Space Flight Mechanics
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Lecture – 14
Classical Orbital Elements - Parameters (Contd.)

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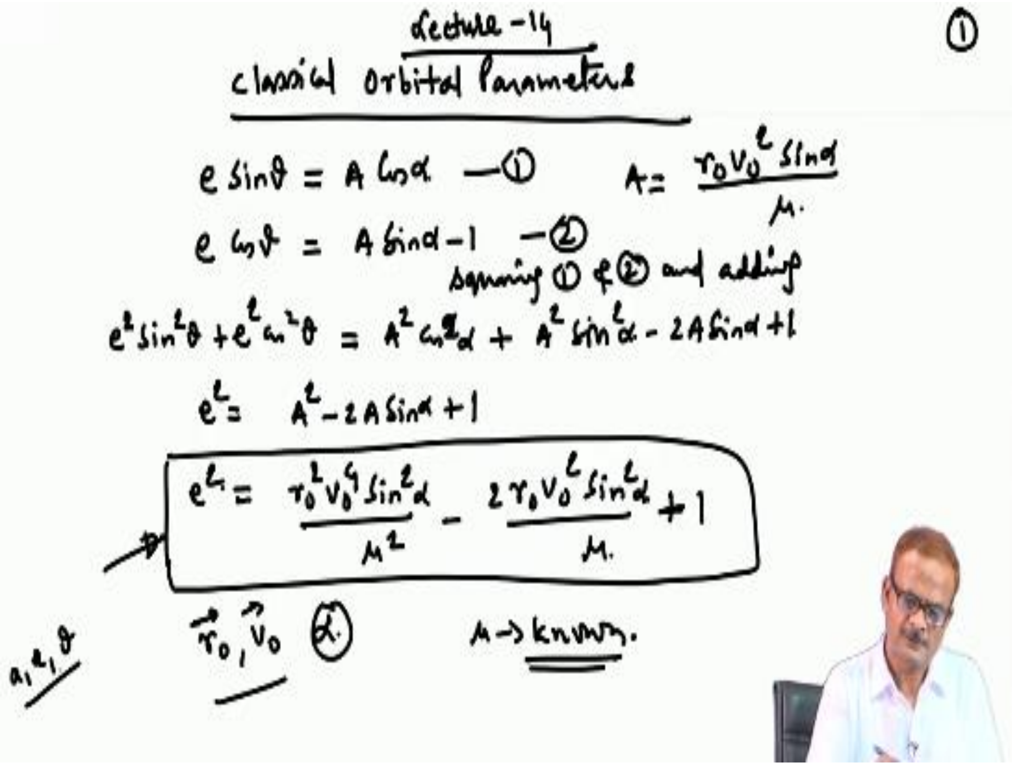

Welcome to lecture number 14. So already we have looked into how the θ is determined so in due course of time we have determined this as

$$\text{A e sin } \theta = \text{A cos } \alpha$$

and e cos θ = A sin α – 1, where A we have written as $r_0 v_0^2$ sin α divided by μ. So if we square and add them this is by squaring and adding 1 and 2 and adding. So this implies $e^2 = A^2 - 2A \sin \alpha + 1$ and A from this place we have this becomes

$$= \frac{r_0^2 v_0^4 \sin^2 \alpha}{\mu 2} - \frac{2\, r_0 v_0^2 \sin^2 \alpha}{\mu} + 1.$$

Okay so this is another expression for eccentricity already we have worked out eccentricity in various ways, but suppose that your r_0 and v_0 is given. So α can be determined if α can be determined and r_0 v_0 is given so therefore e can also be computed in this way and μ is of course this is known. So you can see as I told you that in the space flight mechanics the same problem can be solved in multiple ways. So this is one of the way we can work out the problem. So till now we have worked a, e and θ.

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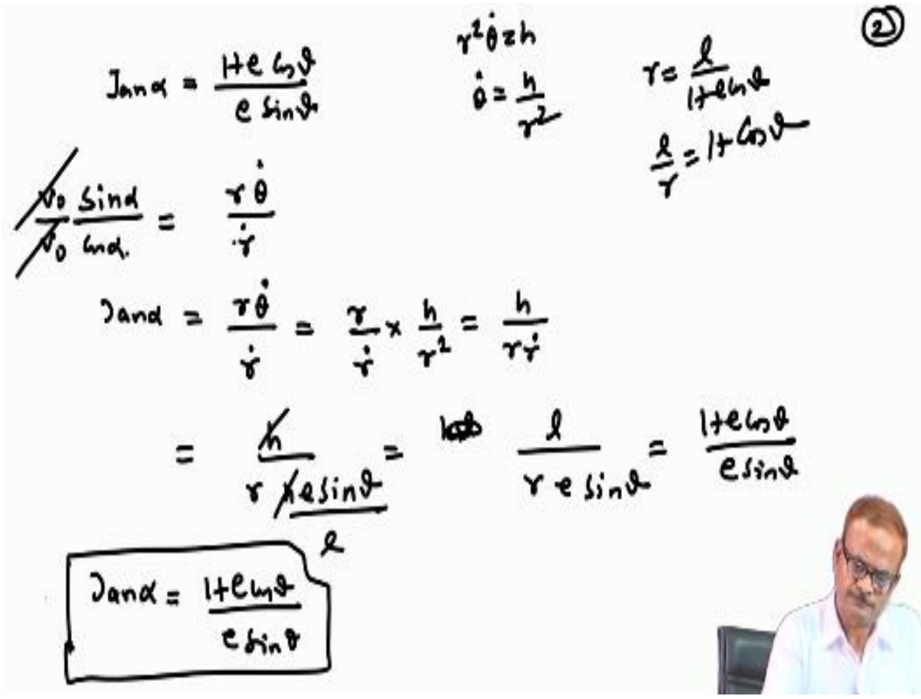

Now we are going to work out the other parameters so for working out other parameters we need to we can do it by the vector method also and the scalar method also. So I will prefer first to work with the vector method and then maybe we can take up the scalar case. Okay one more thing I would like to write here this tan α this can be written as

$$tan\,\alpha = \frac{1 + \mathrm{e}\cos\theta}{\mathrm{e}\sin\theta}$$

So this is obtained by dividing sin α divided by cos α. So sin α and cos α we have to fetch from our earlier work out here this is v_0 sin α = r $\dot{\theta}$ and this is $\dot{r}$ r $\dot{\theta}$ divided by $\dot{r}$. Okay here v0 and v_0 is there so this cancels out and we are left with

$$tan\,\alpha \;=\; r\dot{\theta}/\dot{r}$$

and $\dot{r}$ $\dot{\theta}$ all this things we are aware of so r times r divided by $\dot{r}$ and $\dot{\theta}$ is h /r^2 because r^2 $\dot{\theta}$ this = h so $\dot{\theta}$ is h / r^2 to replace it here.

So this becomes h / r times $\dot{r}$. Now we can insert the value for this r and $\dot{r}$. So r and $\dot{r}$ we have derived earlier so we need to go and look into go back and look into that equation $\dot{r}$ we have derived in terms of see here in this place $\dot{r}$ is we can rearrange this equation and write it so $\dot{r}$ we can write as h times e sin θ divided by l h e sin θ divided by *l*.

So this becomes h l h, h will cancel out and we get here $l/r\ e\ sin\ \theta$ and what is the quantity

$$r = \frac{l}{1 + e\cos\theta}$$

so *l/r* will be 1 + e cos θ. So here we can write it as l / r becomes 1 + e cos θ and divided by e sin θ. So your tan α this is quantity

$$\tan\alpha = \frac{1 + e\cos\theta}{e\sin\theta}$$

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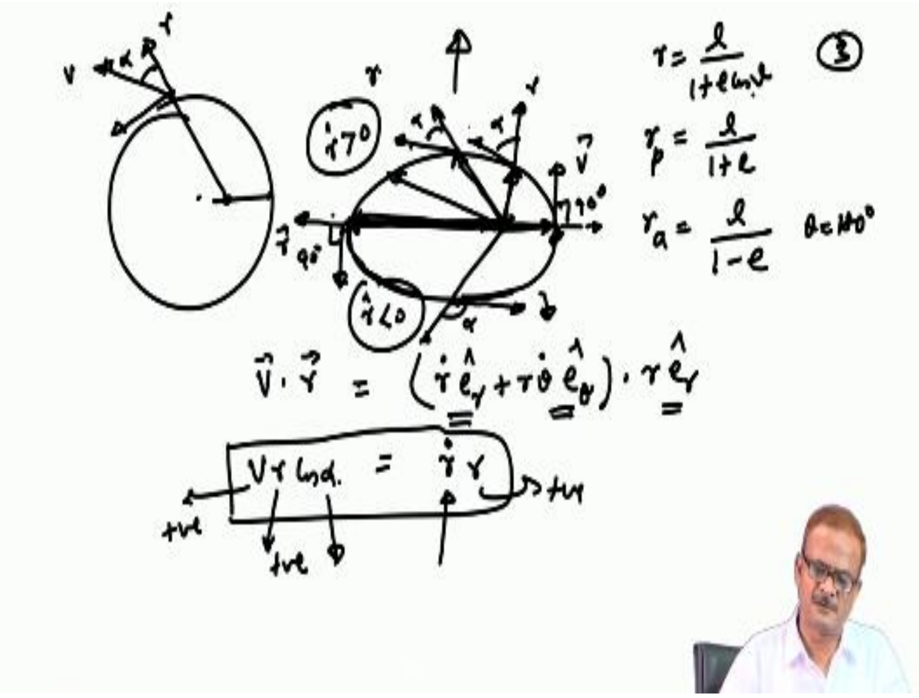

Again one important part I would like to remind you that this is the $\vec{r}$ and $\vec{v}$ is somewhere here this is the local normal this angle we have written as α. So what is the limit in which this α varies so let us take the case of an elliptical orbit. Here we can see that if this is the $\vec{r}$so $\vec{v}$ is here in this direction so angle is 90°. On this side this is your $\vec{r}$ so $\vec{v}$ is here perpendicular.

Okay it will be tangential so this 2 extreme points it is a 90° as you can put here like

$$r = \frac{l}{1 + e\ cos\theta}$$

so once we put θ = 0 when this point θ = 0 so you get r = 1 + e so this is r perigee (r_p) and r apogee (r_a) will be l by once we put θ = 180° so this will be corresponding to θ = 180°. So here also this is 90°, but what in other places. So here in this place or either here in this place what will be the angle between $\vec{v}$ and $\vec{r}$.

So how do we find that so this is the concerned angle α so obviously if we take the dot product between $\vec{v}$ and $\vec{r}$

$$\vec{v}.\vec{r} = (\dot{r}\,\widehat{e_r} + r\,\dot{\theta}\,\widehat{e_\theta}).r\,\widehat{e_r}$$

and in taking the dot product this $\dot{r}$ r this only this part will exist and this part will vanish because they are perpendicular to each other this we are taking as dot product. So v times r cos α is this quantity.

This quantity is positive, this quantity is positive so depending on the and r is always positive so depending on the sign of $\dot{r}$ your cos α is going to be determined and α will vary if you look here now look on this side. So this is your $\vec{r}$ and $\vec{v}$ is somewhere here so this is your α. So α varies between 0 and 180º so where it align. So from this expression you will be able to work out.

And also you can check that in this quadrant while here this is the shortest distance and thereafter r is increasing. So here your r is increasing on this side and till here this is the maximum value is achieved means r dot is positive on this side and $\dot{r}$ is negative on this side thereafter the r starts decreasing and it approaches the minimum value here in this place. So $\dot{r}$ is negative on this side.

$\dot{r}$ is positive on this side on this side this is positive on this side $\dot{r}$ is negative. So you can fix up what is the value of α and there are so many things how to fix the θ in which quadrant it is align so we will use this technique to resolve all those things.

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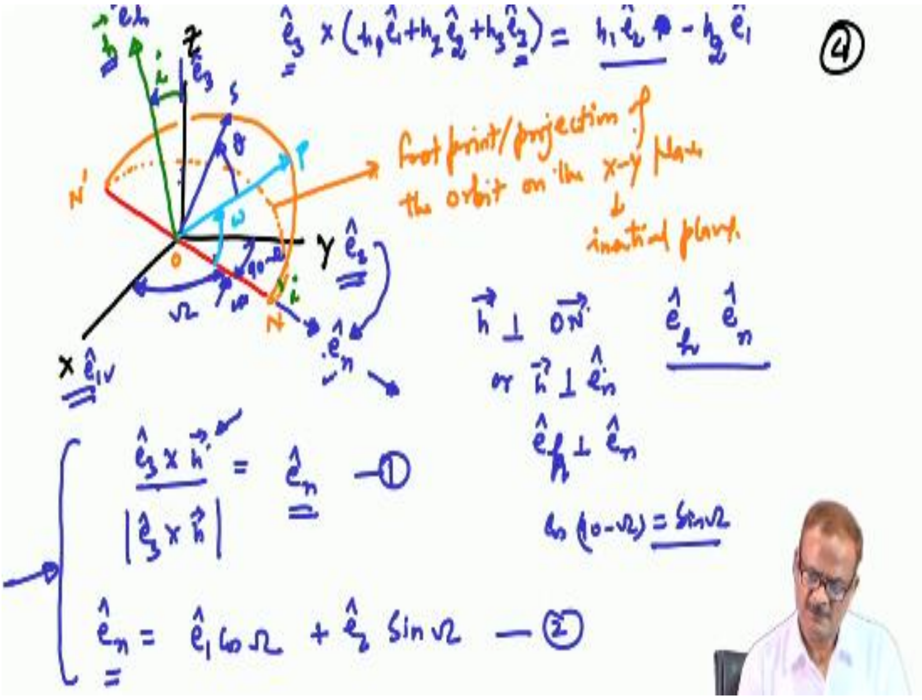

Okay now we go for working out the other parameters. So we resort here to the vector method and again will return back to the scalar method because the vector method it gives you a lot of information and it is interesting also. This is X, Y and Z this is the nodal line. This is the orbit and below which is shown as the dotted line the footprint of the orbit on the X-Y plane so this is the footprint / projection of the orbit on the X-Y plane which is the inertial plane X-Y plane this is the inertial plane this point is o we have written as N and N'.

Perpendicular to the orbit we are writing as this is h vector this angle we have written as i and this inclination here also we have shown it by i and somewhere the argument of perigee and other things we have taken up so I will show that also say this point is where the point P is located and somewhere here in this place the satellite is located. So this angle we have shown to be small ω.

And from here to here this angle we have shown to be θ this is the true anomaly. So in this direction the unit vector we are going to write as $\widehat{e_n}$ in this direction we will write unit vector $\widehat{e_h}$ h and in the Z direction already we have used the notation $\widehat{e_3}$ in this direction $\widehat{e_1}$ and along the Y direction $\widehat{e_2}$. So what we see that $\widehat{e_3}$ cross h or $\widehat{e_h}$.

This is a vector this is the cross product this $\widehat{e_3}$ if we use the right hand rule and you can see that from this direction to this direction once it goes. So this vector will be directed along this direction so this will be nothing, but divided by $\hat{e_3}$ cross h magnitude this will be written as $\widehat{e_n}$ because this vector is $\overrightarrow{e_3}$ is normal to $\overrightarrow{e_n}$ because this is $\overrightarrow{e_n}$ it lies in the X-Y plane.

And therefore $\overrightarrow{e_3}$ is normal and moreover by rotating $\hat{e_3}$ you get to the direction of by rotating $\overrightarrow{e_3}$ about the en direction you get to the direction of h. Therefore, this $\vec{h}$ is perpendicular to the line shown by red one here or we can write here $\vec{h}$ is perpendicular to $\overrightarrow{ON}$ or equally we can write as $\vec{h}$ is perpendicular to $\widehat{e_n}$ or equally we can write $\widehat{e_h}$ is perpendicular to $\widehat{e_n}$.

This is h we do not have to confuse it or we can use some other notation for this then it maybe better that N and h do not confuse us. Okay what we will do that h I will represent it like this so that it remains clear $\widehat{e_h}$ and $\widehat{e_n}$ so there is a difference between 2 notation so we should not have any problem. Now this angle we have shown as a nodal angle Ω.

So what is the unit vector $\widehat{e_n}$. $\widehat{e_n}$ is the unit vector which we can write it as $\hat{e_1}$ cos Ω, $\hat{e_1}$ in this direction so it is cos here up to this point till this point and + $\hat{e_2}$ sin Ω this is 90 – Ω. So cos 90 – Ω this will be sin Ω. So we are taking e_2 projection along this direction so this is this part here. Now if we compare 1 and 2 this is $\overrightarrow{e_n}$ and this is also $\overrightarrow{e_n}$, but this is written in terms of $\hat{e_3}$ and h and $\hat{h}$ is known to us in terms of r_0 v0 etcetera.

So therefore we should be able to work out the whole thing. So I am not going to expand it because I will do the scalar part so therefore I am not going to expand it, but this is one of the most elegant way of working out the problem. If we try to expand and work it out it will be little longer say if we write in a shorter notation let us say that h_1 $\hat{e_1}$ + h_2 $\hat{e_2}$+ h_3 $\hat{e_3}$ these are the components of $\vec{h}$ along the X, Y and Z direction.

And that you are taking cross product with $\hat{e_3}$ so this will be = e_3 times e_1 is e_2 so h_1 times $\hat{e_2}$+ e_3 times $e_2$will give us – h_1 so this is – h_2 times this is e_2 h_2 times e_2 so e_3 times e_2 it is an opposite direction so opposite to the right hand rule so that is going to give us – $\hat{e_1}$. So this is h_2 times – $\hat{e_1}$ and $\hat{e_3}$ times $\hat{e_3}$ the cross product that vanishes so here this is what we get.

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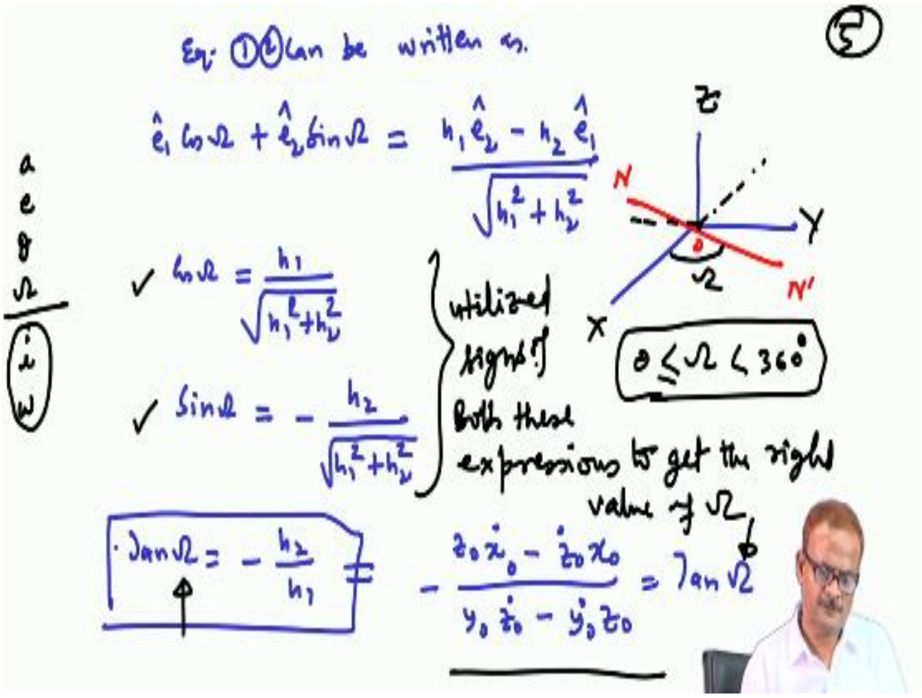

Let me write it on the other page we just wanted to skip it. So what we have on the left hand side here. So equation 1 can be written as $e_n = e_1 \cos\Omega$ equation (1) and (2) $\hat{e_1}$ cos Ω + $\hat{e_2}$ sin Ω this equal to on this side this particular part divided by this. So already we have written $h_1\ \hat{e_2} - h_2\ \hat{e_1}$ and divided by magnitude of the cross product this cross product.

So this is the magnitude of this cross product this is a vector so its magnitude will be $h_1^2 + h_2^2$ under root this is the magnitude. So if you compare so this is cos Ω then from here it appears as

$\cos\Omega = h_1/\sqrt{h_1^2+h_2^2}$ and sin Ω this will appear as $-h_2/\sqrt{h_1^2+h_2^2}$. So again here in this place if we use this so tan Ω this becomes obviously if we divide it so – h_2 divided by h_1.

And if you remember the h_2 and h_1 already we have written h_2 and h_1 somewhere we have written it. So h_2 / h_1

$$\tan\Omega = \frac{h_2}{h_1} = -\frac{z_0\dot{x}_0 - \dot{z}_0\, x_0}{y_0\dot{z}_0 - \dot{y}_0 z_0}$$

sign so this is your tan Ω, but as you know in the figure earlier we have drawn this is N this is N' this point is O and here we have written as X, Y and Z. So this line this angle we have written as nodal angle.

So in the X-Y plane this nodal angle will vary from 0 to 360º, this varies between 0 to 360º. So our lone tan α if you use this equation or either this form you would not be able to work it out.

For the right value you need the signs of both of them what is the sin Ω and cos Ω as I told you earlier and using that then you determine in which quadrant this Ω is located.

Okay just based on this equation you will do the error. So keep care of this that utilize both these signs utilize signs of both these expression to get the right value of Ω. Okay I just intended to skip this part because in a scalar one again I am going to work out this. So this part we have finished. So till now we have determined a, e then θ and Ω rest remaining are i and ω these are the 2 things that we need to work out.

So again we come to this place say along this direction this is the perigee direction and along the perigee direction you are aware that $\hat{e}$ is the unit vector already we have derived $\hat{e}$ this = B divided by μ this is the unit vector here in this direction and in this direction along the r direction we have $\widehat{e_r}$ this is the unit vector. Therefore, if we take dot product of these 2 unit vectors then we will be able to get the angle θ.

Similarly, this unit vector is known $\widehat{e_n}$ and this unit vector is known. So if we take dot product of this two so the ω this ω and this θ we can determine. Now the next step we are going to utilize this fact that the unit vectors are known in this direction and we can utilize them to find out these 2 angles.

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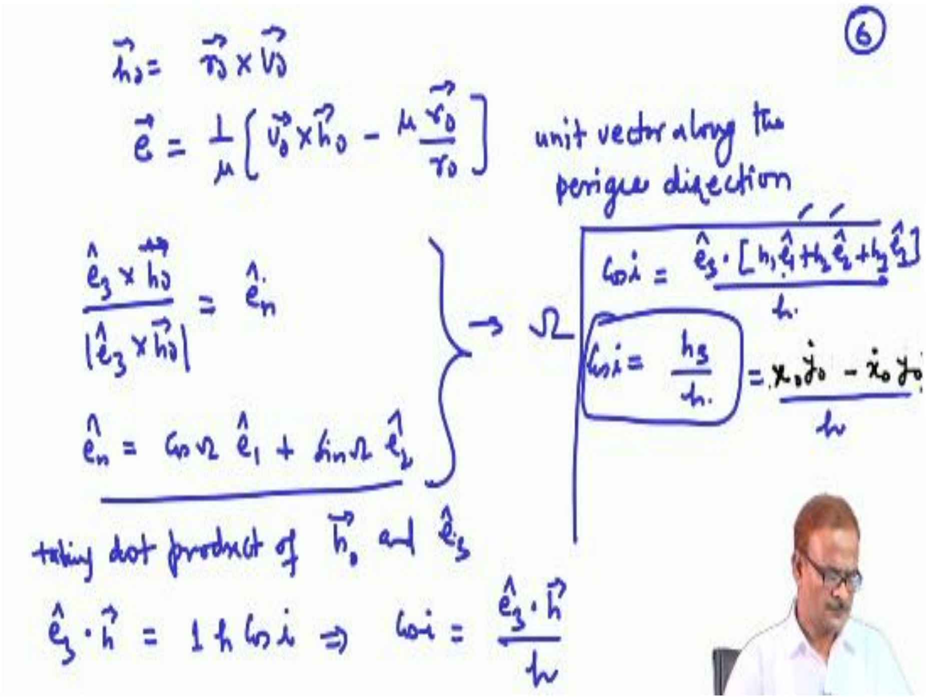

So the expression for and from there we have derived this e = 1 / ; v_0 cross $h_0 - \mu \overrightarrow{r_0}$ / r_0. So this is the unit vector along the perigee direction. We will be solving certain problems based on the initial value of x_0, y_0, z_0 $\dot{x}_0$ $\dot{y}_0$ $\dot{z}_0$ for getting the value of a etcetera. So due course of time I will be solving all this problem. So now we have to take the dot product of this vector r_0 to get θ.

And with $\widehat{e_n}$ to get the angle ω or what we call as the argument of perigee. and what else i is remaining θ we have already got one way, but here also this will be another way of doing it so which we will do again so already done and one more way is waiting and this two we have to work it out. So 3 things we are going to work out here. So $\widehat{e_3}$ cross h_0 this already we have written as this is nothing, but $\widehat{e_n}$is the unit vector in the e_n direction.

And thereafter we have used this $\widehat{e_n}$ as just repeating this to revise it $\widehat{e_2}$. So from here already we have worked out Ω. Now next we have to go and work out the other parts which are remaining. Okay for finding out this i if I take dot product of h and e_3 so what it will give us. So taking dot product of h_0 or h vector and $\widehat{e_3}$ so we write here $\widehat{e_3}$ dot $\vec{h}$ this = magnitude of e_3 vector which is 1.

And this magnitude this will be h, 1 h and angle between them which is cos i. So from there this implies cos i = $\widehat{e_3}$ h divided by h and therefore cos *i* this = $\widehat{e_3}$ dot here h_1 $\widehat{e_1}$ + h_2 $\widehat{e_2}$ and h_3 $\widehat{e_3}$ divided by h and of course you know that this two will vanish only leaving the last one so this becomes h_3 divided by h. So

$$cos\ i\ =\ h_3\ /h$$

Now h_3 already we have written it and just now I wrote it somewhere. So

$$h3\ = \frac{x_0\ \dot{z}_0 -\ \dot{x}_0\ z_0}{h}$$

So if we use this clear I am discussing some more things. This angle i as you see from this place this is the X-Y plane suppose and in this your orbit is right now lying like this I will show it by some other color and then it is moving out. So once it is moving out this is the angle here you are looking at the side view this is the side view of the orbit and this is X-Y plane. So as it moves out so *i* will vary between 0 and 180^o. So this is going from 0 to 180^o.

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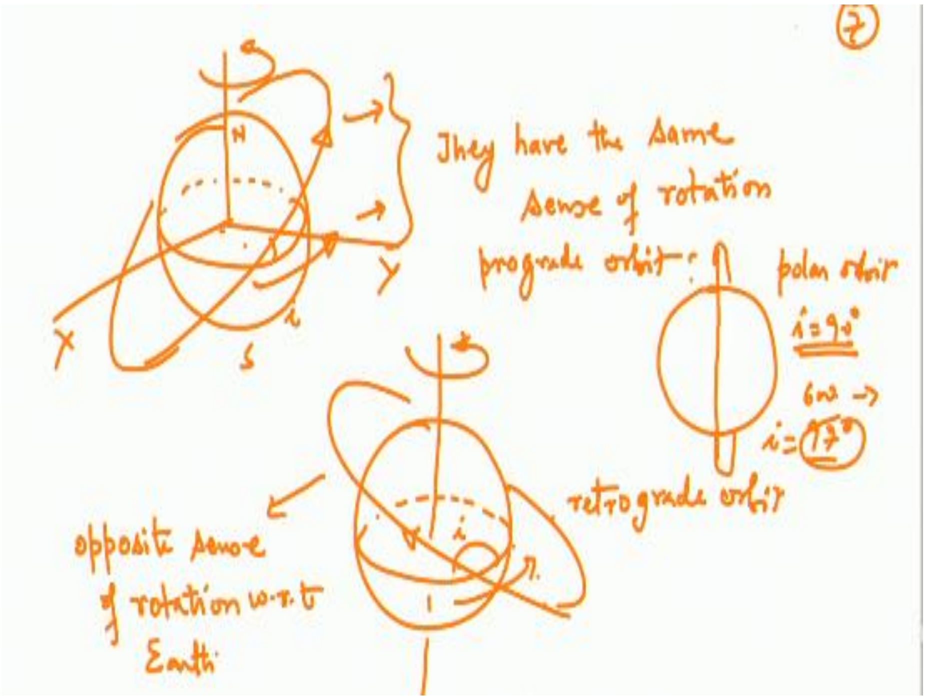

If I have earth here and earth is rotating on its axis in this direction it rotates from west to east and this is the north direction here and this is the south here. If your orbit orientation is like this and satellite is going here in this direction and of course you see here that the earth is rotating this way. So both are moving in the same direction their nature of rotation is the same or what we call it they have the same sense of rotation and this we call as the prograde orbit.

Contrary to this if earth is rotating like this so this is the sense of rotation of the earth, but your orbit is something like this. So you can see that here in this case here if I take this as the X-Y plane so this is the inclination angle i. Here in this case the inclination angle is acute while here in this case inclination angle will become obtuse so this is called retrograde orbit why because this has opposite sense of rotation with respect to earth.

If the orbit is exactly 90º that means t is going up and down like this so this is called the polar orbit. If orbit is little bit more inclined around say for 600 km orbit you are around 97º of inclination i = this kind of orbit, we call as the sun-synchronous orbit because this orbit it keeps rotating and always faces the sun that we will come later on while we do the general perturbation method variation of parameter so at that time we will take care of that.

So here what my emphasis is why I am discussing this because this i is varying between 0 and 180º if i = 0 there is a line in the x, y plane. If you are measuring angle with respect to this i this is not with respect to the equator for that you have to give correction for that then you call the equatorial orbit and similarly the polar orbit also we have to give correction because the plane of rotation of the; this equatorial plane and the X-Y plane they are not exactly the same they differ.

So using this expression you will be able to resolve get the right value of i you do not need any other support to get the correct value of i. So here i lies between 0º to 180º.

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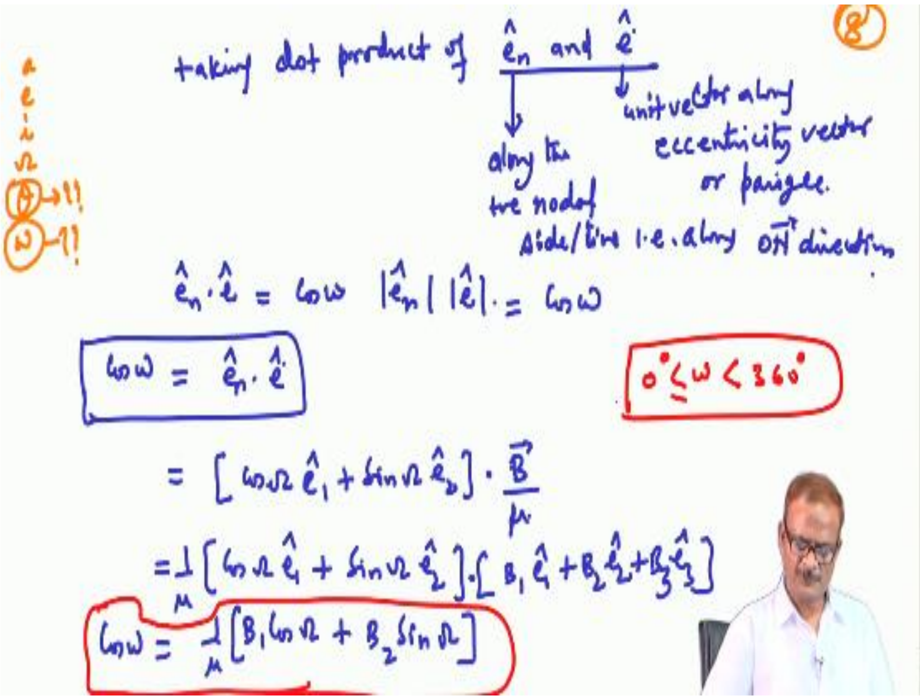

Okay next so right now what we have worked out a, e, i, Ω and θ we have done in one way. One more way we will do ω is remaining so these 2 are remaining. So working out through the vector method we have to do these 2 things and we approach it in the same way. So then again we look back here in this figure and then if we have to look at this evaluate this ω.

So we have to take dot product of $\widehat{e_n}$ and this $\hat{e}$. If we do this ω will be available to work. So taking dot product of $\widehat{e_n}$ and $\hat{e}$ so this is the unit vector along eccentricity vector or periapsis or perigee and this is along the positive nodal side or nodal line that is along ON direction. So we have

$$\widehat{e_n}\ \hat{e} = cos\ \omega$$

because both the $\widehat{e_n}$ and $\hat{e}$ they are of unit magnitude so this is cos ω.

So cos ω ω= $\widehat{e_n}$ times $\hat{e}$ and now we need to insert this $\widehat{e_n}$ so $\widehat{e_n}$ again we have to fetch from the earlier one this is cos ω sin ω cos Ω $\widehat{e_1}$+ sin Ω $\widehat{e_2}$ and dot $\hat{e}$. $\hat{e}$ is the vector already we have written here, $\hat{e}$ is this vector. So obviously you can see that this is little more complicated and if we try to expand it, it will take lot more time and it will get complicated also.

So rather than doing that if we just write it in this format like this is B divided by μ and B will have components sin Ω $\widehat{e_2}$. So B_1 $\widehat{e_1}$ B_2 $\widehat{e_2}$ so this will get reduced to and there is the dot product here and this is B_1 cos Ω 1 / μ + B_2 sin Ω this is cos Ω, but then you need to work out this B_1 and B_2 here which is required and that we have to get from this expression.

So this is nothing, but B/μ so right hand side we have to expand and after expanding we have to combine all the terms together of the similar terms like which corresponding to e_1, e_2 and e_3 and thereafter from there we will have the B_1 and B_2 and utilize it to solve this problem. So this we are not going to do otherwise it will stretch for another 10, 15 minutes. So this suffices for the time being.

Okay we can take as of some tutorial problem as an exercise that will be much better. Okay ω here also it lies between 0 and 360º. So how to know that in which quadrant it is align both ω and θ they have the same problem and Ω also. So we have to work out in which quadrant it is going to lie.

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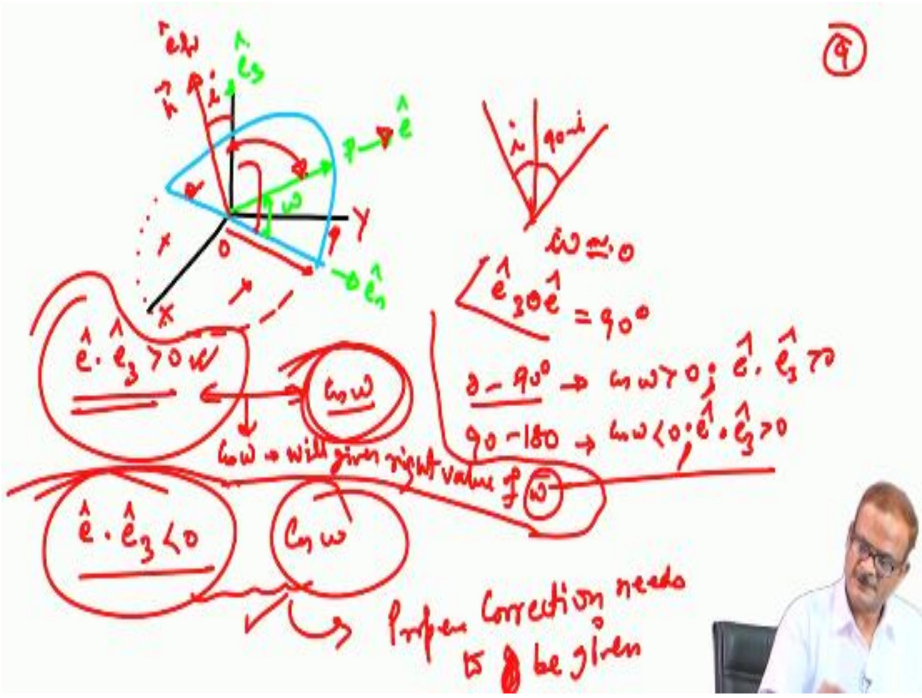

So these are the unit vectors already we have written. Now to resolve that in which quadrant this ω is going to lie as you can see this is cos ω and cos ω obviously whatever the value is there if it comes positive or negative so you have 4 quadrants the cos is positive here in this side, positive here on this side, negative here, negative here so where it is going to lie somewhere you have to resolve it just by getting this value you would not be able to tell the correct value of ω.

So to resolve this we need to consider few things here like what we can see here that if e cap times e3 cap if this is > 0 that means your this part of the orbit it is align above the x, y plane this is x and y plane so it is align from here to up to here. Okay going from this place to this place. So you can see that in this case this quantity is going to be > 0 because this angle the angle between this and this it will be always < 90 ºs.

If you look from the side so this is vertical line and this is the orbit and here this is the orbit normal so this is i this is 90 – i so this angle is always going to be < 90 º. If suppose ω is near about 0 so that means it will be lying along this line. So in that case I will this ω angle will be around ω is nearly equal to 0 so the angle between e3 cap and e cap vector this angle between this two this will be around 90 º.

So if we write this as O so this is the angle so this is the way this is 90 º here. So this is not going to exceed that value. So if your this perigee line is lying above the x, y plane so this is going to be satisfied and you know in that case that if cos ω is positive and this is positive then it is bound to be in the first quadrant means it is less than between 0 and 90 º.

If this is coming out to be negative cos ω is negative, but this quantity is positive so it is we are sure that it is lying between 90 and 180 º. So this is the case when cos ω is > 0 and e cap and dot e3 cap is > 0 and this will be the case when cos Ω this is < 0 e cap and dot e3 cap this is > 0 so this is the situation. The same way we can get the information while this is < 0.

So what is the cos ω value so cos ω value combine it with this one and you will be able to determine whether it is lying in the third quadrant or either the fourth quadrant means it is going below like this going below the x, y plane. So we will be able to sort out this problem. So here in this case if this is the situation proper correction needs to be given while here in this case if this is the case cos ω will give right value of ω.

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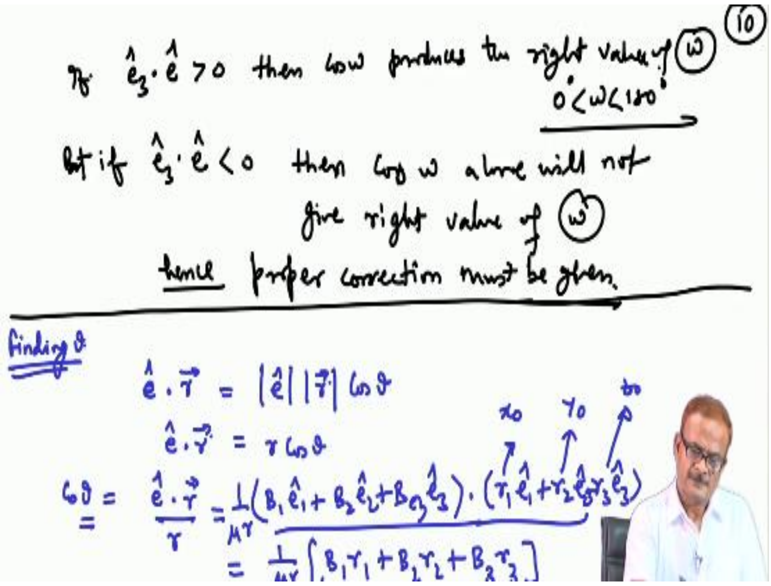

So on the next page I will write it if e3 cap dot e cap is > 0 then cos ω produces the right value of ω that is this will be lying between 0 and 180 º, but if e3 cap dot e3 if this is < 0 then cos ω alone will not give right value of ω and hence proper correction must be given as explained on the previous page. On the last we are left with the θ angle so similarly for finding θ where e cap dot r cap so where we have drawn this here, here in this place.

So this is your e cap and er cap is here so if we take dot product of this two so this angle will be known to us. So we do this now so e dot e cap dot product with r so this will be e cap magnitude times r cap instead of writing r cap we can write here r times cos θ. So this is r cross θ and therefore cos θ this = e cap dot r divided by r and e cap we know that we are writing in terms of B e1 cap, B2 e2 cap, B3 e3 cap dot r1 e1 cap, r2 e2 cap + r3 e3 cap divided by r.

So this will result in B1 r1 + B2 r2 and divided by μ of course here + B3 r3 divided by this divided by μ r. I should write here somewhere in different place let me rub it out this is divided by μ times r so we would write here divided by μ r and this = 1 / μ r times B1 r1 + B2 r2 + B3 and r1, r2, r3 what these are these are nothing but x, y, z. So the quantity this quantities are this is x0, this is y0 and this is z0 the components of the r vector along the x, y and z direction. So again θ this is the angle, this is the true anomaly.

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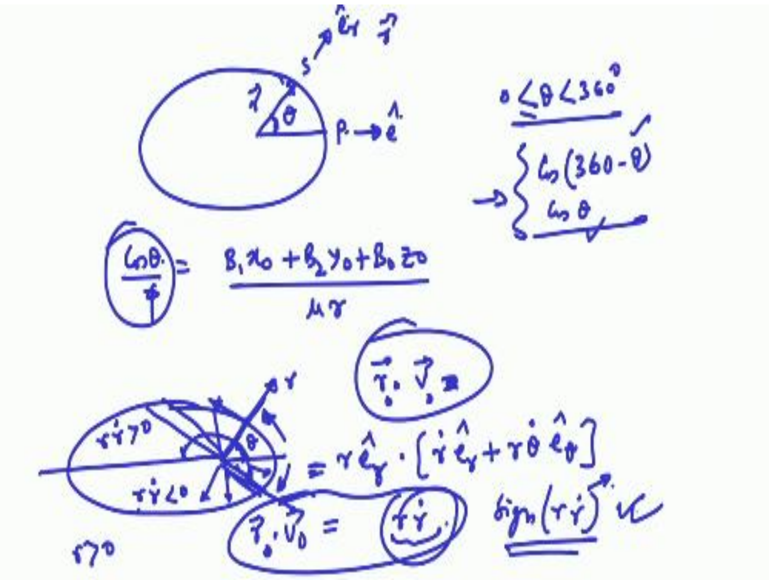

So θ is the true anomaly we are measuring from the periapsis this is the point P and from here we are measuring angle θ this satellite is here and this is angle θ. So along this direction we have taken as the e vector e cap and along this direction we have er cap which we are writing in terms of r so this is the r vector. So this θ angle this again it varies from θ varies from 0 to 360 ^{o}s.

So once you are getting it using the cos θ so you need to be sure in which quadrant it is aligned cos θ this equal to whatever we have got there B1 x0 + B2 y0 + B3 z0 divided by μ times r. This is what we are getting here. So we need to resolve this part also where this θ is going to lie. So for this what we will do we will utilize the technique we have developed earlier if this is here this is your r vector in this direction this is the v vector.

So r vector the angle between r0 or the v0 vector if we take this dot product so this is your r times er cap dot r dot times er cap + r times θ dot times e θ cap. Now this quantity becomes equal to this so this part is vanishing and we are left with r times r dot. Earlier also perhaps we have discussed this so as you can see here in this place that in this part this quantity is going to be positive.

So rr dot is > 0 here in this part and rr dot is < 0 here in this part because on this side the radius vector is increasing. So r dot is positive on this side radius vector is decreasing continuously so r dot is negative, r is always positive, r is always > 0 and therefore using the sign of this. So find out what is the sign of this sign rr dot use this sign which is coming from this place of course that means you have to use that r dot sign what is the value of the r dot.

If you use this and use this equation here so you will be able to determine where you are exactly the θ is located because this will again get duplicated. You will have the same value like your cos 360 – θ and cos θ both have the same value. Okay so how do we know that this is the right one or this is the right one that means your θ is lying here in this point or here it is lying here in this point which one is correct.

It is lying here on this side or it is lying here on this side either on this side or either on this side where it is lying. So this we need to resolve so this can be resolved using this particular technique. So this way we will be able to resolve all this issues and we are going to do the same thing little by, already I have done scalar method also I have introduced, but again I will repeat for few of them.

So that later on while solving the problem it becomes little easier to deal with all this topics and of course I will provide you the hard copy sorry the soft copy of this material printed soft copy, but there it will not be exactly the lecture wise. Let us say it is a one hour lecture it is condensed that way you have to look into that so you can take help of them while the course runs.