Space Flight Mechanics
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Lecture – 16
Inverse Problem of Orbit Determination (Classical Orbital Elements)

Welcome to the lecture number 16. So, we have been working with the orbital elements, classical orbital elements. So, this time so what we did that given the initial position and velocity vector from there we derived the orbital parameters. So, right now we do the inverse problem means given the orbital parameters; find the velocity and position of the satellite. Okay, after that we will take up some problems.

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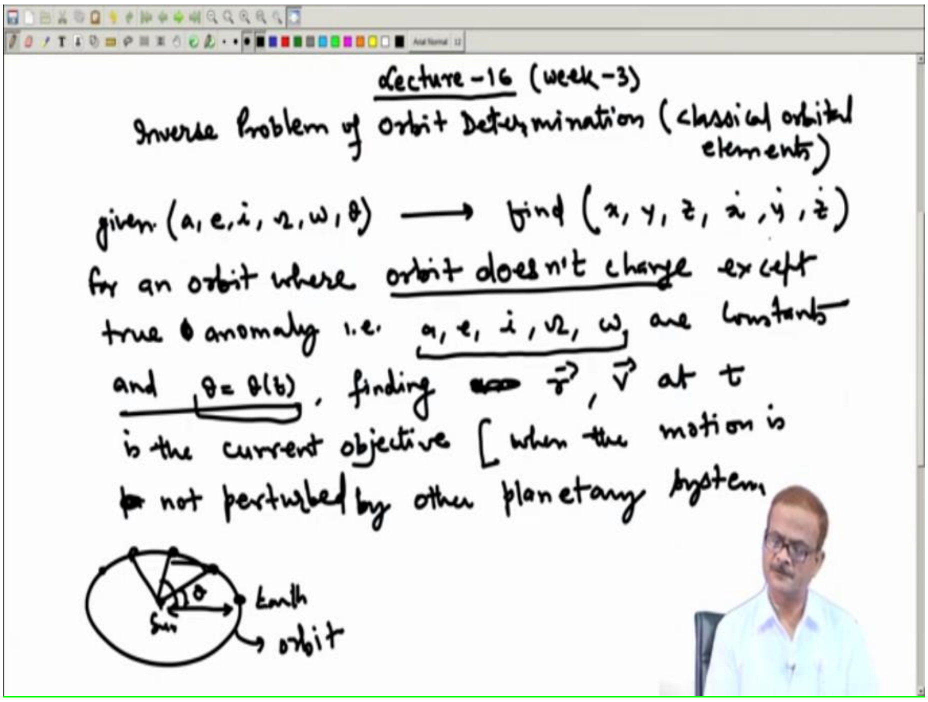

So, we start with, so, given a, e, i, Ω, ω and θ, find x, y, z, $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$. So, if you see here for Keplerian orbit or rather than using this word, I will use here, later on I will introduce this word, for an orbit where orbit does not change except true anomaly is called Keplerian orbit. So, we are working for that, so does not change except true anomaly that is a, e, i, Ω, ω are constants except the true anomaly that is it is an constant and θ is equal to θ (t). It is the function of time.

Finding r and v at t is the current objective. So here; when does this happen when orbit does not change when the motion is not perturbed by other planetary system. Say in the case of the here this is sun and we are taking in the orbit, the earth. So, what we are assuming that there is no other planet. So, only the sun and the earth motion will be considered. In that case, when the other planets are absent then these quantities are going to be constant and only θ.

The true anomaly which is defined by the position of the earth along this is θ in different strength of time, along the transitory, so this we call as an orbit. So, our current objective is to find x, y, z, $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$.

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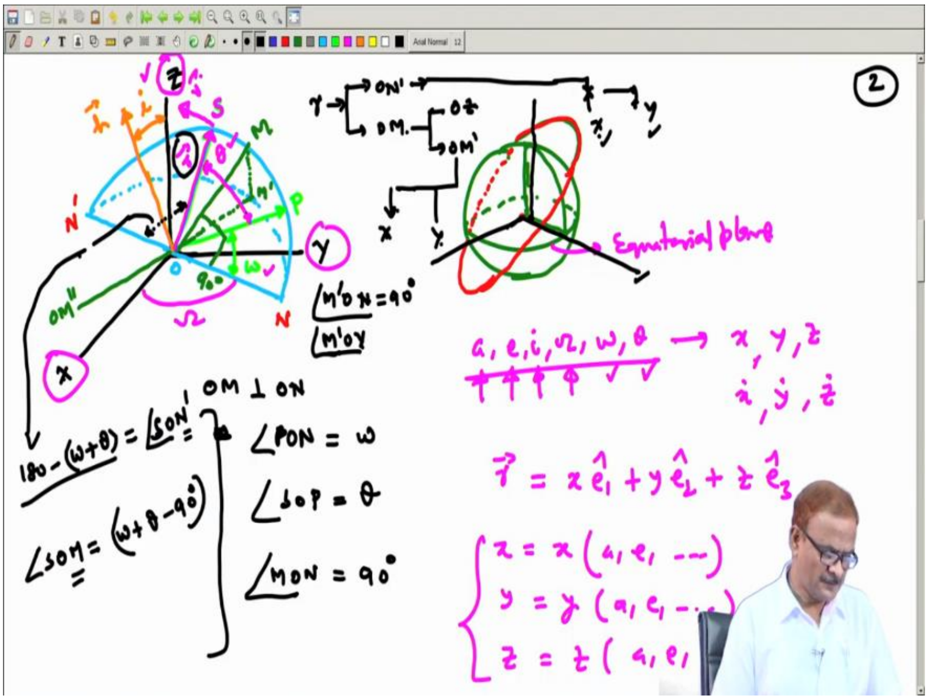

As per our earlier figure, this is the satellite orbit. So, in with respect to the initial X-Y plane, inclination is shown with respect to inertial Z axis. So, we have here, this is the earth and around the earth, the satellite is going on, it is moving. So, satellite is moving in this orbit. So, I can fix a plane centre at this point and that plane, here this is your equator, the equatorial plane of the earth.

So, XYZ plane as earlier I have stated, it does not conside with equatorial plane. It is a little different. It can be shown like that. It is little different; it is not exactly the same. It is a complete domain in itself in astronomy where how the orientation is fixed and how to represent the rotation of the earth with respect to the initial plane. Everything is derived there. So, right now we are not concerned with this because we are more concerned with the engineering aspect.

Later on while we take the reference frame, so that time we will discuss it to some little extent. So OM is perpendicular to ON, ω or we can write angle PON is equal to ω and SOP is equal to θ and angle MON is 90° . So, these are the things given to us and this is the radius vector initial and velocity vector v. So, what is given here is that a, e, i, Ω, ω and θ, this is given. Then a and e are purely the property of the ellipse or whatever it may be, the orbit i is shown already there.

This is your nodal angle, so this is Ω. Near the nodal angle, argument of perigee is shown here and θ is also shown. So, given this, find out x, y, z, and $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$. This is our objective. Here it is our *r* vector. We start by taking components of r along the X, Y and Z axis. So, if we do this, we are able to find out components r along with the X, Y and Z axis. So, let us say we indicated x $\widehat{e_1}$, y $\widehat{e_2}$, and plus z $\widehat{e_3}$. So, obviously, x will the function of all these orbital elements. Similarly, y will a function of and similarly z will be a function of; Once we get it here in this format.

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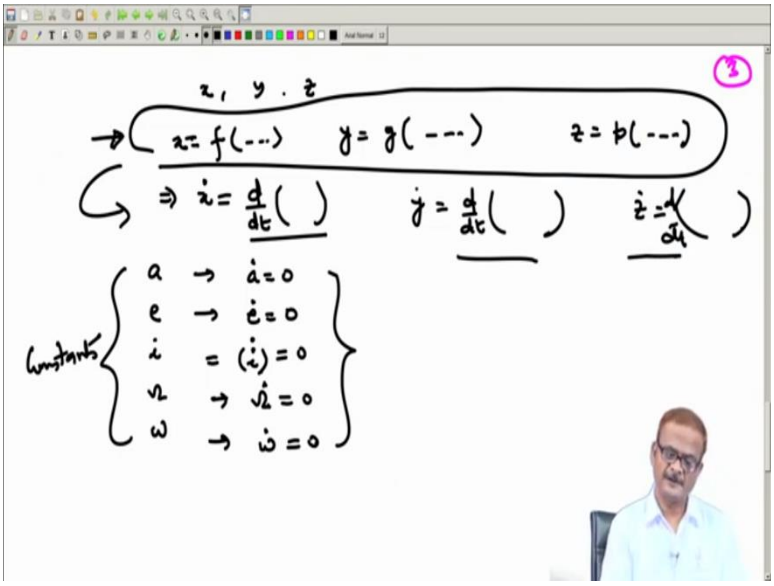

So, after getting this x, y and z, so we are getting x as a function of all other elements. Similarly, y has the function of all other elements; all the elements and z as a function of, let us say, this notation will change and we write this as the function of p other elements. Once, we differentiate it, $\dot{x}_0$, this is the basic principle I am just now writing here. So, this implies that $\dot{x}_0$will be on the right hand side. You have to differentiate. Once you get this, so immediately this will be available to us.

$\dot{y}_0$ will be d/dt and $\dot{z}_0$ will be d/dt some other quantity. We know that because a, e, i, Ω, ω, these are constants. So, we will utilize their properties that $\dot{a}$ is equal to 0 is no changing, $\dot{e}$ is equal to 0, $\dot{i}$ is equal to 0, and $\dot{\Omega}$ is equal to 0, $\dot{\omega}$ this will be is to 0. So, this equation once repeated the derivative, this will get pity simplified. This is our approach. Let us start with the finding x, y and z.

This is our $\vec{\imath}$ and $\vec{r}$ here. This $\vec{r}$ we need to break along different lines. First of all, we will break r along two directions along the OM and ON', as in the figure it is appearing. And, thereafter this vector, the r component. First we will divide r along the ON' and the other one along the OM and this OM will be broken into two directions one along the OZ direction and other along OM' direction. Now, OM' then it will be broken along the x, y and z axis.

So, this will be broken along X, Y axis and Z axis is already in the X-Y plane. So, z is no component. Similarly, ON' what we are getting, this will have component along the X axis and another component along Y axis. So, we will have two components. One along X axis and another along the Y axis because it is ON ' and it has line in the X-Y plane. This is the principal we are going to utilize.

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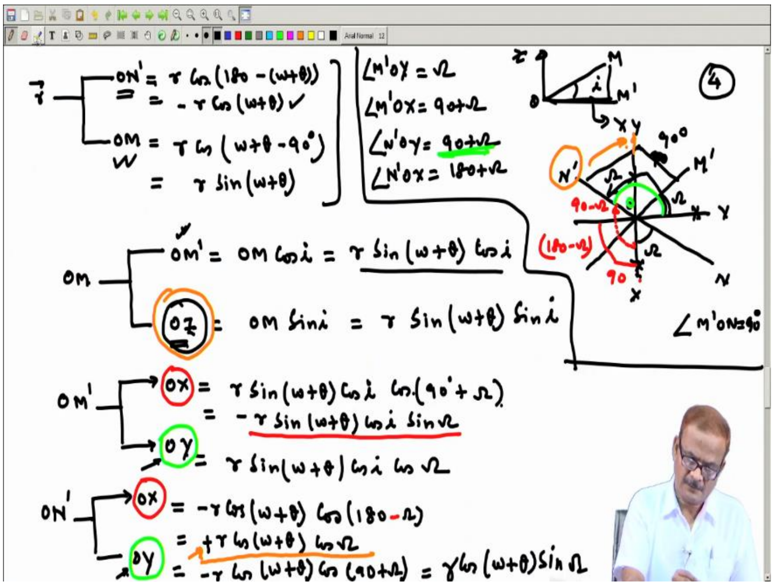

So, r first we divide along the ON' and another one we divide along as told in the OM direction. And there after the ON', we write a; ON' this component is going back here ON' we go here and you see that this will depend on this angle, angle from this place to this place and this angle is 180 - ω + θ. What here is shown by black line, this is the corresponding angle. So, we write here then

$$ON' = r\,cos\,(180 - \omega + \theta)$$

This get reduced to r cos (ω + θ) with minus sign.

We have this whole structure and we have to keep breaking it up one by one and along the OM, similarly we go back along the OM direction, and this angle is ω + θ - 90. angle SOM is equal to ω + θ - 90 ° and, this angle SON‘ and whatever you write. Here, this is M and here this is N‘. So, therefore, this component along this direction becomes ω + θ - 90. OM is

$$OM' = r\,cos\,(\omega + \theta - 90°)$$

So, this will be r sin (ω + θ). So, we have got these two components. Now, as told earlier, ON‘ has to be broken along in different direction and this is one of them. I will not cluster in one place. The next we take OM and this OM then we have to break along two directions OM breaking along two direction which is OM‘ and OZ direction. So, OM‘ this quantity will be OM cos i because i is the angle between, this is your OM and this plane is the X-Y plane and this angle is i and here is your line M‘ so OM‘ becomes OM cos i and OM is

$$OM' = \mathrm{r\,sin\,(\omega + \theta)\,cos\,i}$$

and similarly OZ this will be OM times Z is here in this direction so that becomes OZ equal to

$$OZ = OM\ Sin\ i$$

$$= \mathrm{r\,sin\,(\omega + \theta)\ and\ sin\ i}$$

So, what we are getting here is OM‘ and OZ‘. Now, we take OM‘ and this again we have to break along two directions. Along the OX direction and along the OY direction because as we look back OM‘ is lying in the X-Y plane. So, we can have the component here. Along the X-direction and Y- direction.

Now, what are angles? The angles we have to look into. Angle M‘ ON is equal to 90° and we are looking for the angle between y. Angle M‘ OY, how much this will be? This angle already we have plotted. I will show it here in another figure. Here, I will make the figure. Once we go in this plane, so this is your X direction and this is Y direction. This is N‘ this is N here in mid between O we have written here.

This angle we have shown as Ω. These two are perpendiculars. Once, this X-Y axis is rotated by Ω, this also goes here by Ω and this becomes M‘. The angle M‘ ON is equal to 90°. So, now the angle we are right now looking for is the angle between M‘ O and Y. So, angle M‘ OY this is Ω from this place. Angle M‘ OX is equal to 90 + Ω.

So, utilize this for breaking it into two components in the OX direction. Along the OX, it becomes breaking it along the OX direction so that component come out to be OX will come out to be

$$OX = r\, sin\, \omega \; + \; \theta\, cos\, i$$

This component here and then we will break it along the OX direction, so we have to take cos (90 + Ω), so that gives you

$$OX = -\, r\, sin\, (\omega \; + \; \theta)\, cos\, i \; + \; sin\, \Omega.$$

Similarly, the OY will be

$$OY = r\, sin\, (\omega \; + \; \theta)\, cos\, i \;\; cos\, \Omega.$$

So we are done with this component is broken along the two direction. Now, what is the OZ component will not have any component in the X-Y plane. So, this we do not have to worry about. This we are done with. Now, we take the ON‘. ON‘ will have the component along the OX direction and the OY direction. So, going here in this place, O is here and N‘ is here. Breaking along the OX direction, we get here, ON‘ is equal to - r cos ω + θ this is the component along the ON‘.

So, this component we have to take one component along this direction another component along this direction. So, this angle is 90°, so angle N ‘ OY is equal to 90 + Ω. And, angle from this place is 180. This angle is Ω, so this angle is also Ω from here to here and this whole angle is 90 °. So, how much this angle becomes? Angle N ‘ OX is equal to 180 + Ω.

So, accordingly break it along the OX and OY direction. We have this part already here - r cos (ω + θ) and taking along the OX direction so, cos (180 + Ω) is equal to - r cos (ω + θ) and this one. Here, if you look into, you can also look from this perspective that the angle is from this place to this place. So, if this is Ω, so this angle will be 90 - Ω and this angle from here to here is 90°.

So, that becomes 180 - Ω. So, directly ON‘ this component you take along the OX direction. So, that will be simply cos (180 – Ω). So, both are going to give you the same result, so this minus minus dot will become. So, we have cos Ω -. So, both ways are okay. Either you do this way or that way. Let us make it with the minus sign to be convenient because it looks better from this place.

So, we will put a minus sign here. Either way you will get the same result. So, this equal to - r cos ($\omega + \theta$) Now, 180 minus, so we have to make plus here at this place and then along the OY direction we have, the same thing we have to pick up here. So, this - r cos ($\omega + \theta$). Here, in this case, this angle is only convenient. The angle from this place to this place which is 90 + Ω. Already we have written angle N‘ OY this angle 90 + Ω.

So, this is cos (90 + Ω) is equal to + cos ω. And, we summarize it here. This will be plus sign with cos (90 + ω) and minus minus become plus cos ($\omega + \theta$) and sin Ω. So, these are the things we are getting. Okay, if you look at here in this place, I should mark it positive r cos ($\omega + \theta$) cos Ω, is particularly appearing with plus sign.

ON‘ appears along the OX direction we will have component along this direction which will be negative but this negative sign and also the minus sign is here. So, both the minus sign makes it positive. So, do not get confused, here it will be plus sign and here also this comes as a plus sign. Now, once we are done with this, we have to add all the terms like the OX here and OX here will be summed up together.

Similarly, OY and OY will be summed up together and, we left with OZ and OZ will be written separately. So, once we add it, we get the components along the X, Y and Z direction of the r vector and therefore, x, y, z component.

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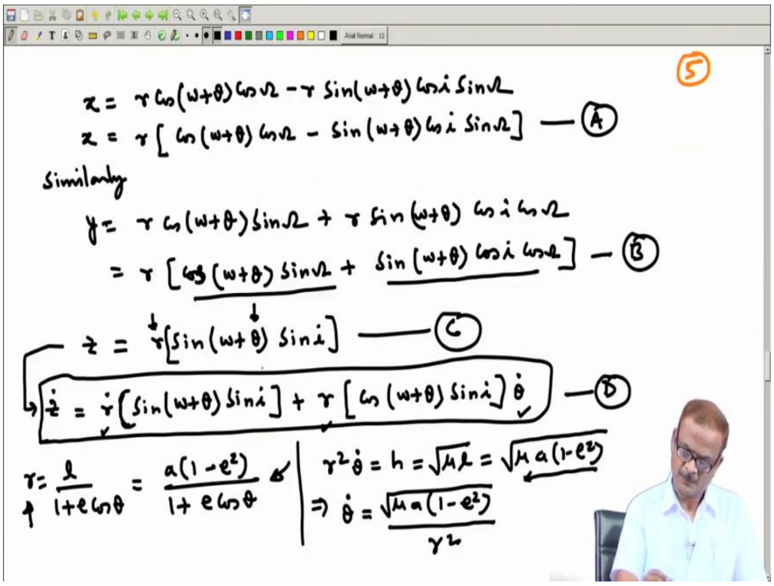

So, therefore

x can be written as,

$$x = +r\,cos\,(\omega\ +\ \theta)\,cos\,\Omega\ -\ r\,sin\,(\omega\ +\ \theta)\,cos\,i\,sin\,\Omega$$

is equal to

$$= \mathrm{r\,cos\,(\omega\ +\ \theta)\,cos\,\Omega\ -\ sin\,(\omega\ +\ \theta)\,cos\,i\,sin\,\Omega}$$

and the other term, whether + or - whatever it is. This is minus sign. So, - r sin (ω + θ). We are picking up this term. So, in the above equation, we take r is common. This is x component of the r vector.

Similarly, the y component will write as adding all the green ones, here r is missing out. We will put here r. So, r cos (ω + θ) sin Ω+ r cos (ω +θ). r cos sin Ω This is this one and then OY from this place.

$$y = r\ (\mathrm{cos\,(\omega + \theta)\,sin\,\Omega}\ +\ sin\,(\omega\ +\ \theta) cos\,i\,cos\,\Omega)$$

and z equal to here it is written

$$z = r\,sin\,(\omega\ +\ \theta)\,sin\,i$$

So, these are the three expressions we are getting. Thereafter things are easy to work out. Now, here in this case if we look, θ is a variable and r is a variable. Rest others are constants and r will be function of time. So, therefore, $\dot{z}$ we take this one first so $\dot{z}$ is equal to $\dot{r}$ sin (ω + θ) sin i + r times once you entered this, this becomes cos (ω + θ) sin i and $\dot{\theta}$ and we need to replace this $\dot{r}$ and $\dot{\theta}$ by proper quantity.

So, rest other things are just mathematical exercise. Already, we have done all the steps. There is not much remaining in this part. What we do that; We know r equal to

$$r = \frac{l}{1 + e\cos\theta}$$

which is equal to a (1- e^2) divided by 1 + e cos θ. So, on the right hand side all the quantities are known at any instance of time. Therefore, this is known. Similarly, the quantity $\dot{r}$ we have earlier worked out. The other quantity is $\dot{\theta}$.

So, $r^2\dot{\theta}$ this quantity we have written as h, and h is nothing but l equal to under root μ times a 1- e^2. So, mu is known to us and all other quantities are known to us. Therefore, h is known to us and this implies that $\dot{\theta}$ is can be written as

$$\dot{\theta} = \sqrt{\mu\frac{a(1 - e^2)}{r^2}}$$

So, $\dot{\theta}$ and r is known to us and $\dot{r}$ we have to write. So, $\dot{r}$ we have written earlier. Again, I will write on the next page.

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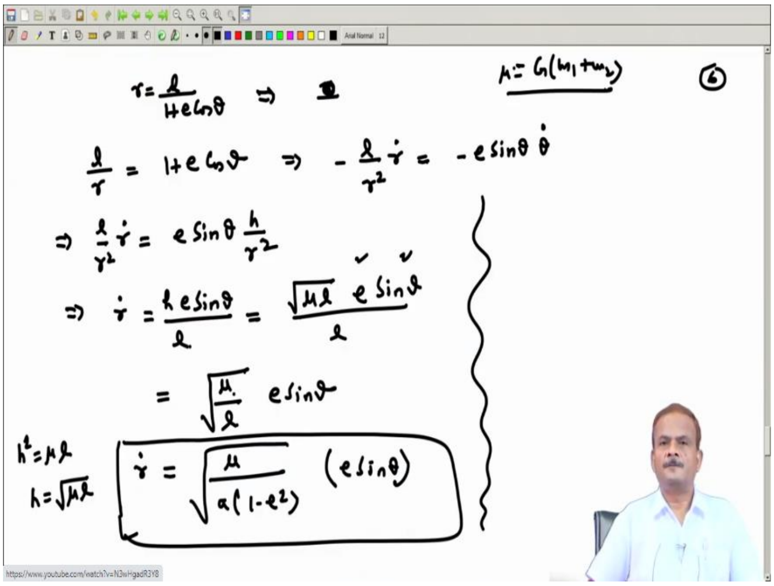

$\dot{\theta}$ is equal to h/r^2 and therefore it cancels out and we get here as h e sin θ by l, h will be there and r will be cancelled out from both the sides. So, once we have got this, in this place we use the relationship h to the power two is equal to μ times l. So, we will replace h, so h is equal to

$$\dot{r} = \frac{\sqrt{\mu l}\, e\sin\theta}{l}$$

All these things are known to us and l is also known because of our previous work.

So, this is

$$\dot{r} = \sqrt{\frac{\mu}{a(1-e^2)}}(e\ sin\theta)$$

So, this is our order. Once, we know these quantities, once you do the numerical calculation so you just have to program on computer and you can get all these values. You can see that how $\dot{z}$ is being calculated from z. Similarly, the x and y terms can be written. I will just complete it.

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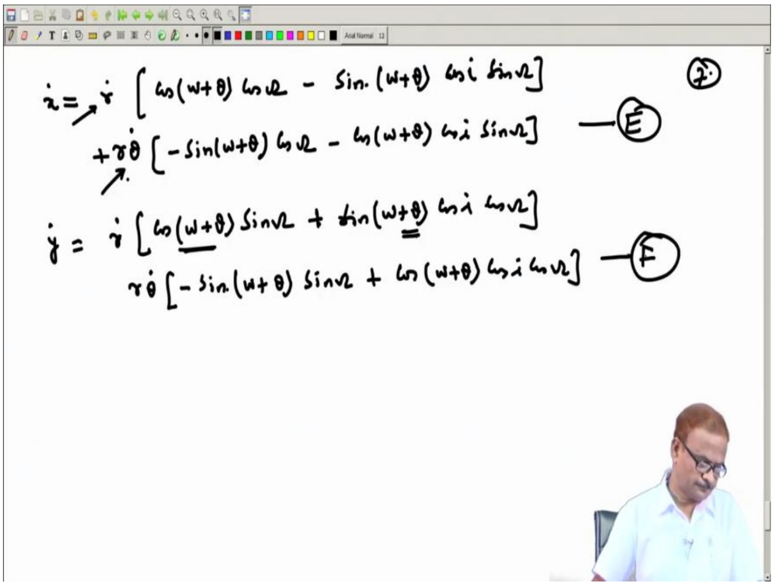

So, x dot I will just write here quickly.

$$\dot{x} = \dot{r}(\cos(\omega+\theta)\cos\Omega - \sin(\omega+\theta)\cos i \sin\Omega) +$$
$$r\dot{\theta}\,(-\sin(\omega+\theta)\,cos\ \Omega - \ cos\ (\omega\ +\theta) + \cos i\ sin\ \Omega\ )$$

. Okay. This r times and the next one we have to differentiate, so this becomes sin ω + θ times cos Ω and $\dot{\theta}$ will take it outside, minus sign this becomes cos ω + θ cos i sin Ω times $\dot{\theta}$. That means θ dot you can put here in this place itself rather than putting on the back side and r times θ dot. So, already we know all these values.

We have worked out and therefore all these terms can be calculated. Okay cos sin and Ω, cos (ω + θ); cos ω; this is fine. Similarly, $\dot{y}$ this we write as

$$\dot{y} = \dot{r}(\cos(\omega+\theta)\sin\ \Omega + \sin(\omega+\theta)\cos i \cos\Omega) +$$

$$r\dot{\theta}\,(-\sin(\omega+\theta)\,sin\,\Omega + \,cos\,(\omega\, +\theta) + \cos i\, cos\,\Omega\,)$$

as we have written earlier here in this place cos sin. Here, we have a mistake here. we have to do the correction, this is θ, $\omega + \theta$, and this is again θ everywhere it is fine. Okay other term will be $r\dot{\theta}$, just like here it is appearing.

Similarly, it will appear and we just have to differentiate the corresponding term. So, this will be - sin ($\omega + \theta$). Once, this term is differentiated, so it will appear like this and $\dot{\theta}$ term will put it outside sin Ω and again here this term is differentiable, so cos ($\omega + \theta$) cos i cos Ω. This is $\dot{y}$. So, this completes the derivation, this is (D), this is (E) and this is (F). Okay, so we are done with this.

It is just the matter of programming and doing all this calculation given on that is the equation calculator if it can be worked out provided. We have to use all the equations that we have written here. So, everything will be known, mu will be known to you because this is planetary gravitational constant. In the case if you are taking two body system, then mu will be simply g times $M_1 + M_2$. So, we have finished this topic here. Now, we will take up some problems. We will solve some problems whatever we have done in the next lecture. Thank you very much.