$

so this is not going to change, this is always it will remain same, if you take it on another side, on this side, so obviously the projection will change but this representation is not going to change, this will be $r\,cos\theta$. So, in that case what we have done; because we are measuring distances from this point and this is your distance x.

So, here we have subtracted from this x, this particular distance which we have written as ae, so this we have written as this particular part, from here to here, $x - ae$ and this is what enter here in this place while here in this case, we are measuring distance from this place, I will use another colour, so we are measuring distance from this place, so we have to look for this distance, how much this distance will be.

So, if you look for this value r_p is from this place to this place, this is your r_p, okay from here to here, this is your r_p, this colour is better and this distance is your x but here in this case you are taking it on the negative side, so we put here a minus sign, okay so this is $r_p - (-x)$, so that becomes $r_p + x$ and accordingly we have written $r_p + x$ and e remains as usual, so this is a very simple thing, you can work it out yourself also.

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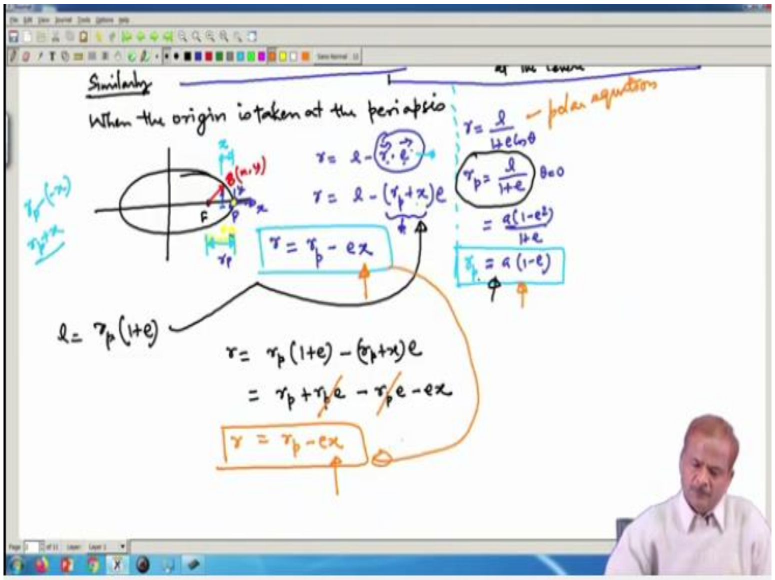

So, according to this particular part ultimately, we can reduce the equation here in this format, okay, so we can go on the next page, so we have taken l equal to this particular part, we can write it this way also and also we can write

$$l = r_p(1 + e)$$

and if we insert in this place, so this becomes

$$\mathrm{r} = r_p(1 + \mathrm{e}) - (r_p + \mathrm{x})\mathrm{e}$$

And expanding it and then these 2 cancel out and this gives us $rp - ex$, so either you can express it in this way also we can express in different ways as we use different equations. So, bisecting the origin from focus to periapsis or either to the centre, we can see that how the form of the equation representing r with changes, so this is our polar equation but here the Cartesian coordinate are involved. Here you can see that x is involved here in this place, here also the x is involved okay, these two are the same equation, so we have to go on the little bit up, so here also the x is involved.

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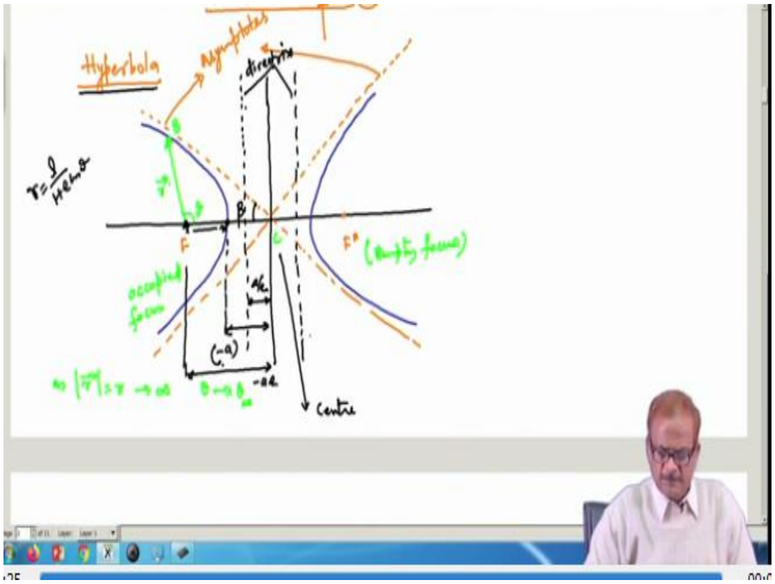

So, we will wind up this topic with the small introduction to the hyperbola, so we have already looked into the fact that how the hyperbola is generated from two cones, these two are asymptote, this angle we will let us say this is focus F and here another focus F star let me show, so any point on the hyperbola as I shown in the same way, this place is called the occupied focus and this is called vacant focus or empty focus.

This angle we show this as theta as your r tends to infinity, so this is your r, so as r magnitude equal to r, this is tends to infinity, so theta will be designated as theta tends to theta infinity, so this point we writes as C and this is your centre and these 2 lines, this is a directrix, as usual from here to here this distance we call as the periapsis distance, okay I will not go into much detail of this because later on we are going to discuss about the motion of the particle.

So, under one such describing a hyperbola, so at that time again we have to take and repeat the same thing, so I am just skipping this particular part at this stage, so the distance from centre to the periapsis, this will be written as minus a, this distance to the directrix, this will write as a/e and the distance from here to here, this will write as - a times and this angle we will indicating as beta and later on also, we will use the same notation.

And the same thing is applicable also on the right hand side of the this axis, so here also we can measure the write in terms of the Cartesian coordinates, the polar notation already we have noted that this

$$r = \frac{l}{1 + e\,cos\theta}$$

So for that you have to just use this coordinate if you want to convert into the Cartesian coordinates accordingly, you have to go through it. So, here we stop and we will continue in the next lecture, thank you very much.