Space Flight Mechanics
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Lecture – 13
Classical Orbital Elements / Parameters (Contd.)

Welcome to lecture number-13, we have been discussing about the orbital element, so we will continue with that, so for the 2-body system, we got the relative motion equation and from there after we have been developing how the orbit will look like.

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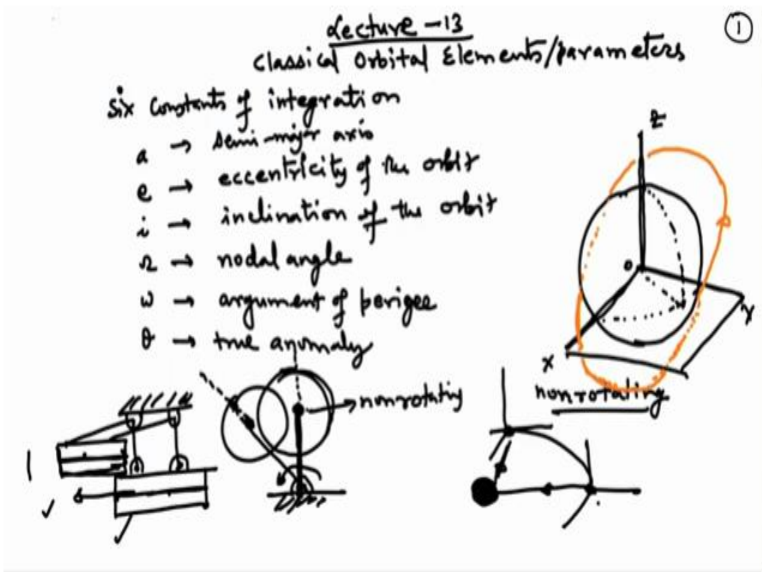

So, we have already discussed that there are 6 parameters for that relative motion equation, the 6 constants of integral and those constants of integrals we have written as a, e; the 6 constants of integrals; the 6 integral constant was; 6 constants of integration we have written as a, e, i, Ω, ω and θ, this we have written as true anomaly, ω as the argument of perigee.

Ω is the nodal angle, inclination of the orbit and this is eccentricity of the orbit and this is semi major axis, so if we discussed the case of earth, so at the centre of earth we can place an inertial reference frame which is not rotating along with the earth, okay and let us say this is X, Y, and Z and this then constitutes the; not the equatorial plane of the earth but the plane of the inertial reference frame, X-Y plane of the inertial reference frame.

So, here this frame this is non-rotating but it is a being dragged along with the earth around the sun and therefore in reality, this is not actually inertial intrusions like what is the difference I will show you here, say a block is hang from the ceiling like this, so if you push it so this will have a purely translational motion, this will not have a rotational motion because this block will; if it is pushed from one place to another place, so this block will look like this.

This position will come here and this will go here, so this is a purely translational motion in this you can say that the orientation does not change, a line joining; a line like this it remains horizontal itself while in the rotational motion, what you will see that if I have a say, here circular chord or whatever it may be, it is fixed with this, at this point it is a fixed, this point is non-rotating.

So, here the chord cannot rotate but here in this place, there is a pin and about this it can rotate, so then you can see that a line like this, if it is here after sometime if it gets displaced, so this line will appear like this, so this line orientation initially was in this director now, it is in this direction, so this has rotated, so this is a rotational motion while this is purely a translational motion.

So, similarly here in this case, this reference frame X, Y, Z, okay this is an inertial reference frame and this is just rotating around the sun, let us assume that the sun is because we are writing equation with respect to the sun okay, so this reference frame if it is right now, it is here in the orbit, so after some time once it goes here, so still the same orientation will be maintain, the orientation will not change okay, so all through the orientation remain same.

But you can see that the centre of this origin of this reference frame, this is accelerating toward the earth, toward the sun, okay because it is going in a curvilinear path, here in the elliptical orbit. So, in true sense this is not an inertial reference frame but right now we can assume that this is equally valid though this is not valid reference frame this way, some corrections are; some corrections need to be given to this kind of formulation.

And there are many books related to this how to give the correction and taking the relativistic effect, there are so many things which can be discuss. So, one book summarises; something if I remember correctly this is by Seidelmann, one book is there, in that book you will find all these details okay and there are many other books on astronomy which will give you this kind of how the; how to do the this correction relativistic correction and other thing this acceleration and.

But for the satellite around the earth, we can assume that the acceleration about the sun is so small that we can assume that we can safely assume that this translating reference frame, it is a serving as good as an inertial reference frame, this is translating and accelerating, okay. So, around the earth, our satellite it goes like this. So, this is the orbit of the earth, satellite around the earth and it will go like this, it will cross this plane, it will go inside this plane.

So, this plane also I will make it here something like this, so this will go inside the plane, so inside the plane I will show it by dotted line and from here from the back it is coming, okay going inside, it is coming like this and going inside this, okay outside you can see it, it is a solid inside here it is a dotted line, so your satellite is going in this orbit.

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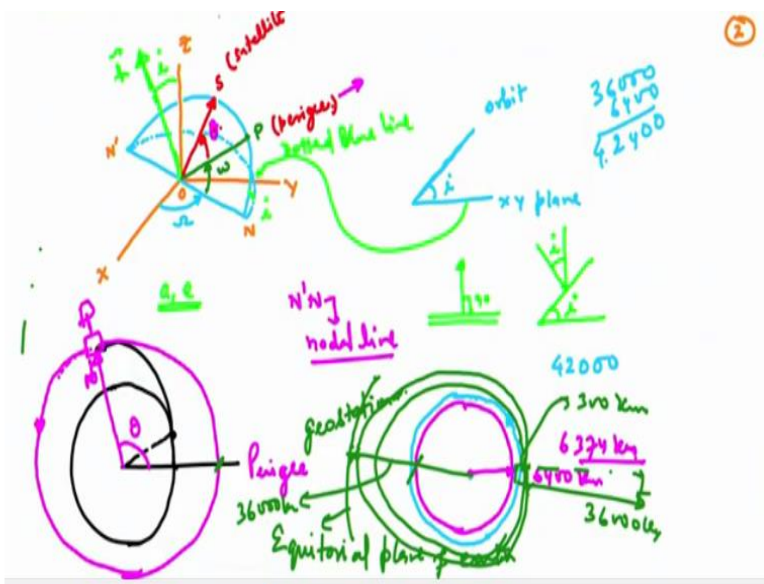

So, this I have shown you; earlier I have shown you the same thing by drawing this figure, this is X, Y and Z, so this is half of the orbit we are showing and then I named it this as the N' and this point as N, this as O, then we indicated one point as P here, this is the perigee and then the satellite was located somewhere here, we have written this as S, it is the perigee and here this is satellite, this angle we have written as small omega.

And this angle as theta, so small omega is the argument of perigee and theta is the true anomaly already we have written and then this angle we have shown as capital omega which is the nodal angle and then inclination of this orbit with the X-Y plane, so we can show projection of this orbit on the X-Y plane like this, so if I show it like this, so this is the projection of the orbit with the X-Y plane.

So, if you see it from the side, okay so the orbit will appear like this, if you see from the side, so the side view of the orbit it is looking like this and the X-Y plane will look like this, so this is your X-Y plane and this is the orbit, so this angle here this is i and equally what we can say that this perpendicular to this orbit which is given by this line in which direction the each vector is directed, okay this angle is also i.

Because here if 2 things are just coinciding and then thereafter you; 2 lines are coinciding thereafter you rotated by i angle and if there is a perpendicular here, this is 90 degree, so this perpendicular will also rotate like this, so how much it will rotate with the vertical, this will rotate by i, so the same thing is shown here, so this is your inclination angle and here it is not very clear but this is also the angle of inclination means, this is the dotted blue line is the projection of the orbit on the X-Y plane.

So, with respect to this so, this is basically your dotted blue line, this is the dotted blue line, so with respect to this the solid line the whatever the angle it is making that we are writing as i which is angle of inclination. So, we have identified here 4 parameters and a and e already we are determined, so we need to work out rest of them because we start at this point discussing about this whole thing.

So, in the little bit more explanation on the earth, you may launch your satellite say here in the case, the Sriharikota is located somewhere here, this is the equator and from the equator it is located somewhere here, so from this place the satellite is being injected into the orbit and this orbit then once it is injected, so this orbit will look like this, it will go around the earth and once it is launched, so basically this constitutes on elliptical orbit.

So, let us say that the perigee line or the periapsis of this orbit it lies here, so that means we measure theta from this position as you have shown here, so this is your perigee line, okay. So, from perigee line, we are measuring this theta, so from here this theta is being measured, okay so this is your true anomaly, so this is one of the parameter that we need to determine the rest others are capital omega.

So, capital omega is what we have already written as the nodal angle and NN’; N’N, this is called nodal line, so your satellite is here injected in this place and then the satellite will move along this orbit but this orbit did not be circular, it depending on the initial injection velocity, it may be highly elliptical or it may be elliptical but it is never the circular is achieved so, basically, the injection altitude, so if I show this as the earth, okay of and earth radius is around 6400 km roughly and 6374 km.

So, if you look for the satellite orbit, so it will look 300 km something like this, okay it is quite close to the earth surface, if you look from outside the earth, so it will look like this, so it is going so close to the earth, okay and thereafter if you are looking for the geostationary satellite, so you will, you are aware of that the geo stationary satellite altitude is around 36,000 km from the surface of the earth, so approximately 42,000 km, we have 36,000 km and plus 6400, this is the radius of the earth.

So, this makes it 4, so 42,400 it is coming, so from the centre of the earth the geostationary orbit it is located at quite far distance, this is 6, so if we look to the scale, this will be far away, okay this will be something like going like this, it is a faraway, okay but we show that let us say that this is your geostationary orbit and this geostationary orbit in this case this will be in equatorial plane of earth not in the inertial X-Y plane, okay.

Because the equatorial plane of the earth and inertial X-Y plane they differ, they are not the same, so if you have to restore this altitude then at the perigee location, first we would start let us say said this is the perigee location here in this place or also I will show it here that this is the perigee location, so from here the satellite orbit is raised okay by giving a number of impulses and ultimately, while it is raised, so this part will approach the geo stationary orbit.

Thereafter from here, the if the impulse is given, so this will be raised, so initially says this distance is 300 km, so this distance will also be raised and on this side, this will go to the total from the this place to this place, this will go to 36,000 km and from here also from this place to this place from the surface of the earth to this place, this will go to 36,000 km.

So both way it will be this, so that the orbit become circular and in the plane of the; in the equatorial plane, okay so this is the basic thing involved in the satellite launching and if it is polar satellite, so you will launch it in the polar orbit not in this equatorial orbit, so we will discuss during course of time, so let us start with first whatever the parameters remaining we work it out.

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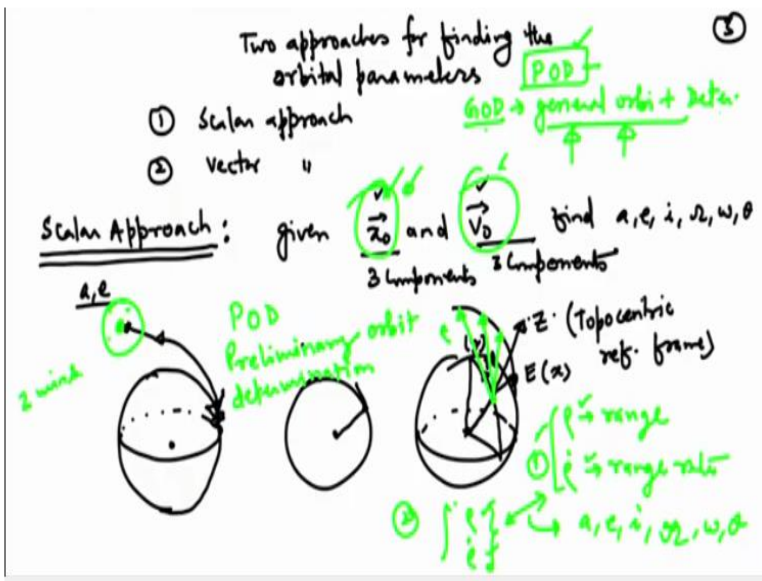

So, there are 2 approaches for working out the parameters of; 2 approaches for finding the orbital parameters. Now, the first one is the scalar approach and the second one is vector approach, so we will look into both of them, both are useful okay, as we were discussing here, so initially one of the satellite is injected into the orbit, so which is shown here by the black line, so this velocity may be here in this direction, okay.

So, inertial navigation system that gives you initial v and initial r that means,

$$\vec{r_0} = x_0\hat{\imath} + y_0\,\hat{\jmath} + z_0\hat{k}$$

this is known and sorry, this is $\dot{x}$, $\dot{y}$, $\dot{z}$ and r equal to $x_0\hat{\imath} + y_0\,\hat{\jmath} + z_0\hat{k}$, so in this format the initial velocity will be known, if this is known, so then you can find out the orbit. So, the problem statement is given x_0 and v_0v0, which is basically 3 components here and here also 3 components, find a, e, i, Ω, ω and θ.

So, this is the objective, so out of that already we have discussed about the a and e, okay rest we have to work out now, one thing I would like to remind that it may happen that these things are not available, okay. So, in that case what is available like if the India is launching its satellite, this is the equator and some guess Harikota is here and says it is a polar satellite is being launched okay or any other satellite and it is injected into the orbit.

So, from the ground station Radar is continuously monitoring the distance, so on the surface of the earth at this point, one reference frame will be fixed, I would better represent it like this, so toward the north, toward this is east, this is north, okay and so depending on the direction we are choosing, so in the case of the satellite we choose it upward, the Z direction if you are writing this as the X, this as Y, okay, so this becomes Z.

And this is called the topocentric reference frame, which is a matter of discussion which we will take up later on, so while the satellite is launched and it is going into the orbit, so Radar is monitoring this distance from here to here, okay and this distance this is written as rho, rho is called range and rho dot the rate of change of the range is called the range rate, so this is measured. Now, angles are also measured which we call as the, those things we are going to take later on.

Because then we would not be able to cover it so fast just giving you initial idea, so this is from one station from another station also, so this is from one station from the second station also, this rho and rho dot will be available to you. So, using this data then you have to determine this orbit where a, e, i, capital omega, small omega and theta, so this we; this is called the POD; Preliminary Orbit Determination.

So, it is not necessary that this is available to you just based on this rho and rho dot from 3 stations; 3 ground stations you will be able to work out okay, if your Radar stations are located over India in 3 different places and satellite is visible, so using initial 2 minutes observation data; 2 minutes, 3 minutes or more, so there are 2 things; 2 terms we are using, one is POD which is preliminary orbit determination and another one is GOD, this is called the general orbit determination.

So, in the GOD, a precise orbit is determined using the least square method or whatever it may be, okay in the GOD, precise orbit determination and POD is not so precise but it is a good enough so that you can in the future you can keep tracking the spacecraft otherwise, the spacecraft may get lost in the space, so now the initial value; your inertial navigation system that gives, it is not precise, okay.

It may be like you intend to put your satellite say here in this place but your satellite may not be placed here in that place but it is a rather placed in the basket, okay and it will have certain amount of uncertainty, so in this basket then initial location is given to you and those location can be used and along with this data, so this is just initial value given here and along with this data rho, rho dot to refine the orbit and do it to a; refine it to extent, so that you can track it near future.

And thereafter you for your actual purpose then you refine it and make it more precise, so this we call as the general orbit determination that part, okay. So, we start with this introduction, we start with our discussion.

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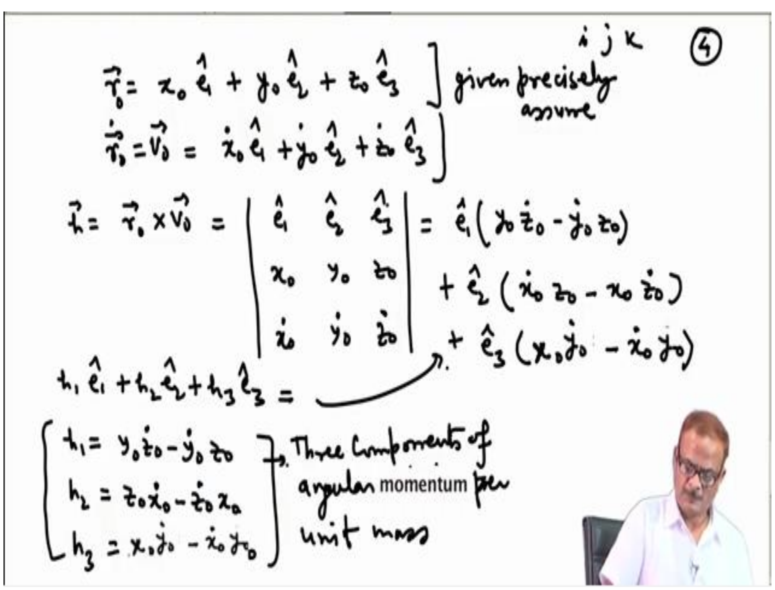

So, given this r equal to $x_0\ \hat{e_1}$, so instead of using $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$, I will be using $\hat{e_1}$, $\hat{e_2}$, $\hat{e_3}$, x_0, y_0, $\hat{e_2}$ and $z_0\ \hat{e_3}$, so this is an initial point and let us assume that this is given precisely, so this part is given precisely, in that case we will be able to determine the orbit, so we are start writing $\vec{h}$ equal to $\vec{r_0}\times\vec{v_0}$ if we take the cross product here, so we can write here

$$\vec{r_0}\times\vec{v_0} = \hat{e_1}(y_0\dot{z_0} - \dot{y_0}z_0) + \hat{e_2}(\dot{x_0}z_0 - x_0\dot{z_0}) + \hat{e_3}(x_0\dot{y_0} - \dot{x_0}y_0)$$

So, this gives you h, so h will have 3 components; $h_1\ \hat{e_1}$+ $h_2\ \hat{e_2}$ and this quantity is given here, so h_1 you can write as $y_0\ z_0$ dot minus; so these are the 3 components of the angular velocity per unit mass, okay.

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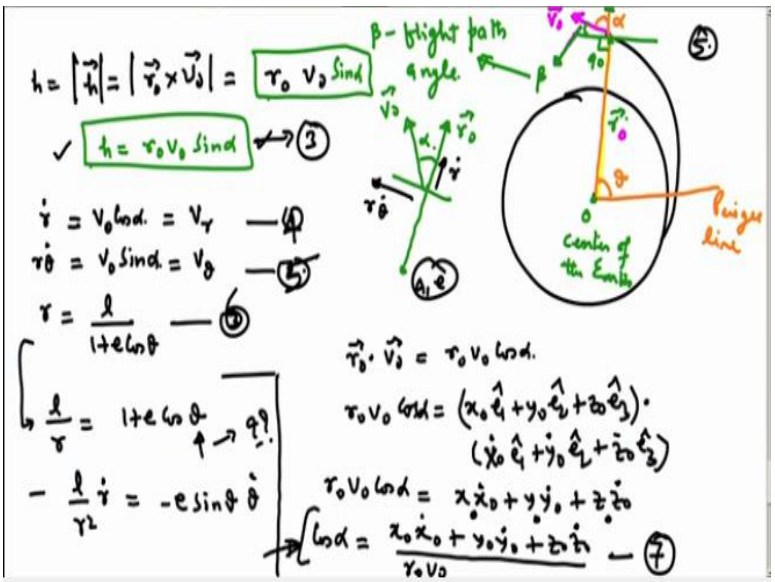

Now, we also we can write it as, say here this is the earth and one satellite has been injected and say this is the injection point, this angle we show as θ, so this is the perigee line, θ is not known, this angle we will write as alpha and here we will draw one line which is perpendicular to this, the radius vector, this is your *r* vector from here to here this is your *r* vector, so this in 90 degree and this angle we will write as β.

So, β we call as the flight path angle, so α is your angle and this red vector which is appearing here this is nothing but your so beta will; move it here and first let me write here, this red vector is nothing but your v vector, v_0, okay and this is here r_0 and this angle, this is this angle is β which we are calling as the flight path angle. So, if you see the angle between the $\vec{r}$ and the $\vec{v}$; r vector is like this and your $\vec{v}$ is here in this direction, this angle is α.

And this is going toward the centre of the earth, so $\overrightarrow{r_0}\times\overrightarrow{v_0}$, angle between them is sin α, so we write here r_0 v_0 sin α, so we write here r_0 v_0 sin α, so h is; also you can notice that v_0 cos α, this can be written as r_0, which is the radial component, in this direction the velocity component and in this direction the velocity component so along this direction $\dot{\vec{r}}$ and here $\vec{r}$ times $\dot{\theta}$, v_0 sin α.

So, these are the 2 components, so these are radial component and this is v θ component and we will number them as let us say for initially we have not written anything, so we will number them as 1, this as 2 and then we also used the equation

$$r = \frac{l}{1 + e\cos \theta}$$

because this is the conic section equation, so we will use this, this information we have and then other part we have already written here, this part.

So, this equation maybe we can write as (1) and here this we can write as (2) and this place we can write this as (3), this as (4), (5) and this as (6), okay from here this equation can be reduced in this format minus e $\theta \times \dot{\theta}$, okay. What we are trying to do here see, we need to determine this θ, this is one of the objective what this is, this quantity is, so I have to determine this a and e we have already worked out, okay and a ω, Ω all those things we need to work out okay.

So, this is one part and then also cos α we can define as and if we do that so, r_0 v_0 cos α this we can write as $x_0\ \widehat{e_1} + y_0\ \widehat{e_2}$ dot product with $\dot{x_0}\ \widehat{e_1}$+ $\dot{y_0}\ \widehat{e_2}$+ $\dot{z_0}\ \widehat{e_3}$, so r_0 v_0 cos α this becomes y $\dot{y_0}$ + z $\dot{z_0}$ and therefore cos α from this place is

$$cos\ \alpha = \frac{x_0\dot{x_0} + y_0\dot{y_0} + z_0\dot{z_0}}{r_0 v_0}$$

So, if your position is given in this format, so you can see that immediately you can work out α.

This equation is there and this equation is there, so you can determine alpha, so we will discuss about this what is the significance of doing like this, so this is (6) and this we will number as (7) and this we will number as; in the next step we will do it, we will do this part on the next page.

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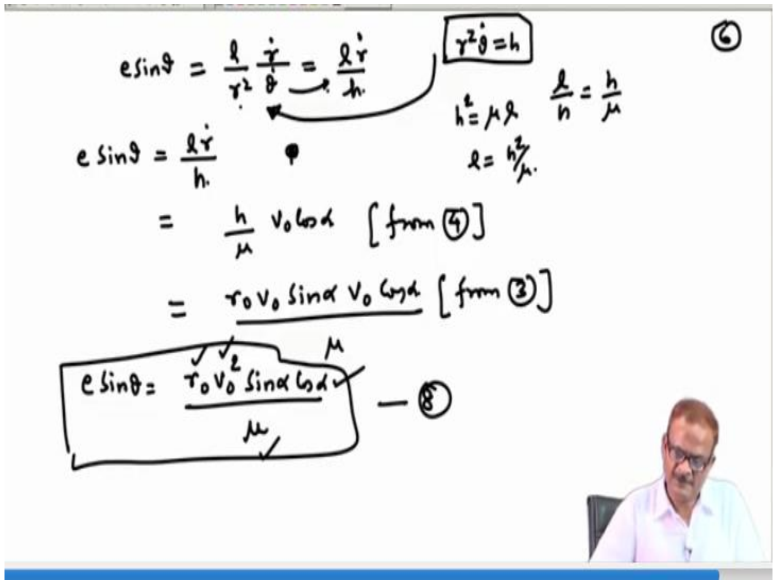

So, this equation can be rearrange as e sin θ equal to this minus, minus sign cancels out; so $e\ sin\ \theta$, this equal to $l/r^2\ \dot{r}$ and then from here this $\dot{\theta}$, minus, minus sign dot gets cancelled, okay and $r^2\ \dot{\theta}$ this quantity is h, so we can write this as; as already we have derived for the central force motion $r^2\ \dot{\theta}$ equal to h, so we can use it here and get this equation.

So, we have got e sin θequal to l times $\dot{r}$ divided by h and so this can be written as *l/h*, some more manipulation can be done, actually what we are trying to do; we are trying to express everything in terms of r_0, v_0 and α, so we can reduce this part, r dot, you are already aware of $\dot{r}$ we have written here, this is your $\dot{r}$, so we can use it, so $\dot{r}$ is v_0 cos α.

And what this there is some relationship between l and h, so we know that h^2 equal to μ times l and if we want to remove this l, so l becomes h^2 divided by μ and we can insert here, okay or simply we could have written here,

$$\frac{l}{h} = \frac{h}{\mu}$$

okay so l/h is here, so h/μ we can remove this h/ μ and also we know so this we are getting from using equation (4), from (4) and then h is also known to us.

And h we have written in the equation (3); $r_0\ v_0$ sin α, so $r_0\ v_0$ sin $\alpha \times v_0$ cos α divided by μ, this is from (3), so e sin θ then this gets reduced to

$$\text{e} \sin \theta\ = \frac{r_0 v_0^2 sin\ \alpha\ \ cos\ \alpha}{\mu}$$

so we will number again, this is (7), equation number (8), and suppose we have another expression e cos θ, so definitely we will be able to find out θ from this place, e sin θ and e cos θ is there, so we can eliminate.

Here we have the α is known, r_0 is known, v_0 is known, μ is known, okay and if we also got cos θ, so we can eliminate e and we can get in terms of sin θ cos θ, so it will be solved for theta in this case okay.

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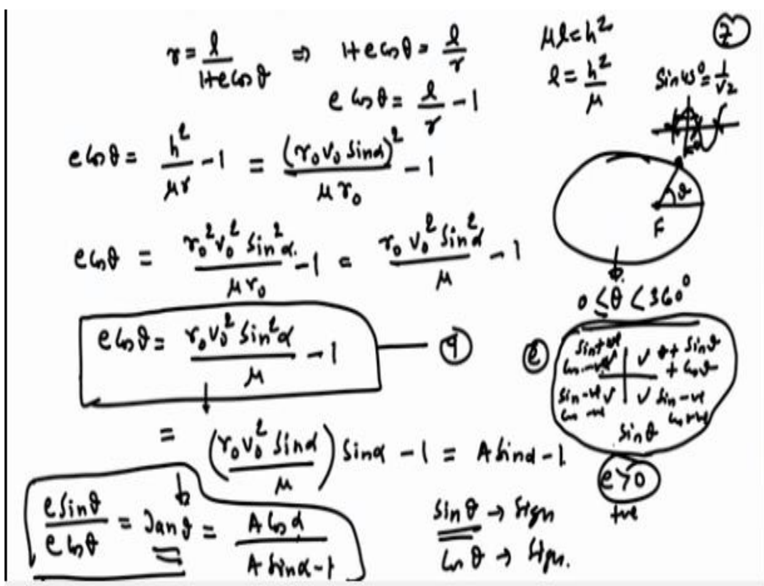

So, we have finally got this now, l by; already we have written,

$$r\ \ = \frac{l}{1 + e\cos \theta}$$

so this implies 1 + e cos θ equal to l/ r and from here, $e\ cos\ \theta$ equal to $l/r - 1$, so therefore e cos θ; now l equal to if we want to express in terms of μ and other things, so we can do that, so μ times l, this equal to h^2 and therefore, l will be h^2/μ, so this becomes $h^2/\mu r$ -1.

And h we are aware of how much this is, this is $r_0\ v_0\ \sin\alpha$, so this will be whole square divided by μr0 – 1, so this gives us

$$e \cos\theta = \left(r_0^2\ v_0^2 \frac{\sin^2\alpha}{\mu r_0}\right) - 1$$

so finally we have $e\cos\theta$ equal to $r_0 v_0^2\ \sin^2\alpha$ divided by μ-1. So, here we have got one expression, e sin θ and here we are getting e cos θ, all in terms of $\alpha\ r_0$ and v_0, this is equation number (9).

And in both places, you will see that $r_0\ v_0^2\ \sin\alpha$ is there and here $r_0\ v_0^2\ \sin\alpha$ times cos α, so suppose if I write this as e sin θ equal to $r_0\ v_0^2\ \sin\alpha$ times cos α and divided by μ in this and the quantity in the bracket I write as A, so this becomes A cos α, this is for just giving it a brief notation and this also can be reduced in this format, so this will be A sin α -1.

And therefore, e sin θ divided by e cos θ that becomes sin θ and we get this as A cos α divided by A sin α -1. Now, the question is whether using this equation you can; using this expression, is it possible to determine θ? You see here in the if this is the orbit and r is always measured from the focus and this is θ, so as the satellite is going around the orbit, so θ varies between 0 and 360º.

So, if you do only using this equation, so you will have problem that whatever the value you are getting, so where it is lying; it is lying between the this quadrant, this quadrant, this quadrant or this quadrant, it is between 0º to 360º, okay and you know that they will repeat their value, so that is resolved using the sin of θ; sin θ sin because e is always positive, okay e is always greater than 0, always positive, it will be 0 only for the circle case okay.

So, therefore we will look for the sin θ sign, sign of this and cos θ sign whether it is a positive or negative, so sin θ is positive here and cos θ is also positive, sin θ is positive here and cos is negative here on this side, here sin is negative and cos is also negative, in this portion sin is negative and cos is positive. So, if you know what is the sin theta and the cos θ value; so using this table you will be able to say in which quadrant that theta is lying.

That means you will be able to determine the exact value of θ, the right value of θ, okay sin cosine they are repeating their value, say the sin 45º, this is 1/√2, that means you are starting with this, you are going like this, so sin θ is 0 here also, this is 0 here, so at θ equal to 0º, 180º equal to 0, this is 0, again it will be 0 here, okay, so till 360º, it is repeating the same value 45º, so 1/√2, suppose it lies here.

So, again it will lie here, means it between 90º and 180º, again it will lie okay, again on this side if you go so this becomes negative, so there is a repetition, so we have to resolve that but here if you look the cos here between 90º and 180º in this range, the cos becomes negative, so this way you will be able to resolve, so we will discuss further on that once we take the vector notation, so how to resolve the sign that we will discuss.

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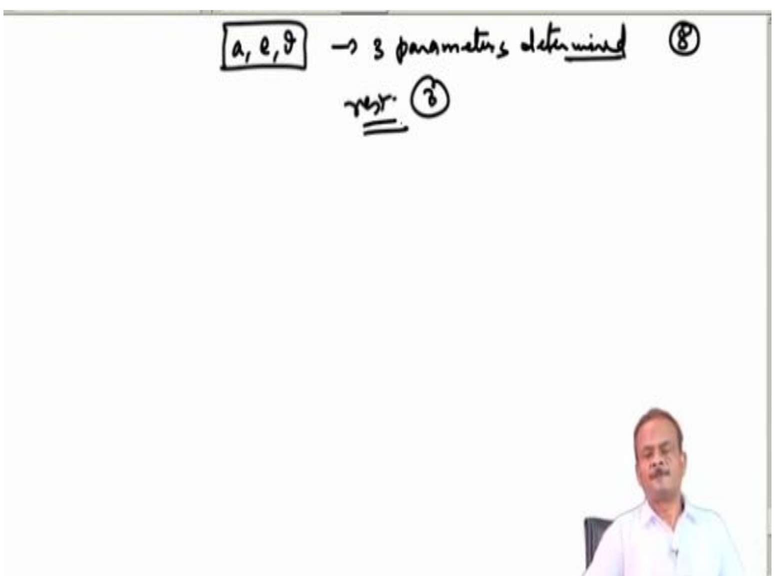

So, this way we have been able to here find out one of the parameter which is θ, so right now we are in a position where we have already determine a, e and θ, so these are the 3 parameters determined and rest 3 parameters we have to work out, so we stop here and then we will continue in the next lecture, thank you for listening.