**Space Flight Mechanics**
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# Lecture – 17
# Problem Solving on 2-Body Problem Related to Orbit and Orbital Elements

Welcome to lecture number 17. So whatever we have done till now so we will solve some problems related to that.

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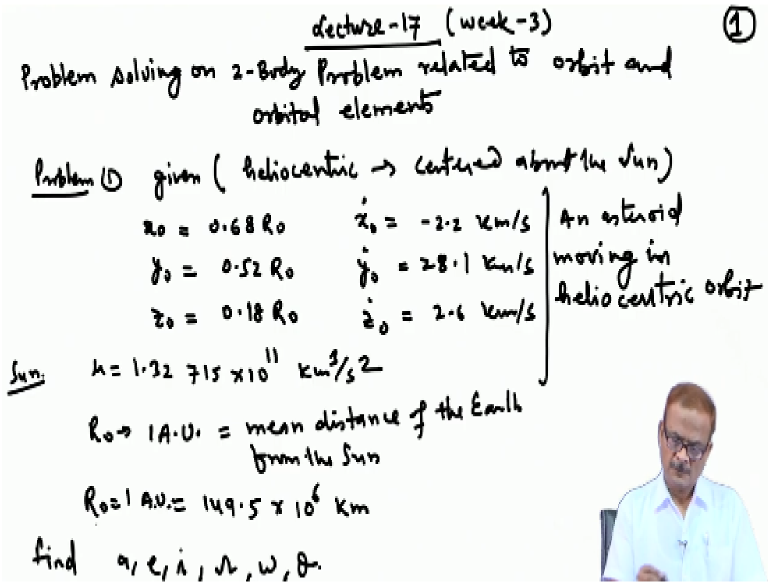


So let us start with the problem. So problem is it is stated like this given $x_0$ = 0.68 r; $y_0$ = 0.52 r and $z_0$ = 0.18 r. Similarly $\dot{x}_0$ is – 2.2 km/s , $\dot{y}_0$= 28.1 km/s. $\mu$ is 1.32715 × $10^{11}$ $km^3/s$. This is the planetary gravitational constants in this case this is for the sun and what the r is appearing here. So basically I will put this as $r_0$ so this is indicating $r_0$ is indicating one astronomical unit.

This is the mean distance of the earth from the sun and this one astronomical unit this becomes equal to this is equal to 149.5 × $10^6$ km that means it is around 15 crore km as we maybe knowing so 149.5 million km and this data is for an asteroid moving in heliocentric orbit. Heliocentric means it is moving about the sun. This is called the centered about the sun.

Similarly, geocentric means centered about the earth. So this is heliocentric data as if it maybe apparent from this place. So what is required that you find out the parameters so what we have to do find a, e, I, Ω, ω, θ. So here in this case we will start one by one writing all the things. So first we find out what is the value of r. So r = $r_0$ or we can use maybe we can use some other notation. I will make it capital $R_0$ so that we do not confuse it with any other term capital $R_0$ so here in this place also.

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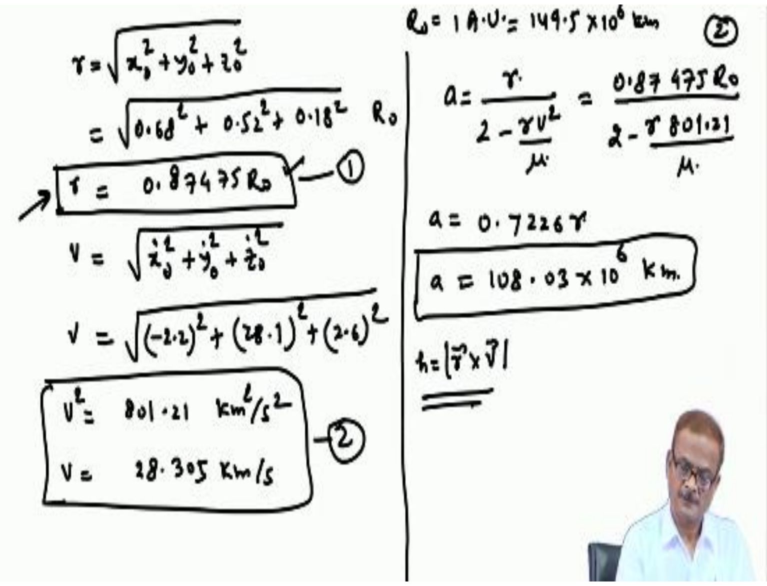


So first we have to find out r so $x_0$ $y_0$ all these things are given. So r becomes

$$r = \sqrt{x_0^2 + y_0^2 + z_0^2}$$

and this is given to be $0.68^2 + 0.52^2 + 0.18^2$ under root times $r_0$ where $r_0$ is one astronomical unit. Okay so this gets reduced to 0.87475 $R_0$. On our small calculator whatever the values have appeared I have just written it because here truncating a lot of error we do so until and unless it is desirable we will not truncate it.

So this is our r and similarly the v becomes

$$v = \sqrt{\dot{x_0}^2 + \dot{y_0}^2 + \dot{z_0}^2}$$

. So $v^2$ we can write as 801.21 $km^2/s^2$ or v, we can write as 28.305 km/s. So these are the quantities we are going to use this is (1). Now these things are available can we find out a, a is equal to already we know that we write it as

$$a = \frac{r}{2 - \left(\frac{rv^2}{\mu}\right)}$$

This is the expression we have used. So here in this case r = 0.87475 $R_0$ divided by 2 – $rv^2$ by μ we have to evaluate $R_0$ we are aware of okay. So $R_0$ this rv we are aware of from this place and $v^2$ is nothing, but 801.21 and μ as usual we know this is 1.3275 something so this is μ. So this r we have to insert and the μ value we have written here this we have to insert in that expression and work it out.

So a this will turn out to be if you insert those value and $R_0$, 1 astronomical unit 149.5 × $10^6$ kms. So this things are given r is also given here in this place so this values are all known. So if you compute this turns out to be around 0.7226 r so a becomes equal to 108.03 × $10^6$ km so this is your semi-major axis. Okay for calculating any other thing x,y,z are given. So for calculating other parameters we need to work out h also where h is the quantity r cross v magnitude so this also we need to work out.

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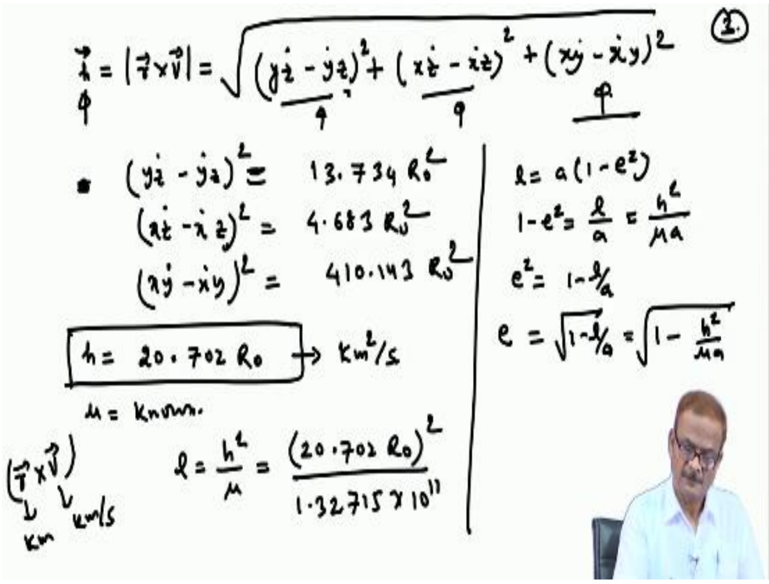


So h = (r × v) magnitude, this quantity can be written as (y $\dot{z}$ – $\dot{y}$z)$^2$ already we have written worked out expression for this. So the same expression I am using here. So this is the x component of this h vector h, this is y component and this is z component. So you can see that corresponding x is missing here y is missing here, z is missing here and this is symmetric in nature.

Yz so $\dot{y}$ $\dot{z}$ and because it is a square so if you write either $\dot{y}$ z first and y $\dot{z}$ afterwards with a minus sign so it does not matter particularly here in this case, but each component itself must be written correctly where $h_1$ is y $\dot{z}$ – $\dot{y}$z this way you have to write otherwise this will not be correct, but here it is okay because we are taking squaring and taking the under root.

So these quantities we have to work out here so separately we write each of the quantities. So inserting the values for y $\dot{z}$ $\dot{y}$ and z we get this as 13.734 $R_0^2$ this quantities is not a small r this is $R_0$. where $R_0$ is this quantity here. As it is appearing here so this is your $R_0$. Okay similarly here this is $R_0^2$ the astronomical unit we are writing in terms of that.

And once we add them and take the square root, so h turns out to be 20.702 $R_0$ and what will be the unit of this? Here h = r × v magnitude so h is written as km, v is written in km/s therefore the unit of this will be $km^2$ per second. So h is in $km^2/s$. So once we have got h and μ is also available to you μ is known so l can be worked out

$$l = \frac{h^2}{\mu}$$

so 20.702 $R_0$ whole square. And divided by μ which is we have written $1.32715 \times 10^{11}$. With this quantity now

$$l = a(1 - e^2)$$

therefore a is known to us l is known from this place so $1 - e^2$ this becomes l/a and

$$e^2 = 1 - \frac{l}{a}$$

and if we take the under root of this is a positive quantity so this becomes

$$e = \sqrt{1 - \frac{l}{a}}$$

. So this is also simply we can write as $h^2/\mu a$. So all the things known here a is already known in terms $R_0$. Here if you look here in this part so this is known in terms of $R_0$ and therefore we can calculate this quantity e.

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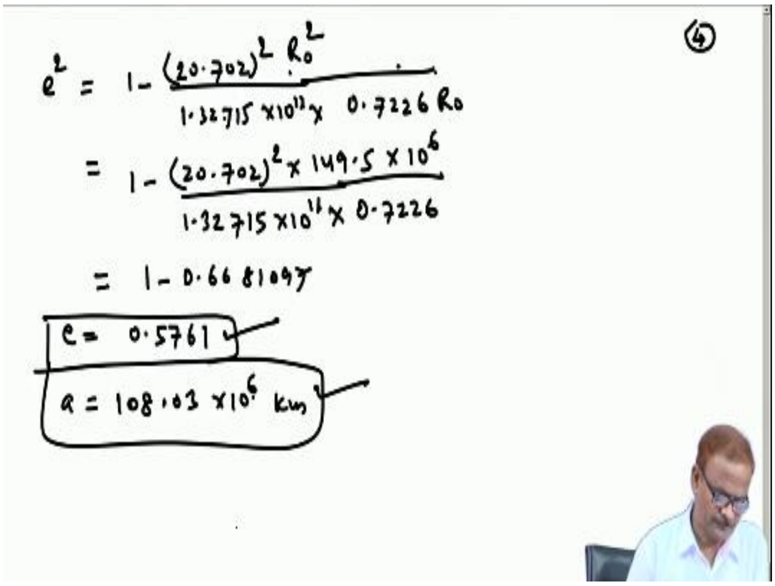

Therefore, $e^2 = 1 - h^2/\mu a$ so $h^2$ is $20.702^2/\mu$ is $1.32715 \times 10^{11}$ and a is the quantity we have written as 0.7226 $R_0$. Okay this quantity we are utilizing so this is $h^2/\mu$ a and ${R_0}^2$ is remaining so ${R_0}^2$ 149.5 × $10^6$ whole square and $R_0$ here actually we could have cancelled.

Okay if we write in terms of $R^2$, if you see here this ${R_0}^2$ is your and a you are writing once we write it $e = 1 - h^2/\mu a$. So a we will write in terms of once we are writing in terms of square so what we will do that one step we will not expand here we will first write it as $R_0^2$. So $R_0$, $R_0$ cancels out one more step we will write $20.702^2 R_0$ divided by $1.32715 \times 10^{11}$ times 0.7226.

Now $R_0$ value this can be inserted here. So we will rub it out and $R_0$ value then we are inserting this equal to 149.5 × $10^6$ and once this is worked out so this will be 1 – 0.6681097 or e turns out to be 0.5761 so this is another orbital parameter. So a we have got as 108.03 × $10^6$ kms. The e is a dimensionless quantity and this is a dimensional quantity dimension of length. Okay next we can determine Ω because in this process that is the most easiest one.

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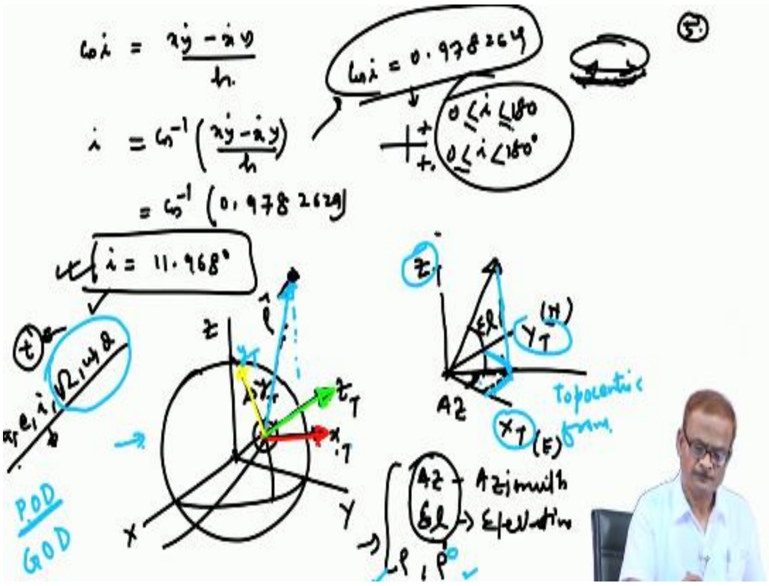


Okay I will verify we have already derived the equation, but I will verify it once that everything goes correctly. Okay

$$\cos i = \frac{x\,\dot{y} - \dot{x}\,y}{h}$$

So this value already we have computed x $\dot{y}$ this quantity 410.143 ${R_0}^2$. 410.143 the square root value so the quantity is available from this place, but here this is squared one so we have to be careful we cannot utilize it. We need proper sign once we take the square root so this we have to take care of.

So i will be cos inverse this computation you can do it yourself because I have done it on calculator and I have not written those values here. So cos inverse of 0.9782629; whatever I have got the value and we are writing here and i = 11.968° that is what we have got here the cos i = 0.978269. So you know that the cos is positive in this quadrant and the first and the fourth quadrant. So why we are not picking up the fourth quadrant value because i is confined to 0 to 180°.

Once 180° it is the same orbit, but see once it is going like this 180° this totally it reverses, but orbit will be the same that means the satellite will be going here in this direction this is the retrograde orbit. So either way you can write this equal to this 180 if you do not want to indicate this, but both are okay. So i = 11.968° so this way we have got the inclination.

So how many parameters we have got parameter this is i here this is the inclination angle e here a here so total 3 right now we have got. The other things we have to find out say with this parameters we can know ultimately what is the objective of getting all this things. Objective is that if we know this parameters on the surface of the earth suppose this is my ground location so from the ground here in this direction we are z and here we write as x.

And in this direction as the y towards the north and satellite is going somewhere let us say it is going here in this orbit from this place. So satellite is somewhere here so if you look into this frame where I have written XYZ. so this frame it looks like $X_T$, $Y_T$ and $Z_{T.}$ this is the north direction, this is the east direction and this readily outward, but there are many correction what altitude you are there and so on.

But for our purpose right now we should not go into all those things. So if we know this values and what is the present time and also we need to discuss all those issues how to look it our Greenwich line. So there are many issues in doing this so we will come to that in the orbit determination chapter. This is corresponding to preliminary orbit determination topic week 9. So at that time we will take up this issues.

So from there once we this is rho vector okay from here to here so this is your rho vector and if I draw up a perpendicular so this I call as the Aziμth angle from here to here and this angle I call as the elevation angle Az is Aziµth and El is elevation angle. So I will be able to know what is the Aziµth angle if I my a, e, i, $\Omega$, $\omega$, $\theta$ all this things are known.

So from there I can go and find out this Aziµth elevation $\rho$ and $\dot{\rho}$. All these things can be determined. So what kind of benefit is that if you know the Aziµth and elevation. So as a satellite is passing over your head see here I cannot show all those things very clearly. So somewhere we have to draw up a perpendicular from this place to this plane and from here then this will come to this place.

So x, z is going radially outward okay remember z is going radially outward so I will make it solid this is not paper. So this is the $X_T$ here this is $Y_T$ so this angle from here to here I will have to make a separate figure for this, this is $X_T$, this is $Z_T$ going outside and satellite is located somewhere there so I will remove this part and $Z_{T.}$ I will show by some other color this is your $Z_T$ and x also I will show by some other color x and this is y and your $\rho$ vector is located in space somewhere in this place.

So you need to draw up a perpendicular from this place to the this is the $Y_T$ and $X_T$ place. So we will drop it on the surface of your as it is shown here this is the $X_T$, $Y_T$ and this is the $Z_{T.}$ So this perpendicular is dropped from this place to this place and then its components are taken. So in this topocentric frame in the topocentric frame we get the elevation and Aziµth and therefore we can orient our radar.

Radar can be oriented and you can track the satellite as you know rho and rho dot also how this rho is varying rho which is the range and rho dot which is the range rate. So this things we can using all this information we will be able to move our radar. So we can program it and radar can be moved automatically by the servo system and satellite will be tracked whenever it is available so at that time the radar will start tracking that.

And all this things can be computerized on the computer it is very easy to do by hand it will be very big task. So our objective is to get all this parameters and ultimately this is used for preliminary orbit determination which we write as POD or the general orbit determination which we write as GOD. So with this 3 parameters rest all this 3 we need to work out.

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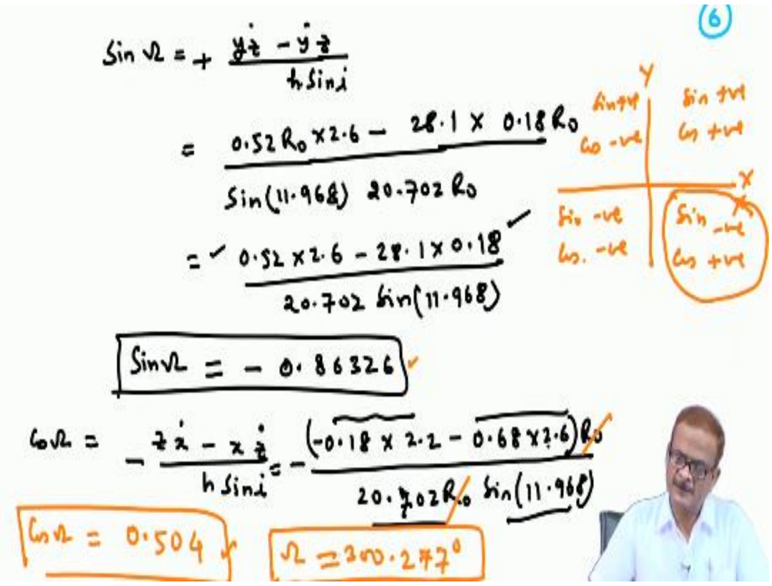


Now sin Ω this quantity is with a $\dot{z}$ x the second one sorry but we are taking here sin Ω we are working. So

$$\sin\Omega = +\frac{(y\dot{z} - \dot{y}z\ )}{h\ sin\ i}$$

and here there is plus sign. Okay sin i and h all these things are known so if we insert this values sin i from the previous part 11.968 and h is also known to you which is a positive quantity though perhaps I have not written anywhere etcetera this is 20.702 $R_0$ and rest others we have to fill in.

So y we know; y value how much we have taken $0.52R_0$ times $\dot{z}$ which is 2.6 km/s and $\dot{y}$ is $\dot{y}$ z, z is 0.18 $R_0$ and $\dot{y}$ is 28.1. Okay so you can see that $R_0$ will cancel out so 0.52 × 2.6 – 28.1 × 0.18 divided by 20.702 sin 11.968. The quantity on the right hand side this part the second term this is greater than this term and therefore this turns out to be negative quantity.

So sin Ω = - 0.86326. Similarly, cos Ω we utilize the relationship and this Ω is

$$\cos\Omega = -\frac{(z\dot{x} - x\dot{z}\ )}{h\ sin\ i}$$

and there is a – sign before this. So the same way if we insert all this values so this turns out to be 0.504 with okay we will write one more step to work it out z = 0.18 r we will take it outside and $\dot{x}$= 2.2 with minus sign so we will place the minus sign before this.

So we will place a minus sign here and – x = 0.68 $R_0$ so $R_0$ we are taking it outside and $\dot{x}_0$ is 2.6 = 2.6 times $R_0$ divided by h is 20.702 $R_0$ sin 11.968. So if we now work it out and one sign also we have to place here and this equal to with minus sign so if we see here this is negative this term is negative, this term is negative so that makes it positive this is a positive term all these are positive $R_0$, $R_0$ will cancel out so this 2 they drop out.

So negative × negative that makes it positive. So we should get a positive quantity. So here in this case we get this as 0.504 so this is cos Ω. So this is x and y now looking into this quadrant cos is positive and sin is negative. So sin is positive in this quadrant, sin is positive here, sin is negative here and sin is negative here in this quadrant and then cos is positive here, cos is negative here in this place and cos is positive here in this place.

So what we have cos is positive and sin is negative so that we have to look for this. So this falls under this category okay so that is this is lying in the fourth quadrant and therefore once you solve it so you get Ω = 300.277°. This may not be very accurate result as I have done on very elementary calculator, but my intention is to show you the principle and how to work it out that is all.

There may be some small error here and there that once you work yourself so you will find the exact result on a better calculator. Okay so this way we have been able to work out 4 elements we are left with 2 more elements.

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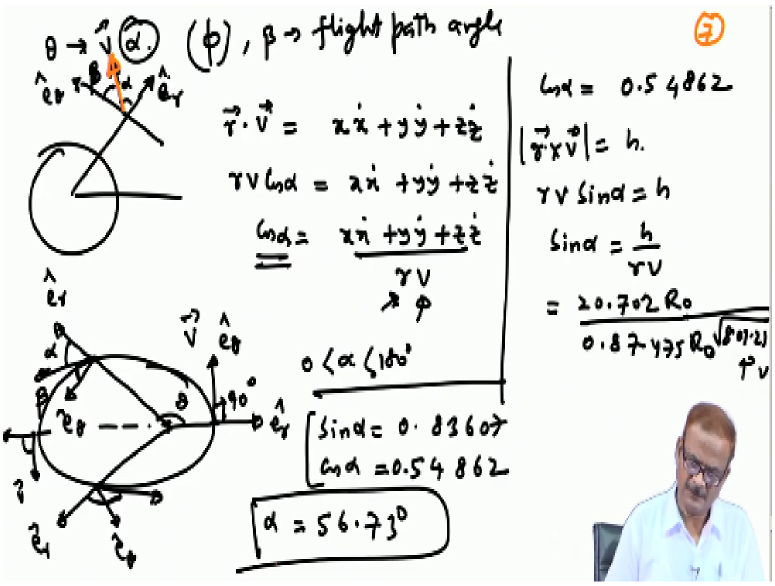


Okay for finding out θ if you remember we need to know $\alpha$ so what is the α here α angle we have used in the context of somewhere this is the velocity vector this angle we have written as α and this angle we have written as $\varphi$ or $\beta$. So β is the flight path angle quite often this is also indicated with $\varphi$. This is the radial direction er cap this is the v direction this one so this is your v.

So r dot v this will be nothing, but x $\dot{x}$ + y $\dot{y}$+ z $\dot{z}$. On the left hand side this will be rv cos α + y $\dot{y}$+ z $\dot{z}$ and then cos α we have written as + y $\dot{y}$ + z $\dot{z}$ divided by rv because r and v both are known to us and x $\dot{x}$ all these things are known to us therefore cos α can be determined and I remember that what will be the value of α this is very important to say this I have an elliptical orbit.

So if I take at this place, this is your θ and $\vec{v}$ is somewhere tangential to this and this is the θ direction $\hat{e_\theta}$ so this is $\hat{e_\theta}$ and here this is the $e_r$ direction. So this is your α and this is flight path angle $\varphi$ or β. So here in this place. this is the $e_r$ direction and this is the $\hat{e_\theta}$ direction. So this angle is the v direction is also here and here so this is 90°.

On this side if you come so $e_r$ becomes here in this direction and this is $\hat{e_\theta}$ while v will be along this direction. So here this is obtuse angle so what is happening here in this case that α it will be limited to 0 to 180°s it is not more than that. It will lie in between α = 0 and 180°. Now let us see here this is 90° anywhere let us say going to lie beyond this range.

Once we come here in this place again we see this is the r vector and v vector is here and r and v this is the 90° angle from here. It increases and again this becomes 90°. So here it is decreasing. So this is not going to exceed you are not getting this value you are getting α between 0 and 180°. If it is in this orbit now here in this case therefore cos α alone it is suffice to determine the value of α because in the range 0 to 180° cos α will give you correct value of α.

So this is the reason I detail it here. So therefore we work it out and write it here so cos α I will write the value you can check it yourself 54862 and also we have r × v magnitude this = h or rv sin α this we have written as h. So therefore

$$sin\ \alpha\ = \frac{h}{rv}$$

and h = 20.702 × $R_0$ / R is 0.87475 $R_0$ ×v, v is 801.21 square root this is v. So therefore sin α we get as 0.83607 and cos α we get as 0.54862 so both are positive.

So as per our discussion therefore the α value will be given by 56.73° this is approximately this value this will be acute angle. As per our discussion here on the previous page where both are positive. So let us say only in the first quadrant so accordingly we have dissolved this value this resolution is resolving this thing is very μch required if you put a wrong angle that means you are creating a havoc in the satellite problem. So till now still we have not done the calculation of θ. So we are going in the next step to calculate θ.

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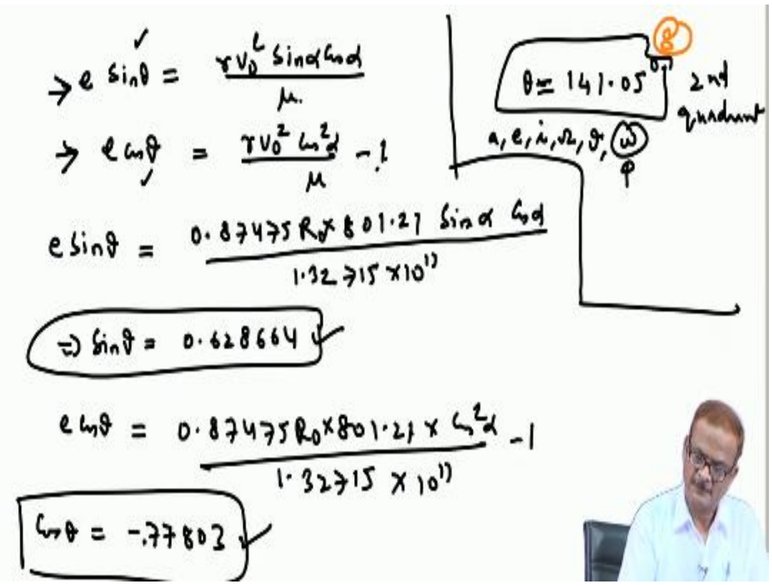


Now we use the relationship derived earlier

$$e \sin \theta = \frac{r v_0^2 \sin \alpha \cos \alpha}{\mu}$$

and e cos θ = r$v_0^2$ cos$^2$α /μ – 1 and if we are looking for sin θ and cos θ is already known to us so we can divide it bring it on this side so maybe I can rub it out and put it here like this. So here this will become 1 – e so we will do it later on.

So we need to insert all this values. So e sin θ then rv 0 square / μ so we need to insert all this values 0.87475 $R_0$ $v_0$ square is 801.21 and then sin α right now we have worked out on the previous page cos α and divided by μ which is 1.32715 × 10  11. So if you divide it by e so sin θ from here let us say 0.628664 something like this we will obtain.

Now next e cos θ these are not minus sign e cos θ = r$v_0^2$ so we put the same way 475 $R_0$ $v_0^2$ is 80 × 801.21 and cos$^2$α divided by μ 1.32715 × 10$^{11}$ – 1 and therefore cos θ from this place this will turn out to be – 7 – 0.77803 so this is a negative quantity. Sin is a positive quantity and cosine is a negative quantity.

So again going back and looking here sin is a positive quantity and cosine is a negative quantity so that means your θ value is lying in the second quadrant. Therefore, once we solve it so we get here θ = 141.05° now this is in the second quadrant. So we have got θ then we need ω so how many things we have already done we have done a, e, i, Ω, θ only thing the ω part is remaining. So for that we need to use the expression we have developed earlier.

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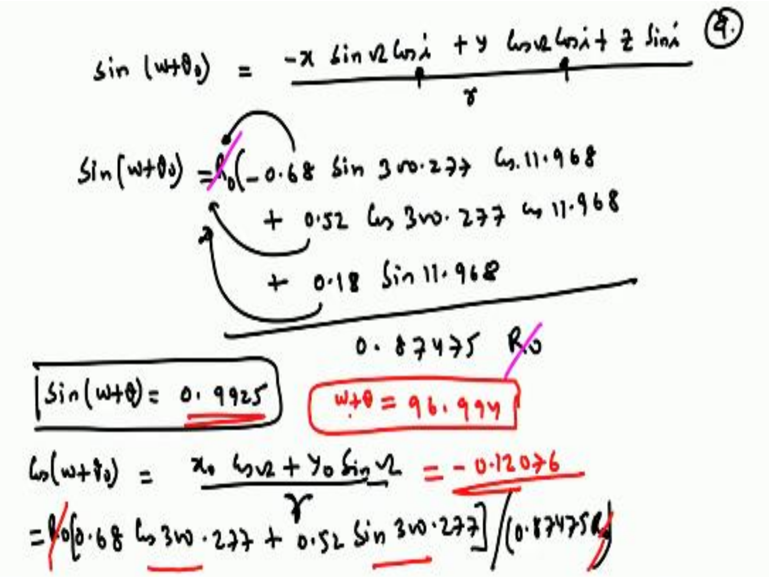

So we use this expression in the last class we have derived it last class means in the lecture number perhaps (15) we have worked out (14) or (15) something like that you can look back into that. So this 0, 0 symbol I will remove from here as I have used earlier and now insert the corresponding values. So sin ($\omega + \theta_0$) that becomes – 0.68 sin 300.277. we have got 300.277 perhaps and cos i we have got as 11 point something;  i is 11.968°.

All these are in degree plus y is 0.52 and obviously $R_0$ is there. So $R_0$ we will be taking it outside $R_0$ is multiplied by this quantity also with this quantity. So I am taking it outside already the bracket then cos from this place Ω is  300.277 cos 11.968 both places cos i is there so not a problem and then + z and z is 0.18 $R_0$ will again go from this place to this place. So we have sin 11.968 and this divided by r which is the position.

So position is we have already calculated that also 0.87475 $R_0$ so this is the value here. So using this value times $R_0$ so this $R_0$ and this $R_0$they cancel out and then we get computing this ω + θ 0 0.9925. Similarly, we can write

$$\cos(\omega + \theta_0) = \frac{x_0 \cos \Omega + y_0 \sin \Omega}{r}$$

So inserting all this values one step more I will take 0.68 $R_0$ being taken common cos 300.277 + $y_0$ is 0.52 sin 300.277.

And $R_0$ the same way divided by bracket close 0.87475 times $R_0$. So this $R_0$ and this $R_0$ this drops out and we get this quantity as – this will be the plus quantity not minus. Let me check here this is negative and 300 this is positive. Okay this will come with a negative sign so – 0.12076. Now we need to work out in which quadrant it is aligned. So you can see that here this is positive and this is negative. So this tells that it is going in the second quadrant.

So that way the ω + θ this should turn out to be 96.994. So already we have observed that the θ value we have computed somewhere the sin θ we have done here θ = 141.05°. So this is the θ we have computed and this is the true anomaly. So once we have done this and in this place what we see that ω + θ is coming less than that, that means what is happening here in this case we go in the next phase.

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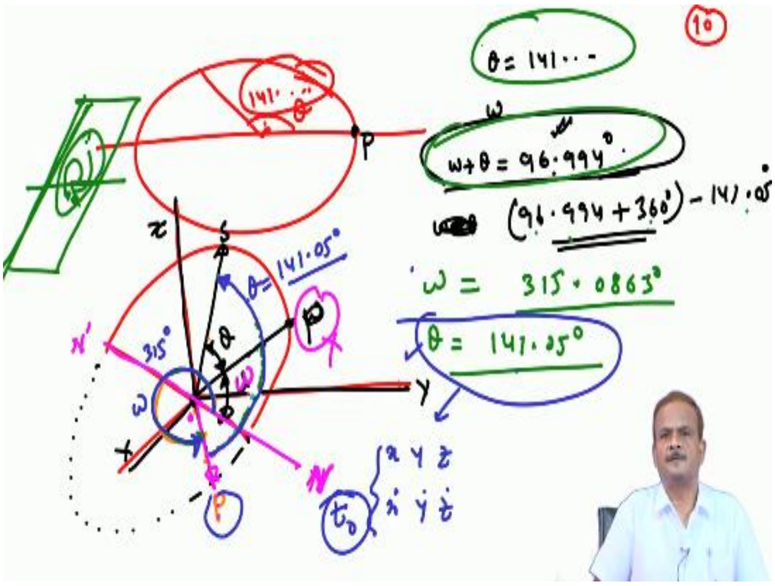


Actually you are looking for this is your θ, θ suppose this is a 90 more than that 141 point something it is going like this it is a 141 point something so this is your θ. Now in the orbit ω is lying somewhere here and θ we have shown somewhere here. This is in the perigee position and this is ω from here to here this is ω and from here to here this is θ this is the satellite here.

So this perigee position P is here what we are showing here in this place in 3D in the plane of the orbit this P appears here in this place. So already θ is 141 point something ° and from here this place ω + θ is turning out to be less than that. So what is the reason for coming like this. So the reason is that your ω is large that means instead of perigee line here now you go below the X-Y plane this is the x and y plane and the Z plane is here.

So go below this. So from here then you will be measuring ω to certain value which is large so let us say that I subtract ω + θ whatever the value we are getting 96.994°. So what I will do that I will add to this ω plus this value I will add 96.994 + 360° and then from there I subtract this θ 141.05° this we will subtract.

So instead of using this we are adding this why we are adding because your ω is not limited to this place, but ω is going somewhere like here this and coming here some to this place and this is then perigee position is lying here and then from here then you are measuring θ angle somewhere and once you look for that. So say the θ angle which will go let me you by other color it goes somewhere else.

And this is your satellite so this is the θ angle then. Now once we are calculating this θ value this I will make it little more clear let me remove this. What we are trying to do here you should understand it properly. So what I told that the perigee position is lying here in this place and to this if I add further θ from here the satellite is displaced. So from this place so the perigee is not lying here as we have assumed this is your ω.

So perigee is not lying here rather it is lying here in this place. So we measure from the nodal line this is my nodal line and N' and this is o here. So from the nodal line we are measuring the angle so this is perigee location so this is going below. This is going below the X-Y plane and always from the perigee position you are measuring the θ angle. So we need to measure the θ angle starting from this point.

I have to start from this point and go wherever the satellite is located so I have to go till that point. So this problem arises because of this particular situation ω + θ it is turning out to be only 96.994° while θ is large. So that means we should add 360° to this and subtract 140° and if we do that so this turns out to be 315.0863°. So this is ω and θ is your 141.05°.

Now you can see that from this place to this place this may be around this plane is inclined. This is something like if we look in the X-Y plane if this is X-Y plane so your orbit is inclined like this. So in this plane we are trying to measure this angle so something it looks something like this suppose this is the inclined plane. So in this plane so in this plane we are measuring the angle.

So perigee is located somewhere here and your nodal line is located somewhere here. Perigee can be located anywhere it is not necessary that it is located here in this place. So we are measuring our angle here in this case from the nodal line to this place and $\theta$ will be measured from this place onwards. So exactly the same thing we have done here we have measured the periapsis line which is coming in this place to this.

This is our $\omega$ and from here then we will go to whatever the value the wherever the satellite is located. So here in this case this turns out to be 315° approximately and this one turns out to be $\theta$ turns this is your $\theta$ this $\theta$ turns out to be 141.05°. So this completes this problem of working out orbital elements. So this way you know what are the orbital parameters.

So out of this parameter this will change because this is corresponding to x,y, z and $\dot{x}$, $\dot{y}$, $\dot{z}$ of a particular time instant let us say $t_0$. So over a period of time this $\theta$ will differ this will not remain the same this $\theta$ is a variable rest others they will vary. So we stop this here. This completes this problem and whatever was required to understand this problem I explained it here bit by bit. So I hope this helps you in solving other problem. Thank you very much.