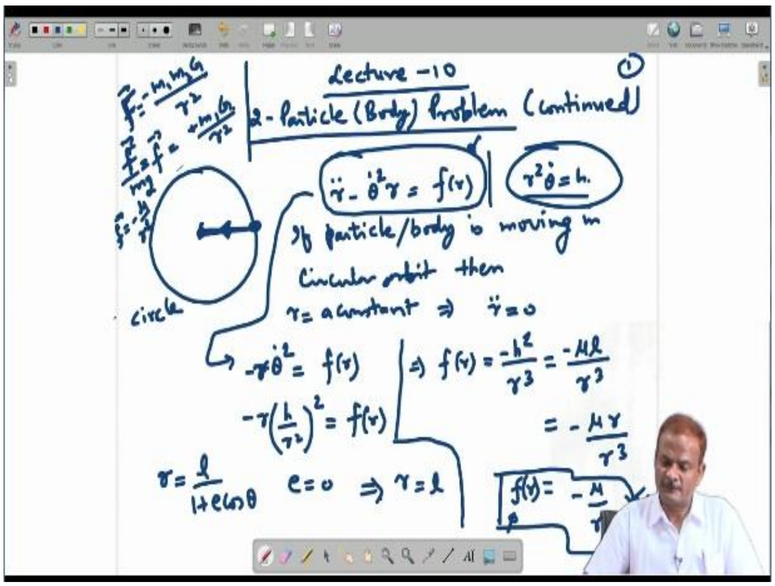
Space Flight Mechanics
Prof. Manoranjan Sinha
Department of Aerospace Engineering
Indian Institute of Technology - Kharagpur

Lecture – 10
2-Particle (Body) Problem (Contd.,)

Welcome to the 10th lecture. We have been discussing about the 2-Particle or 2 body system, so we will continue with that.

(Refer Slide Time: 00:19)



So last time if you remember we were working some; one problem also and then we discussed some theory also. So few things related to that theory whatever we have done, so we will discuss again before we take up the original subject. So if this is a circular orbit and there is a particle here which is moving in this circular orbit, okay. So as per our discussion we have got

$$\left(\ddot{r} - r\dot{\theta}^2\right) = f(r)$$

and r squared $\dot{\theta}$= h, this is what we have derived, okay.

So if particle is moving in circular orbit; orbit then $r = a$ constant and therefore this implies $\dot{\theta}$ this will be 0, so this equation then gets reduced to $r\dot{\theta}^2$with minus sign this equal to f(r). So this is moving under gravitational attraction. So from here we are trying to find out what will be the rule for gravitational acceleration if the particle is moving under this gravitational acceleration in a circular orbit, so this is circular orbit. This is circle, okay.

So theta now will utilize this equation and insert it here, so r $\dot{\theta}$to h/r^2, this is whole square equal to f(r) and this implies

$$f(r) = -h^2/r^3$$

And as we have written

$$h^2 = \mu l$$

so this is; and this will be the minus sign here, minus sign here. Also, we have derived that

$$r = \frac{l}{1 + e\cos\theta}$$

So here in this case, in the case of a circle $e = 0$ and this implies r will be equal to l, so if we insert here in this place then this is $-\mu r/r^3$ so this is $-\mu/r^2$, so this gets reduced to your usual inverse square law.

So that means if a particle is moving in a circle and this force is toward the center, okay and particle is moving in a circle so that is bound to be the inverse square law, which is your gravitational force equation that you write or

$$\vec{F} = -\frac{m_1 m_2 G}{\boldsymbol{r^2}}$$

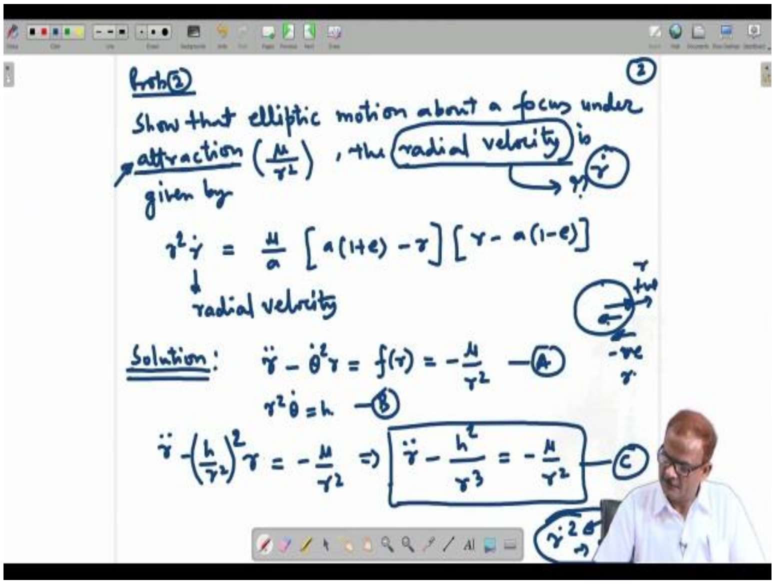
which a minus sign you are writing, so here f / m if we divide so this is that unit which been taken here, this is per unit mass so maybe if we can change it, so $f/m = f$, this gets reduced to m_1 will divide it by m_1 or either m_2, let us divide by m_2, so this is $-m_1$ G / r^2 which is nothing but $-\mu/r^2$.

So this is therefore

$$f(r) = -\frac{\mu}{\boldsymbol{r^2}}$$

Okay, so using this rule we can work out any problem so this constitute, this equation constitutes your basics of the central force motion. This has nothing to do with the, this gravitational force or any other thing, this is just central force motion, and depending on what kind of the function f(r), so the whole thing will your orbit will determine it will; on that your orbit will depend.

(Refer Slide Time: 05:41)



Okay, so in this context we take up one more problem, okay and thereafter we will go to the theory. So we have here, show that, $\frac{\mu}{r^2}$. Here attraction is written so only $\frac{\mu}{r^2}$; actually we should put a minus sign here, so that we will do later on. Once we are writing attraction means it is always; this will be negative. So this is what we need to prove. So what is here, this part is your radial velocity.

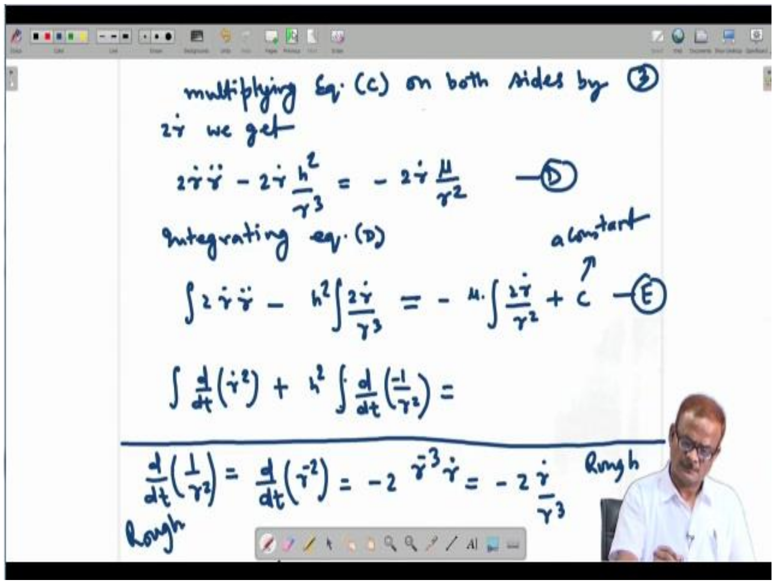
We can divide by this r^2 here and this will be μ/ar^2 so this is the expression for the radial velocity. So we have start with the same condition what we have got $\ddot{r} - r\dot{\theta}^2$this equal to f(r) which in this case given to be $-\frac{\mu}{r^2}$ because this is attraction, okay. Attraction means it is directed toward the center and therefore minus sign will come. This is positive direction of r and this is negative direction of r.

So in this direction this is negative r, positive r, okay. So we are putting a minus sign to indicate this is attraction. And another equation we have $\dot{\theta}^2$ this equal to h so we insert here in the upper equation, so $\ddot{r} - r\dot{\theta}^2$, $(h/r^2)^2$. So r^2; and here r is there so we will put this

$$\ddot{r} - \frac{h^2}{r^3} = -\frac{\mu}{r^2}$$

So see in the question it is asking for computing the radial velocity means it is asking what is $\dot{r}$, so we have to work out $\dot{r}$ But here the expression we are getting after combining this a and b, okay we are getting this expression c. In expression c, nowhere we have $\dot{r}$. See $\ddot{r}$is present, but we need to integrate it ones in order to get $\dot{r}$.

(Refer Slide Time: 10:13)



So multiplying equation, okay multiplying equation *c* by; now *c* here one thing we can guess from this place, if we try to integrate it, so this $\ddot{r}$this must come from $\dot{r}^2$ square, so if we differentiate this so this will result in 2$\dot{r}$times $\ddot{r}$, okay. So what we will do, that we will multiply it with 2$\dot{r}$ and then integrate it so that after integration we get this quantity, okay.

So multiplying equation c on both sides by 2$\dot{r}$, and this technique you should to remember where we are taking, you might be seen quiet often we have been using dot product with $\dot{r}$or cross product with r, here I am taking simply multiplying with 2$\dot{r}$so these are the techniques of solving the problem, okay. If you remember this technique so you may be able to solve a number of problems which will be in your exercise.

So we get

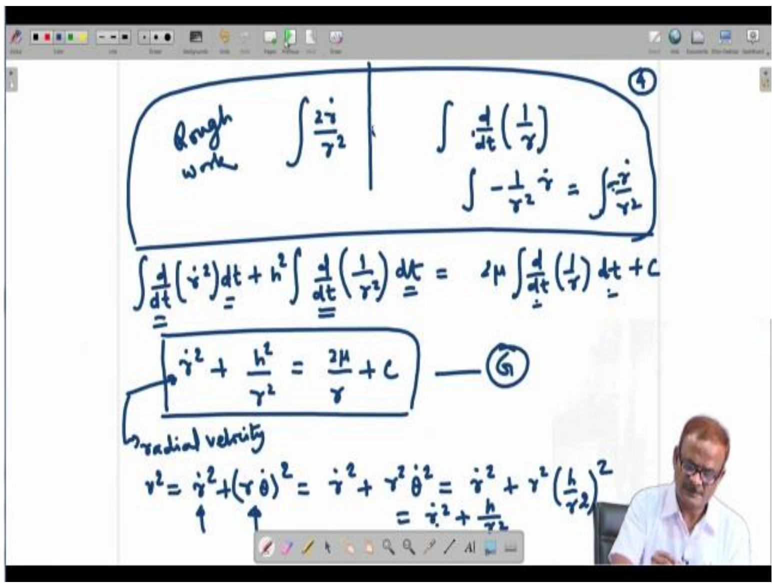
$$2\dot{r}\ddot{r} - 2\dot{r}h^2/r^3 = 2r\frac{\mu}{\boldsymbol{r^2}}$$

So integrating equation (D), so if we integrate it so 2$\dot{r}\ddot{r}$– h^2we will take outside, $h^2$2$\dot{r}$/r^3 and this will be equal to –μ 2$\dot{r}$t/r^2 and plus *c* which is a constant or let us; okay we will keep this as c only, so this is (E) so this is a constant.

So if we integrate it as I told you we can write this as d/dt $\dot{r}^2$. This will be –$h^2$and here if you see this will be d/dt 1/r^2. So we will explore this term and what will the sign here in this place, so we are going to explore it. So we will do some rough work, we can do here on the bottom itself d/dt $1/r^2$so this will be d/dt r^{-2}, so once we differentiate this, this will be -2 here and then r^{-3} times $\dot{r}$. So this gives us -2 $\dot{r}/r^3$.

Okay. So this way this; that means this minus sign should be placed inside, so here instead of putting it minus we put it plus here and minus sign we faced it here in this place. And on the right hand side this is a rough work. Again on this side we have to look into this what the result will be, so for that we need to go on the next phase and work it out and then I will return back to this place. So this is $2\dot{r}/r^2$.

(Refer Slide Time: 16:22)



2r $\dot{r}/r^2$ this integration. So we take two outside, if we write d/dt okay forget this. If we take the quantity 1/r here d/dt (1/r), this two also will remove and then check that what we are getting here in this place if we break it, so this will be $-1/r^2$ times $\dot{r}$t. So this is nothing but your $-\dot{r}/r^2$ while here you have $2\dot{r}/r^2$. So this is you are having rough work.

So here in this case this becomes -2μ; there is a minus sign is appearing here, if we differentiate this 1/r so if you now integrate it, see here in this case what we are doing, that once we have written it like this on the right hand side also we will write in the proper weight -2μ so this is 1/r d/dt (–1)/; +1/r. So if we write it in this way, okay. So from here if we differentiate this is $-1/r^2$ $\dot{r}$, okay 2 is already there μ is already there so this minus sign should get to plus and here we have C. Now we can integrate this expression. Okay.

One more thing is required to do here in this place is that we have to; see what we are doing that we are trying to differentiate this, so ones we have to remove this integral sign so what we need to do. This is the question here. So I will copy the expression from the previous page here and thereafter we will work. This is d/dt integration $\dot{r}^2$square; okay + h^2 d/dt, here so if we differentiate it so that becomes plus, so here the; we have done the mistake here, this is the minus sign.

See once we are observing the minus sign here, okay so; this place what we are doing that if we differentiate it so we can see that this will be $-1/r^3$, we have done this work here somewhere. This is in this place. So once d/dt $1/r^2$ we are doing; so d/dt r^{-2} so this is coming in this way, okay. This is $-2r^3$ divided by times $\dot{r}$, so finally we are getting this expression.

So that means already if we differentiate this so we get a – sign here so I need not take here; here in this place we will not take a minus sign here in this place and in this place we will put again we will put a plus sign. It is not that we observe it here and then putting that; now you can see, the quantity here which is present here, if I differentiate this so this is the way we are writing so this minus sign here appears, this is the – sign, so this – sign will ultimately manifest here in this place, okay.

$$\dot{r}^2 + \frac{h^2}{r^3} = 2\frac{\mu}{\boldsymbol{r}} + C$$

So now this is okay. And on this side also we have written it the same way so once we differentiate it so this quantity then the – sign will appear in this place. So; and one more thing we will do; let us first remove this. This integration sign first we are removing, okay. And this c also we will remove, to make it more consistent this is removed. So you can look now that this part, this particular expression, this can be written like this. Okay.

Similarly, this with; this whole with – sign can be written as this one, okay with whole this plus sign, okay because once we differentiate it this integration sign also will not be present this we have to remove, okay and here also we are removing the integration sign, this *c* also we will remove. So once we are working out like this so you can see that if we just operate this differential operator okay, so if this quantity will result, the upper equation E will result so this equation we will name as; this is your equation number (E), this is (F).

Now we can do the integration with respect to dt, okay. On both sides we can operate with respect to dt, this is the proper way of doing it. So now we go to the next page. So on the next page then we have again with this sign first here integrate it with respect to dt , + h^2 $1/r^2$ dt h^2, so this quantity we are integrating then this is 2μ integration sign d/dt 1/r, d/dt 1/r dt and then the constant sign will be introduced.

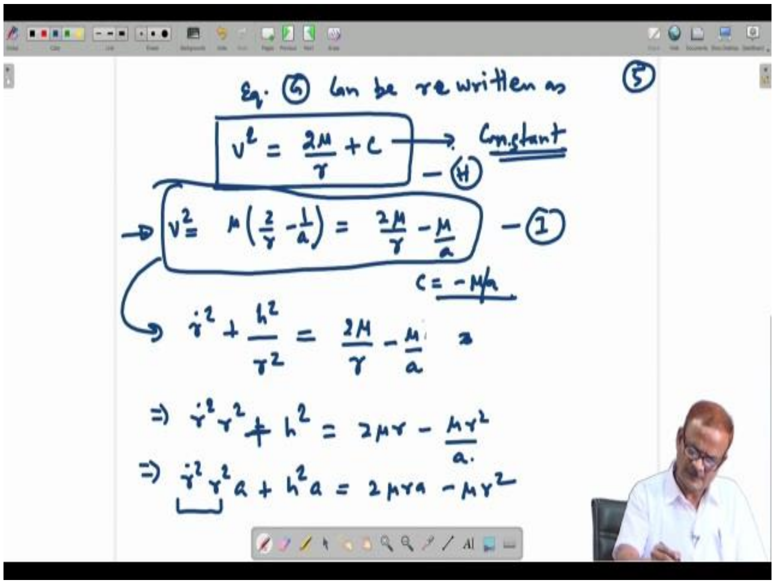
So this is the way of doing it, okay. So this dt, dt can be removed from this place, this place, and in simple way you can write as $\dot{r}^2$+ h^2/r^2, this equal to 2μ/r + c, so this is the integration. So this is (F) and the equation number (G). So now this r dot this is appearing as the radial velocity. So remember the step to work it out. Here it is; I was trying to do; I could have put here dt itself and putting the integration sign.

Because we have to integrate it with respect to *dt* but I have taken one more step for working it out, so that is better. In doing the shortcut quiet of in we do some mistakes so it is a better that we go step by step. Now this is the quantity we were looking for, okay. So can we evaluate it? Once we can evaluate it so our job is done. So now we have

$$v^2 = \dot{r}^2 + r\,\dot{\theta}^2$$

This is the radial velocity and this is the normal velocity as we have discussed earlier, so this is $\dot{r}^2 + r\,\dot{\theta}^2$, okay. And here we can see that this is $\dot{r}^2$and we can replace $\dot{\theta}$/h/r^2, our expression we have used earlier so this gets reduced to $\dot{r}^2 + h/r^2$,. This is square term. This whole square. Okay, so with this we can recognize that left hand side is nothing but here v square. So we go to the next page.

(Refer Slide Time: 28:01)



So equation G can be rewritten as

$$v^2 = 2\mu/r + C$$

And if this C, the constant C is known, if this constant is known so we will able to work out the whole problem. Now also we will utilize one relationship this v^2 which we are going to prove it later on perhaps till now I have not done it, $v^2 = \mu\ (2/r - 1/a)$ means this is $2\mu/r - \mu/a$. So if we compare this two, this is equation number (H), this is (I).

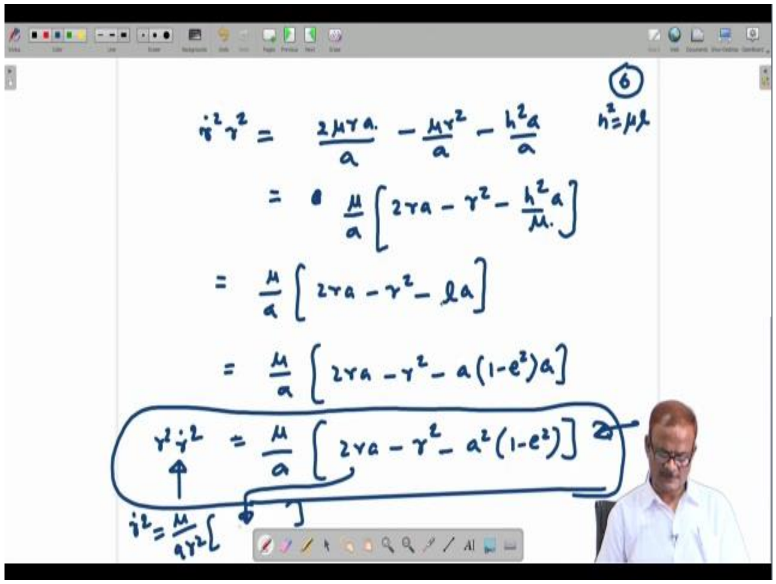
So if we compare this C turns out to be from this place C = - μ a. Okay, so therefore we have now; so either we could have; if we use this equation so we can start at this stage itself, so now pickup from this place $\dot{r} + h^2/r^2$, $\dot{r}^2$ therefore this is 2 μ /r and c = - μ a. Sorry here, this part is we have written the 2 μ/ this is 1/a, so this gets reduced to μ /a. So this is μ /a, c = - μ /a. Here I will correct it.

So this gets reduced into the form. So directly if we know this equation, if we remember this equation, okay we can directly work out from this place. So here this is then this implies $\dot{r}^2$, r= h^2 = 2 μr, taking r^2 on this side – $\mu r^2/a$. Or,

$$\dot{r}^2 r^2\, a + h^2\, a = 2\mu r a - \mu\, r^2$$

And the rest of the thing we will take on the right hand side. So $\dot{r}^2$; what we are looking for, this expression, so $\dot{r}^2$, r^2 as in our question this is.

(Refer Slide Time: 32:44)



So $\dot{r}^2$ r^2 this then becomes 2μ r a /a – Mu r^2 and then $–h^2a$, here it will be $–h^2a/a$, so if you look back μ has been taken as a common throughout; we are missing some term here, h^2, this is okay. Here we will take outside and μ also we will take outside so μ/a as it is required in the problem, so 2ra – r^2 and here this is given to be μ we are taking it outside so h^2a, a cancels out okay so h^2a is there in the previous side, in this h^2 a.

So this *a*, *a* cancels out and we get here; but we are taking a here, so we will keep a here in this place like this. So let us work like this, 2ra – $r^2$and h^2; in the problem if you look h square is nowhere mentioned, if you go back on the previous page here, here nowhere h term is appearing in this equation, so we have to eliminate h from that place. So h^2 now as we remember $h^2 = \mu l$.

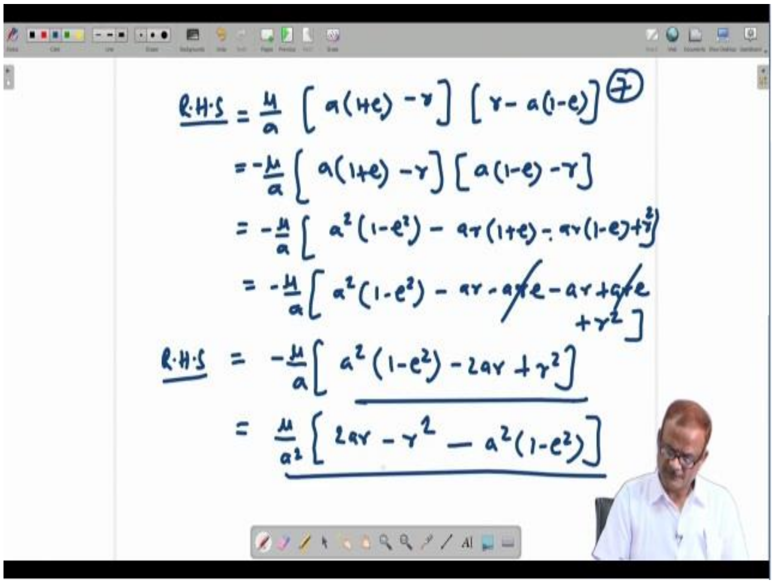
So we write here μ, okay and μ we have taken outside so we have to divide it by μ here in this place, okay. So this, once we work out a, a cancels out h^2, this is fine. So you can check yourself, this satisfies the; this can be reduced to this format. So now h^2/μ, this is l, so remove this part from this place, so this is la, l times a. So this is μ a times 2ra – r2.

Now l as we have worked out here, this is

$$r^2\dot{r}^2 \;=\frac{\mu}{a}(2ra - r^2 - a^2(1 - e^2)$$

So this is$r^2\dot{r}^2$. But still the right hand side it does not look like what it is given in the problem. So what we will do rather than breaking it and spending our energy we will just take the right hand side of the problem itself and reduce it to see whether it matches with this one or not.

(Refer Slide Time: 36:55)



So in the actual expression the right hand side is

$$= \frac{\mu}{a}(a(1+e)-r)(r-a(1-e))$$

Expand it. So we will reorganize it. 1+e – r so that we can multiply it in a better way, a times 1-e – r and minus sign we will put here for the time being, so this becomes a square 1-e^2– ar times 1+e – ar times 1-e and + r^2. 1-e^2– ar –; a + e times r so they ar + 1 + e so this is okay, so ar – are from here this place minus, this is ar, and + are and + r square.

So that terms here, this one and this one they will dropout leaving us with – μ /a, a^2 times 1-e^2 – 2ar + r^2. So this is your R.H.S. If we take the minus sign inside, so this gets reduced to 2ar – r^2 – a^2 1-e^2. So now we will match with the; what we have derived earlier. So here μ /a 2ar – r^2 – a^2 times 1-e^2, so the same thing is here, okay.

$$= \frac{\mu}{a^2}(2ar-r^2-a^2(1-e^2))$$

So therefore whatever we wanted to prove this has been proved. So the radial velocity is related by this expression so little bit of change in this particular part so $\dot{r}^2$, square you can write as μ/ar square and times this whole thing the bracket thing should be copied here in this place, so $\dot{r}^2$, = μ/a^2 and this quantity appears here. So we have started solving this problem in the two particle system, so this was our problem here.

And ultimately we have got the result, but at one stage we have used one information which I have not done still, okay

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

And this particular part, this is also called Vis-viva integral. And this will be quiet often utilized in solving your problems. So we will stop here.