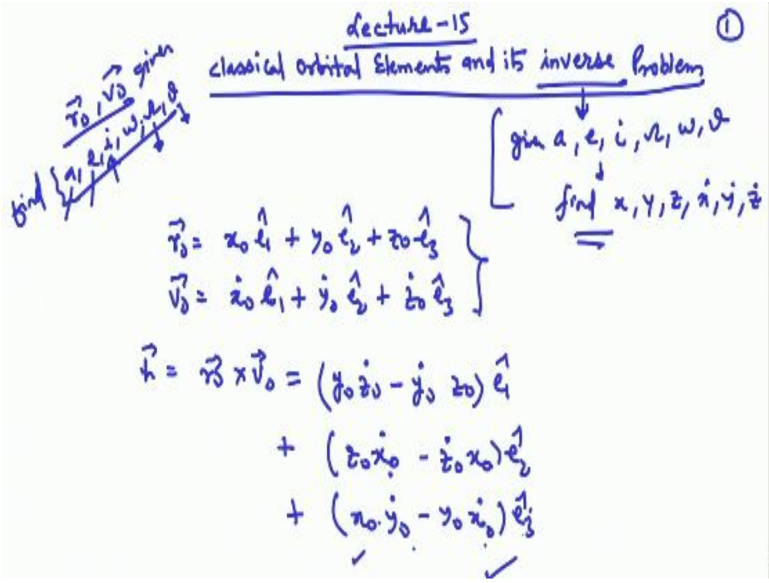
Space Flight Mechanics

Prof. Manoranjan Sinha
Department of Aerospace Engineering
Indian Institute of Technology – Kharagpur

Lecture – 15
Classical Orbital Elements and Its Inverse Problems

(Refer Slide Time: 00:13)



Welcome to lecture number 15 so we are discussing about the classical orbital elements so we will continue with that and also what we have done that from the burnout data. Burnout data means already I have stated that r0 and v0 is given so find a, this is given find a, e, i, ω, Ω and θ and then there will be the reverse inverse problem of this.

So, inverse problem of this will be given a, e, i, Ω, ω and θ and from there, find x, y, z $\dot{x}$, $\dot{y}$ and $\dot{z}$ so these are the two problems. So as I told you probably mentioned you earlier that we will be doing by the scalar method also finding these elements. So already we have done all this things and also while working out I have done it by the scalar method.

So few things this also I have done by the scalar method so ω and θ already we have done by a scalar method also so only thing remains about the ω. So I will quickly workout this and then we will go to the inverse problem. So we have written

$$r_0 = x_0\hat{e_1} + y_0\,\hat{e_2} + z_0\hat{e_3}$$

And

$$v_0 = \dot{x_0}\,\hat{e_1} + \dot{y_0}\hat{e_2} + \dot{z_0}\,\hat{e_3}$$

and from there then I have to derive this each of we have already derived which is r_0 cross v_0.

And that we have written as this expression see it is difficult to remember, but if you know the basic principle you can always work it out okay anytime we can work out. So for saving time I have to refer back okay I am not finding those materials anyway this we can write as y_0 $\dot{z_0}$– $\dot{y_0}$ z_0 $\widehat{e_1}$+ $\dot{x_0}$ just look back into the previous lecture I have done here this part I am writing this part there.

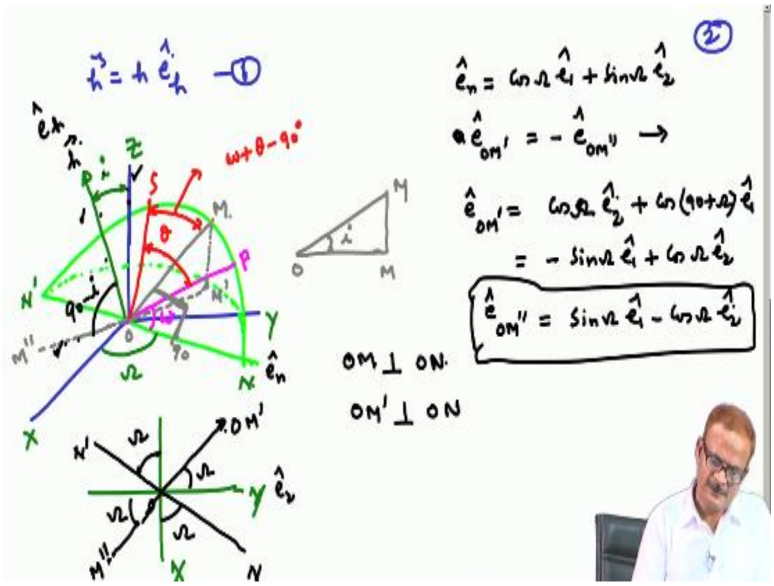
So if we utilize this and so we are rewriting the whole thing there for ready reference $\widehat{e_2}$ and then x_0 times $\dot{y_0}$– y_0 times $\dot{x_0}$ $\widehat{e_3}$. We will write in a proper sequence this is z_0 times $\dot{x_0}$– $\dot{z_0}$ times x_0 the same sequence we are following. Okay whatever I have written here so I will verify with our earlier derivation it will take a few minutes y_0 $\dot{z_0}$and $\dot{y_0}$z0 $\dot{x_0}$ z this is okay x_0 so x_0 here the same thing has got repeated.

See the mistake we have done here e_2 is x_0 $\dot{z_0}$ this is fine and here this is e_3 is x_0 $\dot{y_0}$ this is h_0 dot so we have to correct it $\dot{x}$ = $\dot{x_0}$ y_0 so this is a big error and this will be this is a big error data scripting that is why I wanted to verify by writing copying from one place to another place we when this kind of error creeps in x_0 so here this is okay we have done mistake in h_3.

Okay so this is fine h_3 is appearing anywhere here h_3 is appearing so this place we have to do the correction this is h_3 is x_0 $\dot{y_0}$ and $\dot{x_0}$ times y_0 so this is a correction required so I will note it down so that I can correct in later on. Okay so this correction is required here so already I have told you that this type of error creep in while writing where the attention gets diverted while thinking of the subject.

So I hope we do not have any other place this mistake so this is this all are all okay this are all okay this we have done the correction here the correction is there this is y_0 and this is y_0 $\dot{z_0}$ y_0. Okay so we have been here in this place so this is fine this part is okay y_0 $z_0\dot{y_0}$ z_0 z_0x this is okay.

(Refer Slide Time: 11:01)



Okay so now we I told you that we will do the same thing through the scalar method. So scalar method whatever we have done earlier it is almost the same there is nothing new in that, but still some possibilities are there to explore it further and let us go through that quickly so that in this lecture we are able to finish it. So h we are writing as h times eh cap and already with a equation for this we have written now we need some diagram to work it out.

The projection of this we will show again by a dotted line so this is a projection of the orbit there is a line we are showing somewhere here this is P and so this angle this is ω and this angle we will write there as M and this angle we will show it as cap 90º. So showing like this means it is a 90º angle and then we have the satellite here in this place.

So this angle we are showing from this place to place as θ so angle between this two then this becomes ω + θ –90º. Now projection of this M on to the X-Y plane this we will write as M' and then we can join it by a line like this and this line we will extend it on this side and we will write here M". This is o because this angle here this is 90º.

So therefore this wedge-shaped part we have drawn here this is OM'M this angle also this is *i* and normal to the orbit this angle is also *i*. So here I am not showing, but because of this figure will get much more complex it will look bad. This angle is from this place to this place this is Ω this we have written as N this as N'. So if we take the X-Y plane this is the X direction Y direction.

Okay this is the X and Y direction okay nodal line it is appearing between X and Y so the nodal line is appearing like this. So this is your N' and here it is N and this point is o. So here OM' it is perpendicular to the line OM is perpendicular to ON and also OM' this will be perpendicular to ON. So here we will have OM prime and on this side we will have here M". Okay so this is the construct now this angle we have written as Ω.

So naturally this angle then this becomes Ω and this angle also gets the Ω this angle will be Ω and rest accordingly this angle will be Ω – 90°, 90 – Ω. Now we can work out this part so $\widehat{e_n}$ which is the unit vector along this direction we have written as cos Ω $\widehat{e_1}$+ sin Ω $\widehat{e_2}$.

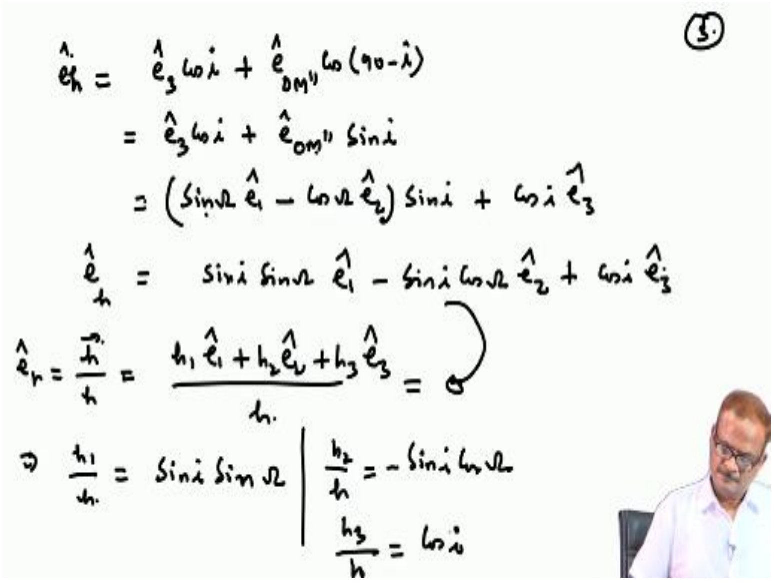
This already we have done also we have $\hat{e}_{OM'}$ this unit vector along the $\hat{e}_{OM'}$ in this direction this will be = - $\hat{e}_{OM''}$ and how we can write the $\hat{e}_{OM'}$, vector unit vector this will be $\widehat{e_2}$ is along this direction. So cos Ω $\widehat{e_2}$ and from here this place this is +cos 90 + Ω $\widehat{e_1}$so this becomes – sin Ω $\widehat{e_1}$+ cos Ω $\widehat{e_2}$.

$$\hat{e}_{OM''} = sin\,\Omega\,\widehat{e_1} - cos\,\Omega\,\widehat{e_2}$$

So therefore $\hat{e}$ there is a OM' here it is $\hat{e}_{OM''}$ this will be equal to minus of this as we have written here from this place. So this will be sin Ω $\widehat{e_1}$– cos Ω $\widehat{e_2}$. So why we are trying to do like this there is a reason behind this which will be shortly visible to you. This angle is *i* therefore this angle is *i* therefore this angle is 90 – *i*. Okay this is $\vec{h}$ here.

So $\vec{h}$ we can break along this direction also we can break along this direction. It can be broken along this direction and this direction or either we can take unit vector component along this direction, unit vector from this component along this direction to get unit vector along the h direction so this is the process of working.

(Refer Slide Time: 19:25)



So already we have written $\hat{e}_{OM}$ this. So we have eh what I was telling that this can be broken along this direction or either we can calculate unit vector along this direction which is $\hat{e}_h$. So $\hat{e}_h$ unit vector will be $\widehat{e_3}$ cos i + $\hat{e}_{OM''}$ cos 90 – i. So e_3 cos i and then + $\hat{e}_{OM''}$ sin i. Now insert the value for the $\hat{e}_{OM''}$ so $\hat{e}_{OM''}$ is sin i sin Ω $\widehat{e_1}$ – cos Ω $\widehat{e_2}$ and + cos i $\widehat{e_3}$.

So this is the situation here sin Ω $\widehat{e_1}$ – cos Ω $\widehat{e_2}$+ cos i $\widehat{e_3}$ and here this is multiplied by cos i we have written here one part we are forgetting so I will have to rewrite it. This part we have written first so sin Ω e_1 this part then this multiplied by sin i this is the missing part and then cos i times $\widehat{e_3}$ so we expand it sin i times sin Ω $\widehat{e_1}$ – sin i times cos Ω $\widehat{e_2}$ and cos i times $\widehat{e_3}$.

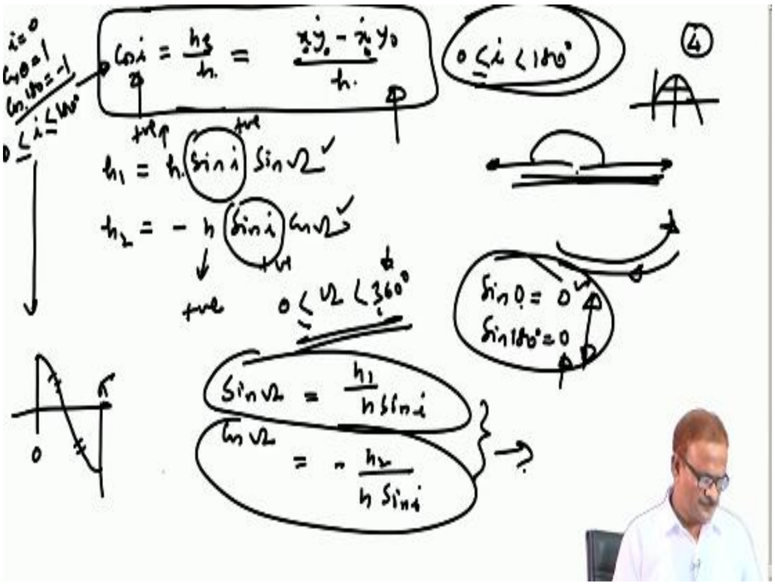
So this is $\hat{h}$ the unit vector along the h direction. So once we have got this now $\vec{h}$ already we know this $\vec{h}$ $\hat{e}_h$ will also be = $\vec{h}$ / h so h we have already written this is

$$\hat{e}_h = \frac{h_1\, \widehat{e_1} + h_2\, \widehat{e_2} + h_3\, \widehat{e_3}}{h}$$

. So this must be equal to this quantity here and therefore h_1/ h this we can write as sin i times sin Ω h_2 /h = - sin i cos Ω and

$$h_3/h = \cos i$$

(Refer Slide Time: 23:52)



So first we take the

$$\cos i = h_3/h$$

and h_3 if we go back h_3 is the quantity this one $x_0\ \dot{y}_0 - y_0\ \dot{x}_0$ X-Y. So already I have told you there is no ambiguity in getting i because of from here the i varies between 0 and 180º. So exactly we are going to get the right value of the i using this expression so there is no problem in this. This is the same thing we have got from using the earlier method only thing there we wrote it wrongly while writing it.

So that correction I have done right now you have seen it, but also I will introduce it in the video that I will have to edit the video. So then the other two we have here h_1 and h_2 using this sin is common to sin i is common to both of them. So using this we will be able to determine sin Ω this way the Ω angle. So this is the same thing what we have done here.

So h_1 = h times sin i times sin Ω and h_2 = - h times sin i times cos Ω. So this is what is here. Now look here in this part this part is always positive because i lies between we can see here 0º means your orbit is just in the line like this along this direction and 180 means it is just going in the opposite direction. So we can put here also the equality sign it is not a problem.

So it varies between 0 that means in one case the satellite is going here in this direction in another case satellite is coming here in this direction so here equality sign is not a problem. So it varies between 0 and 180º for i equal to hundred; the problem will be that sin 0º = 0 and sin 180 equal to also 0 this is the problem. So which one then it is referring to that you have to decide.

And that is why I told you that if we just keep this part for this ambiguity will not be producing. Okay so we do not have to produce the ambiguity so in that case while i = 0 so we are getting this 0 value and when i less than 80º so still it will be positive, but it will have certain value that means our figure it goes like this. So I was discussing something I was strayed somewhat here in this problem once I put i = 0.

Okay so this we got as cos 0 = 1 and if I put i = 180º so we get this as – 1 so there is no ambiguity this is the point where I was telling. So for that case we can put here in this place that i lies between 0 and 180ºs it will not produce any ambiguity as we can see from this curve this is the cos curve which is going from 0 to π. In this portion this is positive in this portion this is negative and therefore no ambiguity here.

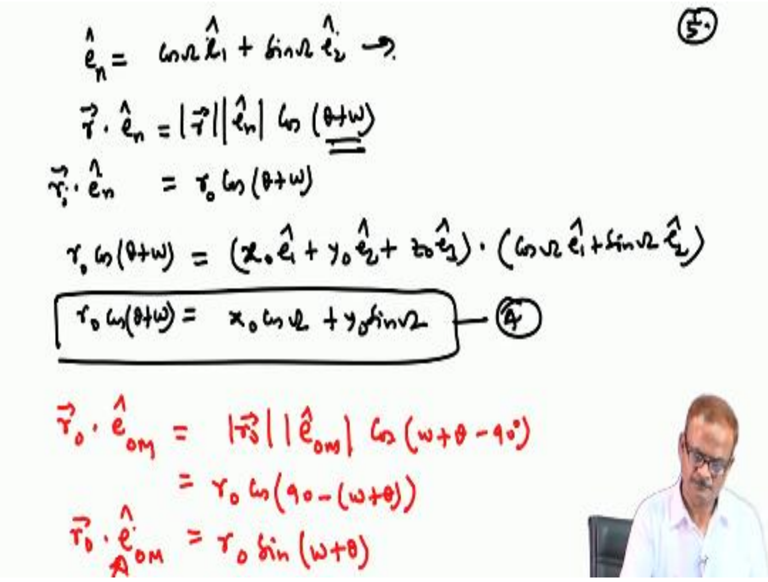
But for the sin, sin is always positive between 0 and 180º so this is always positive. This quantity Ω it varies between 0 and 360º. So we have to fix it up so how do we fix up we know this quantity is positive here this quantity is positive so if we know from this place sin Ω and cos Ω sin which will be nothing, but h_1 divided by h sin i and – h_2 divided by h sin i.

So depending on the sin of both of them we will be able to resolve in which quadrant they are going to lie okay it is very simple resolving it. So this is the objective so you can see that by the scalar method or by the vector method here this is also a vector method, but the work is little different the way you see that how we have proceeded. Okay we have taken an unit vector along this direction and then we have worked out.

So this is bit longer, but no doubt it is longer but it is another way of solving the same problem. So as I told you in the mechanics we solve the same problem in multiple ways the possibilities are there and moreover also I would like to remind you that whenever you detect any error during your course because while writing this error is bound to creep in. So if you find any error you can report it during while the course is running.

So later on also we can edit it. So we have done this part now what we are going to do we know this vector and let us assume that this e cap vector which we were earlier knowing this is not known suppose this is not known. So in that case what we will do if this vector $\hat{e}$ is not known so there are other ways of doing this. So in that case what we do we take this and we take this part and also this is the vector. So utilizing this then we will be able to solve the problem.

(Refer Slide Time: 32:06)



So let us see how do we do this so I will finish in this lecture this part. So en cap we have written as cos Ω $\widehat{e_1}$ + sin Ω $\widehat{e_2}$. You can see here this is θ +ω. So the angle between angle from this place going from this place to this place all the way. This is ω and this is θ so ω + θ. This is the angle between this vector and this vector.

So here we have written this so this gets reduced to r cos (θ + ω) and this is $\dot{r}\ \widehat{e_n}$ and we know the $\widehat{e_n}$ and $\vec{r}$ also and therefore we will be able to solve it. So we can write here as r_0 cos θ + ω this = $x_0\ \widehat{e_1}$+ $y_0\ \widehat{e_2}$ + $z_0\ \widehat{e_3}$times $\widehat{e_n}$ is cos Ω $\widehat{e_1}$+ sin Ω $\widehat{e_2}$. So therefore this gets reduced to

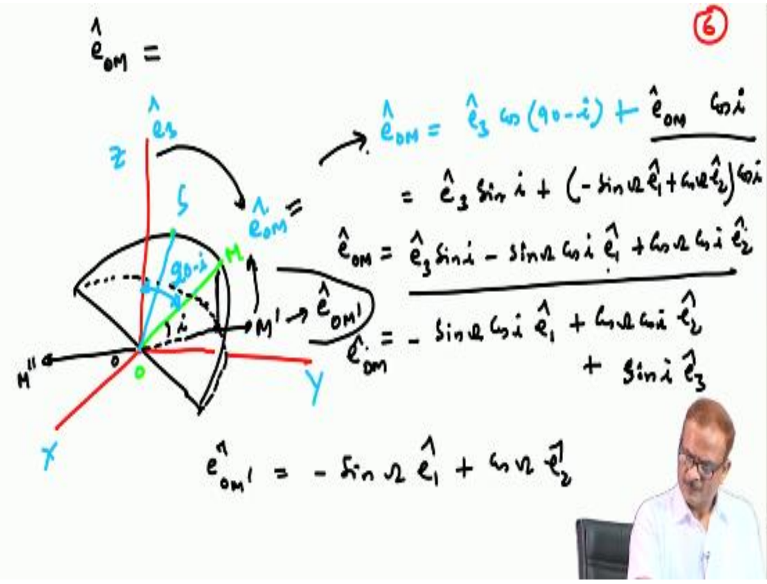
$$r\ cos\ (\theta\ +\ \omega) = \ x_0\ cos\ \Omega\ +\ y_0\ sin\ \Omega$$

So this is r_0 cos θ + ω so this is one equation let us name this as equation A. Now unit vector along the OM direction along this direction this we can get and if we know the unit vector along this direction. So from here to here this angle can be computed θ angle is it is wrongly shown here θ angle is from this place to this place this is θ angle θ angle is not up to this point.

So θ angle is from the argument of perigee to this point this is your θ angle. So here this part perhaps I have shown only this part here so this is θ and this angle is ω + θ which is ω angle is from this place to this place this is ω. So ω + θ – 90 º. So repeating on the same figure the figure gets complicated. So ω + θ – 90 º so this angle if we know.

And the vector along the OM direction if we are aware of we will be able to work out another way. Here again we write that $\dot{r_0}\ \hat{e}_{OM}$ okay so this will be r_0 magnitude $\hat{e}_{OM}$ magnitude and angle between them is cos here θ + ω or ω + θ – 90 º. So this becomes r_0 cos 90 – ω + θ. So this is r_0 sin ω + θ and on the left hand side this is $\dot{r_0}\ \ \hat{e}_{OM}$. So we need to know this $\hat{e}_{OM}$ if we know this then we will be able to work out this problem.

(Refer Slide Time: 37:07)



So we need to know $\hat{e}_{OM}$ vector this we have to determine. So again going back coming to this figure so you can see that along this direction this component will come from $\widehat{e_3}$ the angle again I will go and draw on the next page this figure without that it is becoming very complicated. Okay I will draw the figure here this is the M direction this is o somewhere here satellite is in this place so this is S.

So we are looking for $\hat{e}_{OM}$, $\widehat{e_3}$ is along this direction this is X, Y and Z so you can see that $\hat{e}_{OM}$ vector this can be composed of this angle is 90 – i so $\widehat{e_3}$ cos 90 – i and plus then we have the footprint of this orbit on the X-Y plane. So this footprint is from this place to this place and this we have here written as this point as M and in this direction we have written as this is M this is M' and this is M".

So we need the OM' vector and this angle we know that this angle is i so we have to write here $\hat{e}_{OM'}$ cos i and then we have to write this value how much this is. So this we have already worked out and this vector is basically – cos i sin Ω $\widehat{e_1}$+ cos i cos Ω $\widehat{e_2}$+ sin i times $\widehat{e_3}$. From where we are getting this see $\hat{e}_{OM'}$ we already have here and $\hat{e}_{OM'}$ is also there $\hat{e}_{OM'}$ is here.

And $\hat{e}_{OM'}$ is also there $\hat{e}_{OM'}$ is here okay $\hat{e}_{OM'}$ this is available to us in this place. So we use this

$$\hat{e}_{OM'} \;= -\,sin\,i\,\widehat{e_1} + cos\,i\,\widehat{e_2}$$

I will write here sin and cos Ω $\widehat{e_2}$. So $\hat{e}_{OM}$ what we have written here this vector component and this vector component we are taking multiplying it. So this is e_0 I will erase it this line let us erase and work out properly there is error in here so this part is okay.

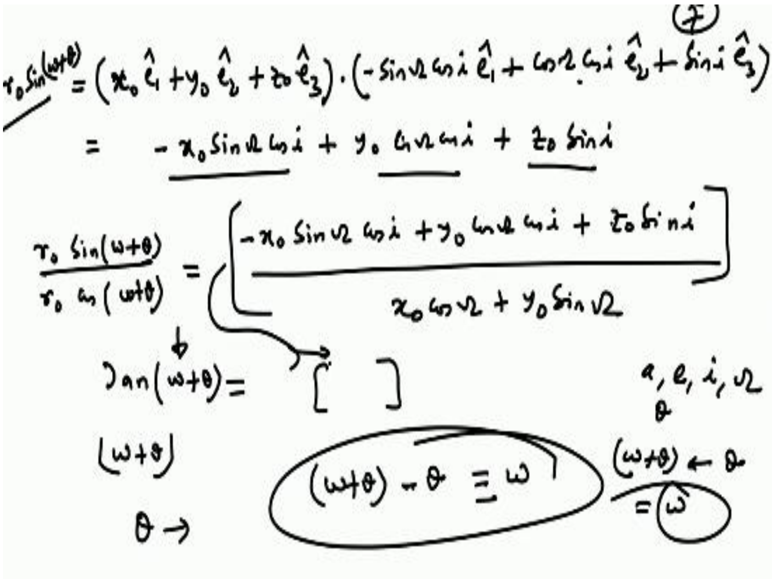
Here along this direction we have the vector $\hat{e}_{OM'}$ so component of this vector along this direction is $\hat{e}_{OM}$ this component and this component so $\hat{e}_{OM}$ will be defined which we are defining here so $\hat{e}_{OM}$ cos i so we write here $\widehat{e_3}$ sin i + $\hat{e}_{OM'}$ which is here which we are writing from the previous page $\hat{e}_{OM'}$ – sin Ω cos Ω.

So – sin capital $\widehat{e_1}$+ cos Ω $\widehat{e_2}$ and this is not a dot product here this is simple thing and then cos i. So once we expand it so this is $\widehat{e_3}$ sin i and then expand it again – sin Ω cos i times $\widehat{e_1}$ + cos Ω cos i times $\widehat{e_2}$. So this is $\hat{e}_{OM}$. So now once we know this vector; so this is what I wrote earlier so I will rearrange it write first this one.

$$\hat{e}_{OM} = -\sin\Omega\cos i\,\widehat{e_1} + \cos\Omega\cos i\,\widehat{e_2} + \sin i\,\widehat{e_3}$$

So this is in proper order so this is $\hat{e}_{OM}$. So once we have got this now it becomes easy to work our problem. So what we were doing that we have written this equation. So you see that on the left hand side r_0 is already known so this is also known now.

(Refer Slide Time: 44:34)

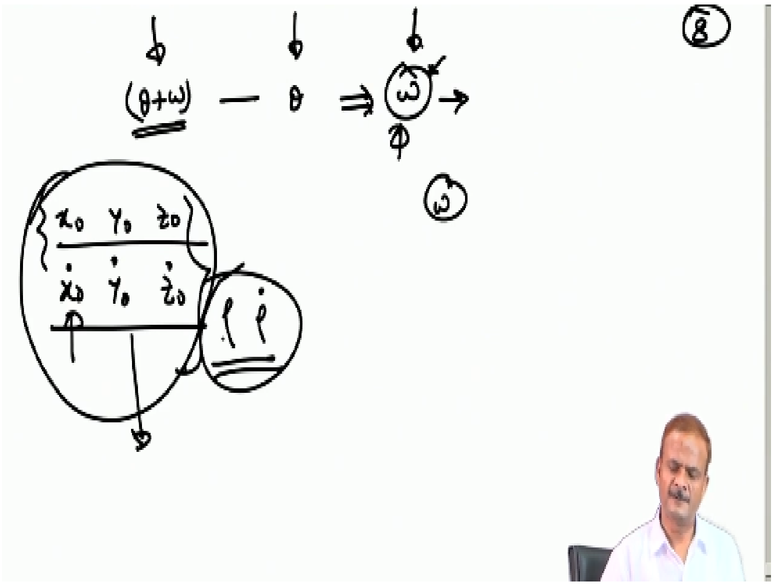


So in the next page we can write r_0 is $x_0\ \hat{e_1}$ + $y_0\ \hat{e_2}$ + $z_0\ \hat{e_3}$ and dot product with – sin Ω cos i times $\hat{e_1}$ + cos Ω cos i times $\hat{e_2}$ and sin i times $\hat{e_3}$ and this is nothing but this quantity r_0 sin (ω + θ). So this is r_0 sin (ω + θ) equal to this quantity. So right hand side we rewrite it, it is a dot product so it is easy to work out – x_0 sin Ω cos i + y_0 cos Ω cos i + z_0 sin i.

Okay so this terms are correct once we have done this so r_0 sin (ω + θ) is available and r_0 cos (ω + θ) is also available. Now if we divide r_0 sin (ω + θ) divided by r_0 cos (ω + θ) then this gets reduced to – x_0 sin Ω cos i + y_0 cos Ω cos i + z_0 sin i divided by r0 cos (ω + θ) that we have to bring in. This is x_0 cos Ω + y_0 sin Ω.

So this implies tan (ω + θ) this will be the quantity in the bracket here. So this way we have been able to determine ω + θ now you know that θ already we have determined. So θ already we have determined so ω + θ – θ that gives you ω. So this way once we have obtained this what we have done a, e, small i, Ω and θ already we have determined. So in this method we are determining together ω + θ and from there we are then subtracting this θ to get the value of ω. So ω can be determined in this way.

(Refer Slide Time: 48:56)



Now there is a problem with this while we try to do this θ + ω estimation and θ estimation and from there then you were subtracting this and this is from here you are trying to get this ω so what happens that you know that x_0 y_0 and z_0 these are not known exactly. These are never known exactly this is you can take it for granted because these are obtained from rho dot information after processing from multiple ground station.

So they will have certain amount of error. So if you try to use this so the error creeps in this value and also error has crept in this value so the error is also coming here in this place. So sometimes they get combined and the error in this ω this gets magnified and therefore this method using this method it is not very good that we determine θ + ω and from here then we determine ω.

Whatever we have done earlier we have separately determine ω by taking the angle between this en cap and along the perigee direction perigee direction I have not drawn here in this place, but we can draw this here also so I will do it in the pink. So somewhere let us say that this is the perigee direction so this is P. So this angle from here to here then this will be your ω.

So between this vector and this vector if we are trying to find the angle ω directly so this is much superior and thereafter between P and S whatever the angle is there from this point and this point then we can get the angle θ otherwise using this method the scalar method that I have followed they actually have prone to this error problem because x0 y0 z0 x dot y dot and z dot they are never known exactly.

And therefore we have to be careful while doing the orbit determination problem. So once you get into this area and we work on all this problem so you will come to know what are the problems you are facing. So I hope that for today this is end of I will stop here the inverse problem is remaining. So I thought of finishing it by 15th lecture, but I will have to go to the another lecture for the inverse problem that is getting a, e, i, Ω, ω and θ all these are given from there you get x y z x dot, y dot, z dot. So this we will do in the next lecture. Thank you for listening. Thank you very much.