Space Flight Mechanics
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Lecture 19
Kepler's Equation / Kepler's Problem

Welcome to lecture 19. Today, we will start with a new topic Kepler's equation or Kepler's problem, which we are going to discuss. So orbit determination we have already done. The basic problem while you try to do the orbit determination, you have to propagate the orbit also. It appears as one of the requirement. Say, right now you know the position and velocity vector of the satellite, so what will be the position and velocity vector of the satellite in the future.

Or maybe it is a radius vector and this is the true anomaly. So this problem, it falls under this category. So orbit determination is one problem, another one we have the orbit propagation. So here, basically we are going to deal with orbit propagation, but a simpler one. We have more sophisticated one, in which we take into account the effects of all the planetary system, even the sun and everything is included in that.

For that I am talking about the satellite or satellite, so effect of the sun and other planetary system, moon all these will be included and then it is propagated numerically. So that is another issue, which we will not discuss in this course, because it is not only beyond the scope, that much of time we do not have for this course. So we will confine ourselves to the two-body system, in which with respect to another body, we are describing the system equation and how the satellite or the planet will move with respect to the other one. So in that context, how the true anomaly and its radius vector or the radius vector and velocity vector it evolves with time, so that is our topic for today.

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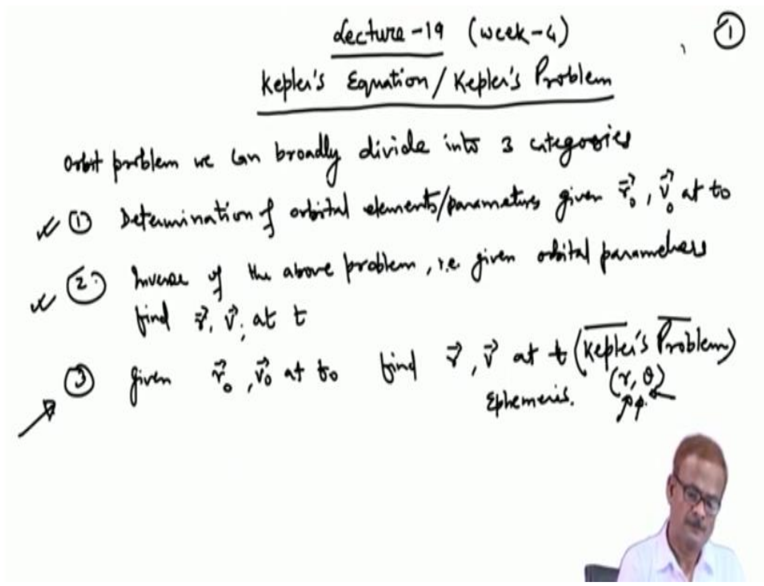

So we have in the orbit problem, we can totally divide into 3 categories, determination of orbital elements, which we have already done, elements or orbital parameters, given $r_0 v_0$ at t_0. So this is one of the problem; it is already done. So the second problem is inverse of the above problem. So inverse of the above problem, that is given orbital parameters, find r, v and t and third problem, this also we have already done.

The third problem is given r_0, v_0 at t_0 find r, v at t. So this is the problem we are going to discuss in this lecture and this we typically call as the Kepler's problem. So this is typically called Kepler's problem. Kepler's problem, especially it is described in context of r and θ that you find the r and θ or the ephemeris. Many times you will see that you are interested only in the location of the satellite, where it is located in the sky.

Because Kepler dealt with planets, so in that case where the planet or the moon, all they are located. So that refers to your θ. So the ephemeris, we are especially talking about, so in that case it is referring to r,θ.

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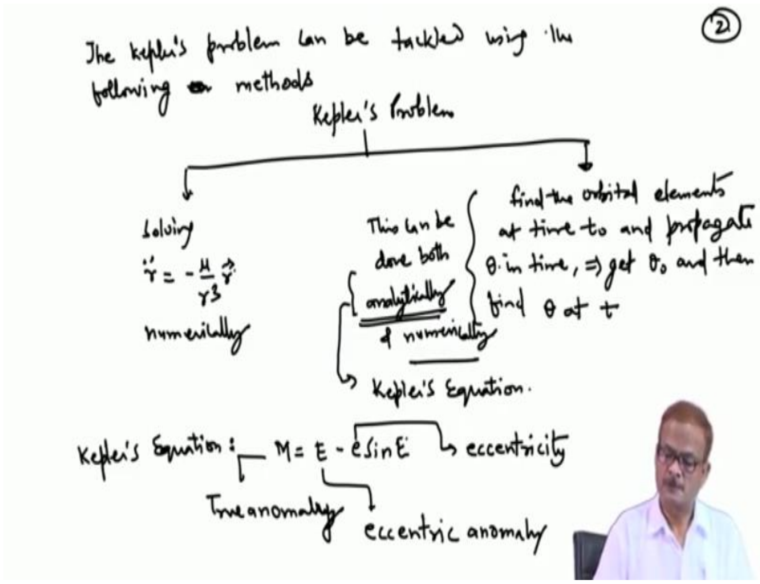

So Kepler's problem can be tackled using the following methods. So here we have Kepler's problem and this can be dealt using, we are doing for the simple case, so solving

$$\ddot{r} = -\frac{\mu}{r^3}\,\vec{r}$$

and numerically. So this is one of the way. In the case, you are considering the effect of other planets, so you will have the perturbation term, which will be present here in this case.

Once we go to the three-body problem, at that time we will take into account all those things. The second way is to find the orbital elements at time t_0 and propagate θ in time; means this implies that gets θ_0 and then find θ at t. So in the future you are going to do this. So this part can be done both analytically and numerically. So we are going to discuss this part, the analytical way of doing this and once we do this that particularly we call as the Kepler's equation.

So shortly, we will write the Kepler's equation, we will derive it. So Kepler's equation, it appears as

$$M = E - e\, sin\, E$$

where M this is called the true anomaly. This one is called eccentric anomaly, and as you know this e is eccentricity.

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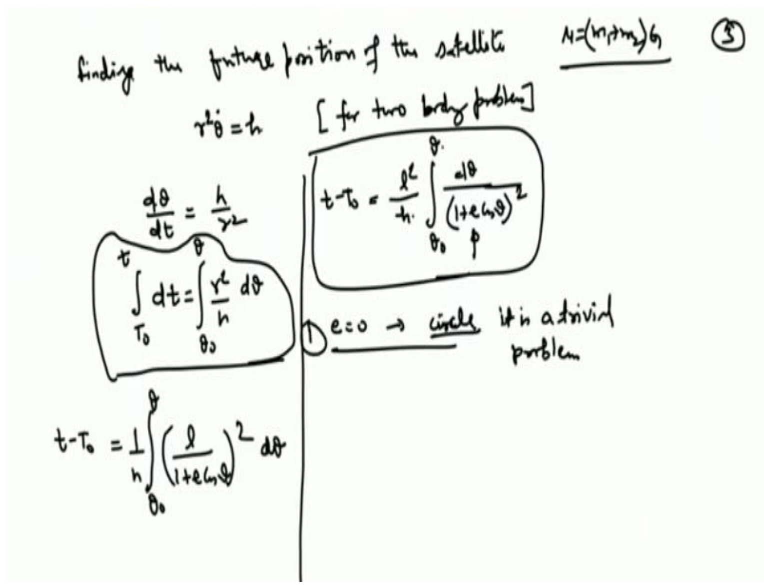

So steps to be carried out, it is very straight forward, finding the future position of the satellite. We have the equation

$$r^2\,\dot{\theta} = \,h$$

for two-body problem. It appears as if one body problem, only thing that the μ, which appears that μ gets modified as m_1 + m_2 times g, otherwise it is similar to the one body problem, we have taken right in the beginning. So from this place, we can write

$$\frac{d\theta}{dt} = \frac{h}{r^2}$$

And from here we can see that dt will be equal to dθ r^2 divided by h and integrate it from say for t_0 to t and here θ = θ_0 to θ. So once we solve this problem, our job is done. So for doing this, we will utilize all the developments we have done earlier. So say, here in this case, the left hand side we can write this as t – t_0 and the right hand side as we know this is

$$r \,=\, \frac{1}{1\,+\,e\,cos\,\theta}$$

this will be whole square and 1/ h, of course this will come outside.

θ = 0 to θ dθ, dθ divided by 1 + e cos θ whole square. So this is the basic equation for propagating θ. Now we have multiple cases here. Once e = 0, you know this is simply circle and it is a trivial problem. Because here in this case, this term gets reduced to 0, we get 1 and simply integrating it, there is nothing much in that. The second case is, this is the first case.

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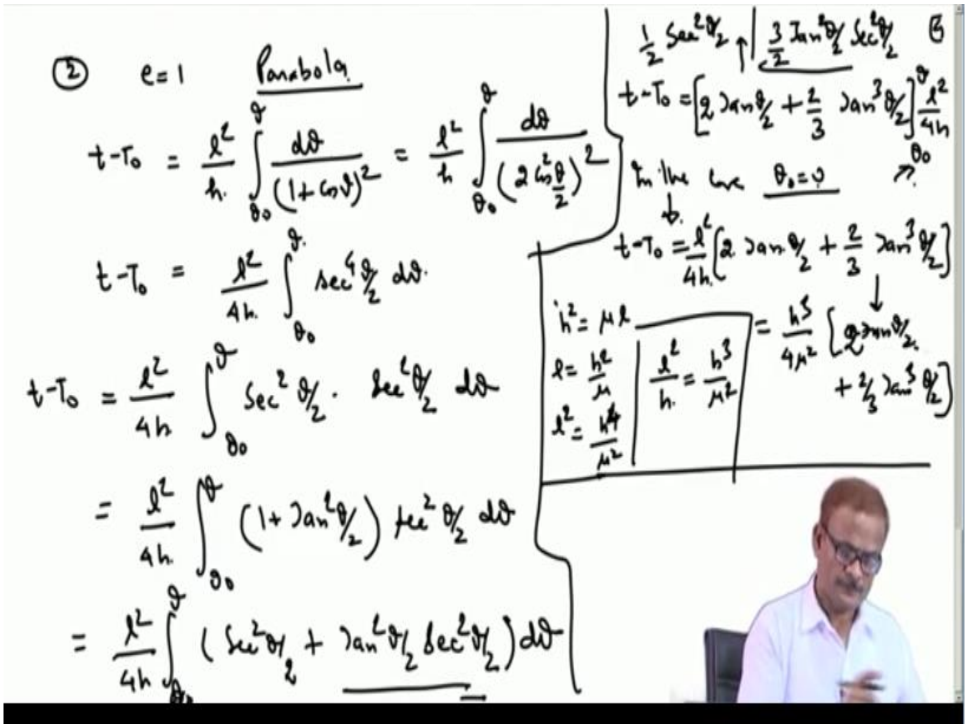

The second case is that of say, the parabola = 1. In that case, t – t_0 that gets reduced to l^2 divided by h dθ 1 + cos θ whole square. Now dθ 1 + cos θ because e = 1, so we are directly written 1 + cos θ, so we can write here 2cos θ divided by 2 whole square. So here in the case of the parabola, this is also little simpler to work with, but once it goes to the ellipse, in that case because of the presence of e, the whole thing gets complicated and then we need to apply a special method to treat all these things.

So one step we will write more and we will try to reduce this in terms of, l we will remove in terms of this l^2/h is there, so we will remove and make it in terms of h right now or we can carry also. Let us forget about this. We will carry and later on we will treat it. So t – t_0 equal to l^2 divided by 4h. Now $\sec^2$ θ, $\sec^4$ θ, we know we can write this as $\sec^2$ (θ/2) times $\sec^2$ (θ/2) dθ.

θ = 0, 2θ and then in the next step we have l^2 by 4h, θ = 0; 2θ, this we write as 1 + $\tan^2$ (θ/2) and $\sec^2$ (θ/2) dθ and then we integrate this part by breaking it one more step we will have to write. These are the standard techniques of doing any problem. Therefore, t – t_0 the first term it can be written as, this is simply tan (θ/2) and the second term. Now for that, here (θ/2) is there. So we have to take care of the 2 also.

If it is only θ, then it is straight forward, but here 2 is there and therefore we account for that also. If we differentiate this, so we can see that this differentiation will be $\sec^2(\theta/2)$ times 1/2. So that means, we have to put here 2 to get to this $\sec^2(\theta/2)$ value. As we differentiate this, so 1/2 will appear and this will be canceled by these 2 and therefore we get this quantity and similarly for this part, we can see this is a result of $\tan^3(\theta/2)$.

Once we differentiate this quantity, this will appear as $3\tan^2(\theta/2)$. This is the first thing, we are differentiating first the $\tan^3(\theta/2)$ and thereafter we differentiate for tan, so that becomes $\sec^2(\theta/2)$ and then 1/2 especially it appears. So this is the situation and here it is given only $\tan^2(\theta/2)$, so that means that we should multiply here by 2/3 to eliminate all these terms and finally we put here $\theta = 0$ to θ as an integration limit.

This is the way we do this problem in the case $\theta_0 = 0$ means you are measuring the position from the periapsis. So in that case $t - t_0$ that gets reduced to $2\tan(\theta/2)$ +, here in this case also $2/3\tan^3(\theta/2)$. This is only for the case where you $\theta_0 = 0$. Otherwise, it is not. Otherwise we have to take into account for this factor also and then we have this term also. Here we are missing other term, $l^2/4h$. So $l^2/4$.

We know that $h^2 = \mu$ times l, so l then becomes h^2 divided by μ and l^2, this is h^2 h to the power 4 by μ^2 and this implies l^2 by $h = h^3/\mu^2$. So we can insert there and that becomes our final result. So this quantity then gets reduced to $h^3/4\mu^2$ times all these terms in the bracket, which is $2\tan(\theta/2) + 2/3\tan^3(\theta/2)$. So this way we have been able to very simply solve this for parabola. So we will do it for the parabola and then we will do it for the ellipse and also for hyperbola.

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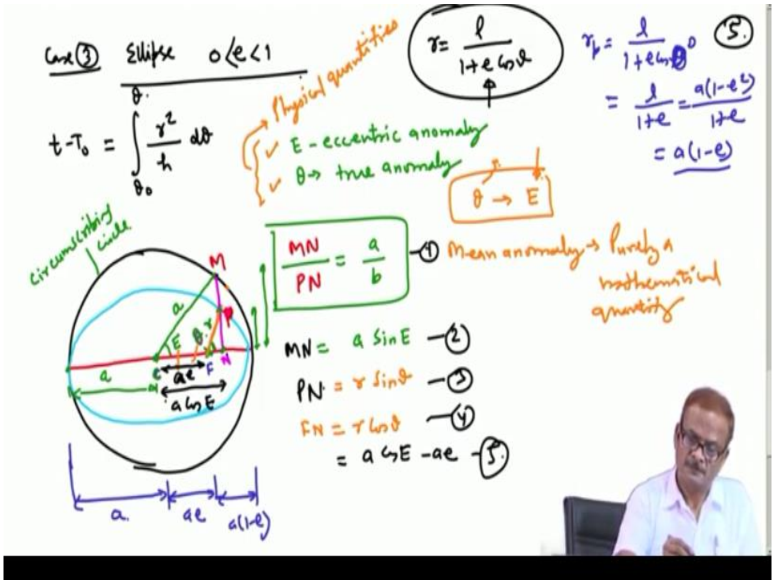

Case 3, already we have done for the circle, this was trivial. There is nothing to do in that and then the other one, we have done the parabola. Now this is an ellipse. Ellipse is the case when e lies between 0 and 1. So in this case, what will be the result? Following the same procedure, we have written this as r^2 by h dθ. Now here in this case problem will be because of the presence of 1 + e cos θ

$$r = \frac{l}{1 + e \cos\theta}$$

So once we insert here in this place, because of presence of this e the integration gets complicated. So for solving this problem, what we call, we change the variable, so that this gets into a simpler format. So what do we do here, that say this is a circle and inside the circle an ellipse is described and somewhere the focus is located here. this part we call as the auxiliary circle or simply we can call as circumscribing circle, which is wounding this ellipse.

Then, we have the radius vector here, so if we extend this to this place, this goes and touches the circle here in this place. Somewhere this point is your center and this point we call this as the focus. This is center and the focus. Here, on this side we have the half of this is semi-major axis. The quantity from here to here as per our conic section discussion, this is ae and this quantity from here to here this is a times 1 – e.

Because that is the way we write

$$r_p = \frac{l}{1 + e \cos\theta}$$

and for that θ becomes equal to 0. So this is 0° and therefore this is l/1 + e and l is nothing but a times $1 - e^2$, this divided by 1 + e this gets to 1 minus. So here in this case, we drop a perpendicular from this point to this point and then extend this beyond this line to this place and let us say that this point we call as the P and this point we call as M.

This angle we show as θ, which is a true anomaly and here the point on the bottom we call as N. If we join the center of this ellipse and point p, then this angle which we show here this is called e. So e is called eccentric anomaly and θ this we are calling as true anomaly. This is θ. So what we have done, that this point on the ellipse we have extended vertically, which cuts the circle here in this point at P. So the radius of the circle will be a, because this length is a.

From here to here, this length is a and therefore this also becomes the center of the circle and center of the ellipse also. Now we use a property for this auxiliary circle and the ellipse, which we write as

$$\frac{PN}{MN} = \frac{a}{b}$$

This is the special property of this circumscribing circle. It bears this ration. This point is here, this point here and this point here. So this length and this length they bear this ratio PN/MN = a/b. This will help us solving this problem.

So PN will also be equal to

$$PN = a \sin E$$

while the quantity this is your r. That is shown by the orange line, from this place to this place, this is r, the angle is θ here. So

$$MN = r \sin\theta$$

Similarly,

$$FN = r \cos\theta$$

So using these details, we will be able to replace this θ in terms of E. We do not want to work with this θ, because once we are working with this θ, so E is appearing.

So our objective is to eliminate this E and in that context we have to introduce another variable; I am calling this as eccentric anomaly. So the true anomaly and eccentric anomaly, both are physical quantities. While earlier, I have told you the mean anomaly. Mean anomaly is purely mathematical quantity. We cannot show that in the picture, while the true anomaly and the eccentric anomaly, we can show it like this, this one and this one.

So mean anomaly, this is purely mathematical quantity. With this restriction, now we proceed further. So already we have written. On the previous page, we have written. The name we will change like we will do little more deification here. We will write this as P and I have used this notation. So I will write here as M. This is M, this is P and N. So therefore, ratio is the same, only thing the notation I am changing here, nothing else.

So

$$MN/P = a/b$$

It is the same thing, only thing the symbols I have changed. So therefore, I will change here in this place also

$$MN = a \sin E$$

MN is quantity from this place to this place and PN we will write as r sin θ and rest of the things are the same way. So this way we have MN/PN = MN/ PN = a/b.

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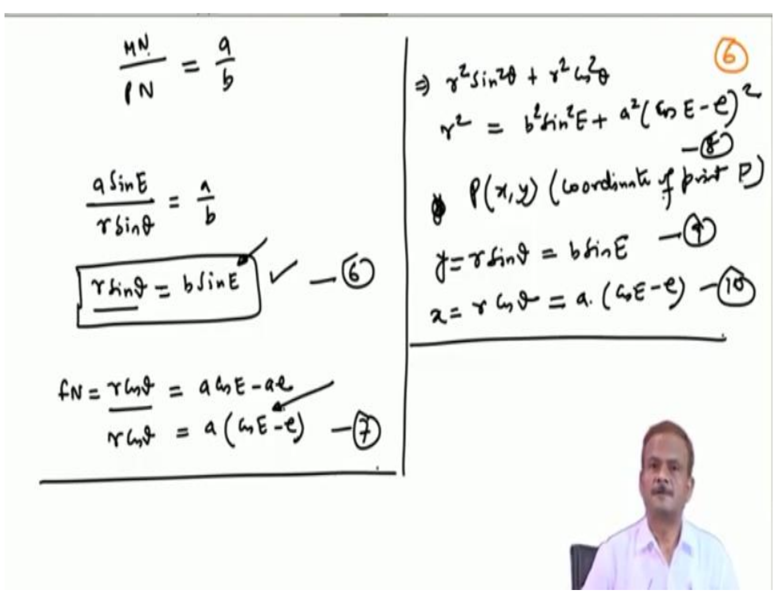

Now we solve this. So MN we have written as a sin E, therefore a sin E and PN we have written as r sin θ. This is

$$r \, sin \, \theta \; = \; a/b$$

so a and a cancels out and we get here r sin θ. On this side, this goes and this is b sin E. Similarly, what we can see here, this quantity is known to us, from this place to this place. This is nothing but a cos E and also this distance is known to. This we have written as ae. So FN this quantity also this becomes equal to

$$FN = a \, cos \, E - \, ae$$

So FN this is nothing but r cos θ, this will be equal to a cos E – ae, acos E – e. So what we can observe that this r sin θ and r cos θ, they are described in terms of this eccentric anomaly. Here also, the eccentric anomalies are appearing. We have been able to eliminate. We will be able to describe r in terms of a, b and e. That is the semi-major axis and semi-minor axis. This we write as equation number (1) here, this is (2), (3), (4), and (5). This is (6), this is (7).

The next step, thereafter its logical step will be

$$r^2 \; = \; b^2 \, sin \, 2E \; + \; a^2 \, (cos \, E - \, e)^2$$

This is one way of writing it. So we will continue in the next lecture, little bit of the coordinates also we will write, because we have done already so much, so I will write the coordinate. The coordinate of the point P, so P its coordinate is (x, y).

Directly we can write

$$y \; = \; r \, sin \, \theta$$

and therefore from here this is nothing but b sin E and similarly

$$x \; = \; r \, cos \, \theta$$

and r cos θ we have written as a cos E – e. So we do till this extend and the next time we will be working further with this. This we will number as (7), (8), (9) and (10). This is coordinate of point P. We end the lecture here and next lecture we will continue from this point only. Thank you very much.