Space Flight Mechanics
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Lecture – 20
Kepler's Problem (Contd.)

Welcome to Lecture 20. So we have been discussing about the Kepler’s problem. So, we finished the parabolic orbit and then we have started with elliptic orbit. So, we will continue with that, but before that we will take one small problem which is related to week 3 whatever the discussion we have been doing.

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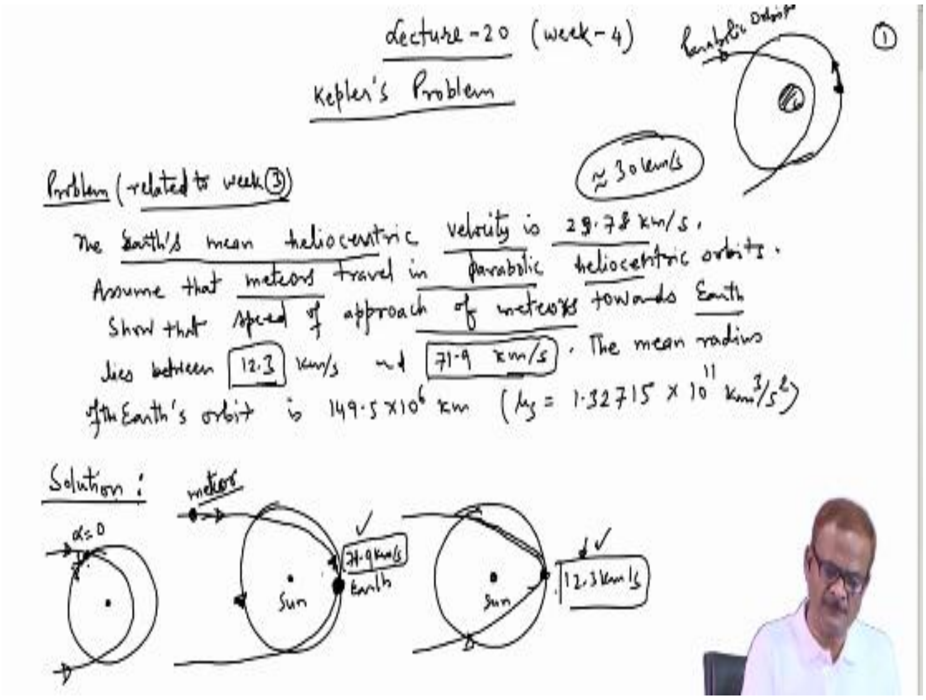

So let us look into this problem. So, here Earth’s mean orbit mean heliocentric velocity it is given it is around 29.78 or around 30 km/s. So Earth is moving around the sun around 30 km/s. So assume that meteors travel in parabolic heliocentric orbit. Heliocentric means moving about the sun so this is the sun and around that Earth is moving so this is your Earth it is going here in this orbit.

Your meteor it is coming from somewhere else and it is going in parabolic orbit, so this is your parabolic orbit. Show that the speed of approach of meteors towards the earth lies between 12.3 km and 71.9 km/s. So, let us say that this parabolic orbit here in this case what I have shown here this is the orbit of the Earth and if your meteor it is coming about this orbit it is coming like this and going like this.

So this is going in a parabolic orbit and Earth is going moving here in this orbit. So, once they come here so at this point if it is lying exactly it is a trajectory line such that it is going to hit the Earth so its velocity will be the 71.9 km/s if it is coming from this direction. On the other hand, if the Earth is going around like this, this is the sun and this is your Earth and meteor is coming from this side.

So, if meteor is going from this direction say and they are meeting here in this place they are crossing each other here in this place. So, the Earth will go here and meteor will also go here so the velocity of approach will be here in this case 12.3 km/s this we have to show and here in this case 70.1 km/s. So, with this much of high velocity 71.9 km/s this is very high velocity.

This again it is a very high velocity while the Earth is moving around the sun with 30 km/s. So, you can see that if the Earth is impacted by the meteor so what disruption will take place. So we have to calculate this value that indeed this will be between this two values. So it is not necessary that exactly here this is crossing here in this point we can consider it at any other point because we have to show the range.

And we can assume that it is such that here if they are in the same direction like if the Earth is coming from this direction and here at this point this meteor is coming from this direction and it is such that angle between them is zero. This angle say this is here if I write this as α so α = 0 so that impact velocity will be 12.3. On the other hand, if the impact take place if the meteor is going from this place and here if it meets here in this place.

So it will be in opposite direction and then the impact velocity will be 71.9 km/s. So this we have to show and this is a very simple problem, but understanding is required.

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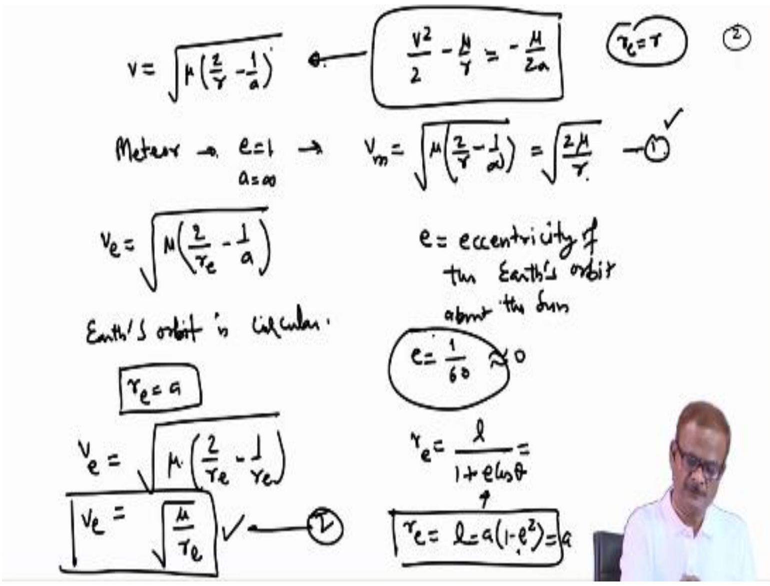

So we use the equation

$$v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}$$

So this is valid for all the time as we know that we have derived it from μ / r this equal to – μ / 2a. From here we have rearranged it and we got it. So, the velocity of the Earth and velocity of the meteor we can calculate from this place. For parabolic for meteor first let us consider. In parabolic orbit e = 1 and therefore v meteor (v_m) we can write as μ times 2 / r and a = ∞.

So 2 / r – 1 / ∞ under root so that becomes 2 μ / r under root where r is the impact where it is taking place? It is taking place at the radius of the Earth's orbit so this is r Earth orbit. So, instead of writing re we simply write it as r so this is the meteor velocity at that point. What will be the Earth velocity in this so v-Earth again this is

$$v_E = \sqrt{\mu\left(\frac{2}{r_E} - \frac{1}{a}\right)}$$

And let us assume that see the eccentricity of the Earth orbit e we will write it as the eccentricity of the Earth's orbit about the sun is e = 1 / 60 around. So this we will assume to be 0 that means we will assume this to be a circular orbit. So if we assume it to be a circular orbit. So, Earth this is simplification Earth orbit is circular. If we do that, that means r-Earth will be equal to a.

And therefore which we know that r-Earth this will be

$$r = \frac{l}{1 + e \cos\theta}$$

in the case of if we assume it to be circular so this e = 0 in that case so therefore r_e becomes equal to l and this equal to a $(1 – e^2)$therefore e = 0 and this gets reduced to a. So, therefore we are writing here r_e = a and once you have insert here in this point so this becomes μ $(2/r_e – 1/r_e)$ under root.

So this becomes

$$v_e = \sqrt{\frac{\mu}{re}}$$

which is the velocity in the circular orbit as we know it is very easy to calculate. So, velocity in the circular orbit for the Earth this is known and for the meteor this is also known.

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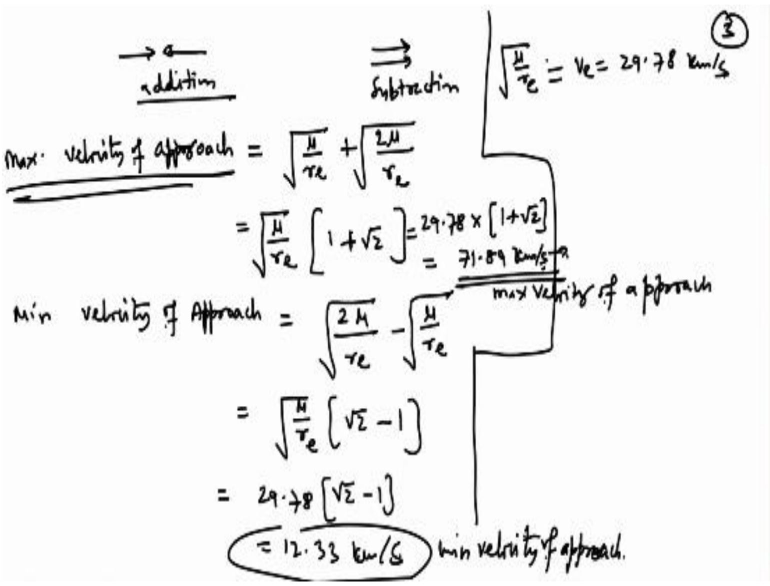

So now if we add them so we get the impact when they are opposite to each other and if we subtract so we get the impact velocity when they are in the same direction. So this is on subtraction and this is on addition. So, simply therefore the velocity of approach this will be given by μ/r_e under root + μ / now here in this case for the meteor we have written here 2 μ/r so this we need to reduce because the impact will take place at the radius of the Earth orbit.

So, therefore we need to replace it by 2 μ/r_e. So therefore here also we have to write it 2 μ/r_e and μ value for the sun is given and therefore we can calculate it. So this is

$$= \sqrt{\frac{\mu}{r_e}}\ (1 + \sqrt{2})$$

another case once they are moving in opposite direction so we have to subtract it. So in that case the same velocity in the case of so this is the maximum velocity of approach and minimum velocity of approach will be minimum velocity of approach this will be given by because this quantity is larger.

So put it first here – μ/r_e under root. So this is μ/r_e under root times root 2 – 1. Now μ / r_e we have to compute and insert here in this two places and we will get the result. So μ / r_e is given to be in the problem itself it is stated Earth's mean velocity is 29.78 km/s so the things are given. So, this quantity is given to be 29.78 km/s. Therefore, once we insert it so ∞ 29.78 and $\sqrt{2} - 1$ this yields 12.33 km/s.

And if you insert here 29.78 into 1 + root 2 so you get the result 71.89 km/s. So this is the way maximum velocity here, this is the maximum velocity of approach and this becomes the minimum velocity of approach. So you can see that using this simple principle what we have developed. We can work out the things which are of so immense importance and many times you might have heard that NASA has predicted that this meteor is moving towards the Earth or some asteroid is moving toward the Earth.

And if it impacts the whole Earth will get destroyed. So, the reason is very simple because of this high velocity and from here you can calculate how much energy it will add if the mass of that meteor or the mass of that asteroid is known. The whole kinetic energy it is going to be imparted. So, while it reaches the Earth and impacts it at such a high velocity so energy will be enormous and the whole Earth will depending on the size even the whole Earth can get destroyed.

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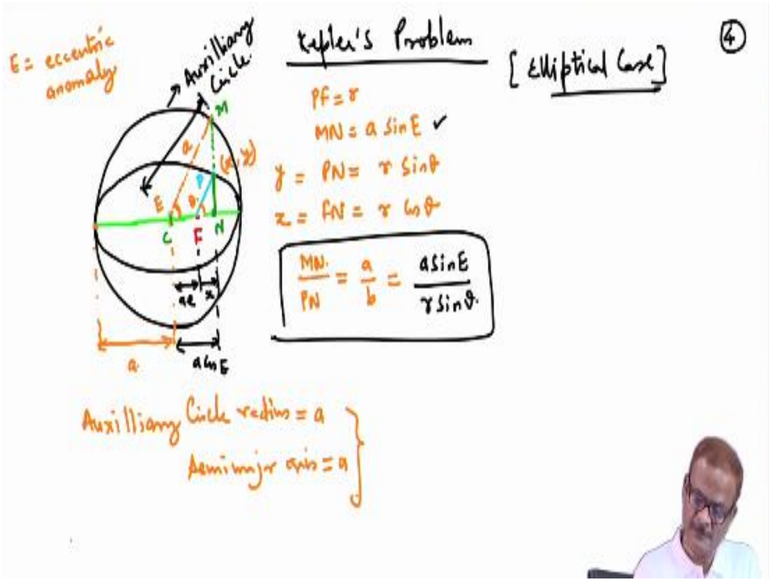

So, we go to the Kepler's problem as we have been discussing earlier so we start with that. So, here now we have been discussing that inside the circle we took one ellipse we were discussing Kepler's problem here for the elliptical case and this we have written as auxiliary circle or circumscribing circle and this is our ellipse, this is the semi-major axis of the ellipse, this is the center of the ellipse here, this is the center, focus is lying somewhere here.

And then what we did that I will shift little bit focus on the left hand side to make it more convenient so I will write here the focus is F is located here. So, thereafter we extended this upward from here vertical this is a 90° here. Okay sorry I have to revise this first we draw the radius vector and from there we have to extend. So, first we draw the radius vector from there first we took the radius vector to any point and this we noted as P this point.

And thereafter we draw up the perpendicular from this point only X-axis so this is the perpendicular here. This angle we have written as θ so this is θ and then this was extended up. So this was extended up so this point we have written as M and this point we have written as N and also we joined the point C and M. So, because distance between this and this point this is a means the radius of the circle.

The auxiliary circle radius = a, and semi major axis is also a this is also a and therefore this quantity is a and MN then we have written as a and this angle we have taken as E which we have written as eccentric anomaly. So

$$MN = a \sin E$$

and PN this we have written as r because PF = r. So PN this becomes equal to r sin θ and similarly FN this is nothing, but x and this is nothing but y the coordinate of P so the P has coordinate (x, y). So FN this is

$$FN = r\,cos\,\theta$$

So this we have written thereafter we have utilized one property of the ellipse that

$$\frac{MN}{PN} = \frac{a}{b}$$

So from this place MN becomes MN is nothing, but a sin E, and PN is r sin θ. So, we are going to get relationship between them. So this we have been working with that. Sorry PN is r sin θ. So this relation we have been working out so now writing the whole thing. So distance FN already we have written this is your coordinate from here to here this you are writing as x from this point to point from F to this point.

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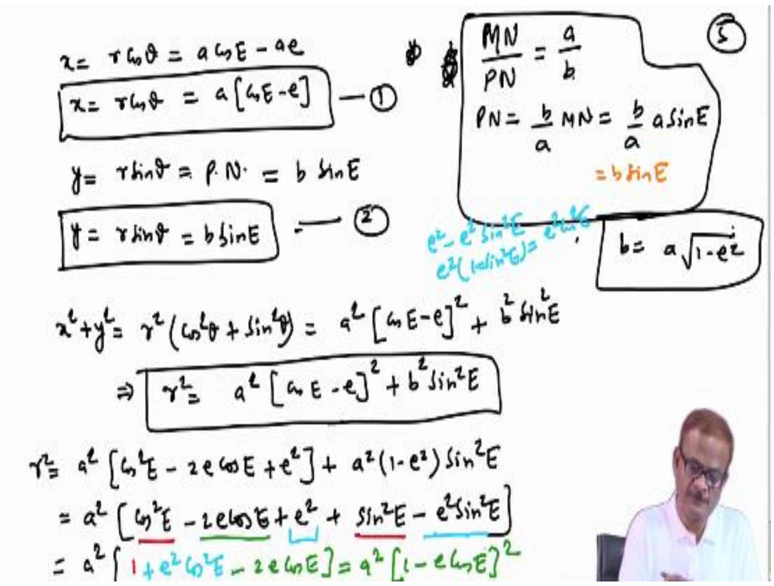

And

$$x = r\,cos\,\theta$$

as we have written earlier this will be equal to a cos E – ae a cos E is what distance this distance is up to this point because this is the radius here of this point this is a so therefore this becomes a cos E and distance between this and this, this is your ae. So we are subtracting to get x. So also x can be written as

$$x = r\cos\theta$$

$$= a\cos E - ae$$

$$= a(\,cos\,E - e)$$

And y equal to already we have written r sin θ and for that we use this ratio equation. So in this case PN is your y this quantity is nothing, but your y. So

$$y = r \sin \theta$$

and we use this relationship this equal to PN and PN = b / so the MN / PN this equal to a / b we have written so therefore

$$PN = \frac{b}{a} MN$$

and MN is nothing but b/a MN if we go back so this is the distance from this place to this place MN is a sin E.

So already we have written here in this place so this equation just we are rearranging so this is a sin E. There is a difference between this r the r always we measure from this point r is always measured from the focus where r is the any point on the ellipse while the radius of the Earth is being indicated by distance from this point to this point or either the distance between this point to this point. So this is the radius of the Earth.

So, you have to be careful about this. So getting this, this is b sin E so

$$PN = b \sin E$$

So what we get from here

$$y = r \sin \theta = b \sin E$$

So this is the same thing here from here also we could have written r sin θ = a / a times b sin E = b sin E. So till this extent perhaps we did last time. So now we have this two equations where(x, y) these are given. Let us say this is equation number (1) and this is equation number (2).

Therefore, if we add them so this gives us $r^2 \sin^2 \theta$ this becomes a $\cos^2 \theta$ the first term is related to x is $\cos^2 \theta$ so we put here the $\cos^2 \theta$ and related to the y this is $\sin^2 \theta$ so we will put here $\sin^2 \theta$ and then squaring the other terms on the right hand side adding them $(\cos E – e)^2$ + b sin E.

So this implies

$$r^2 = a^2(\cos E - e)^2 + b^2 \sin^2 E$$

Okay whatever we are working now so this we are doing by one method which is totally mathematical in nature little bit of this geometrical relationship we have used here. Later on we will do it by graphical method also because it is very important to do by graphical method, understanding the things the basic principles involved.

So, the next step is to expand them so this becomes cos ^{2} E – 2 e cos E + e ^{2} and b ^{2} because b is written as

$$b = a\sqrt{(1 - e^2)}$$

So, therefore b^2 will be a^2 (1 – e ^{2})$\sin^2$ E. So a^2 we can take it outside and rest of the things we can work out so this is $\cos^2$ E -2e cos E + e^2 + $\sin^2$ E – e^2 $\sin^2$ E.

So now the term cos ^{2} E + sin ^{2} E this added together that becomes 1 so we write here 1 and then rest of the terms we have to write here. This term and this term combined together so a^2 can be taken outside and we can write here + e^2 1 – $\sin^2$ e so that becomes $\cos^2$ E. One step I am skipping okay I will write here e^2 – e^2 $\sin^2$ E this equal to e^2 (1 – sin ^{2} E) this equal to e ^{2} cos ^{2} E.

So this I have written here in this place. So this term and this term they are taken care of so last term is remaining as – 2 e cos E so this – 2 e cos E and if you can see that this is nothing, but 1 – e cos E whole square.

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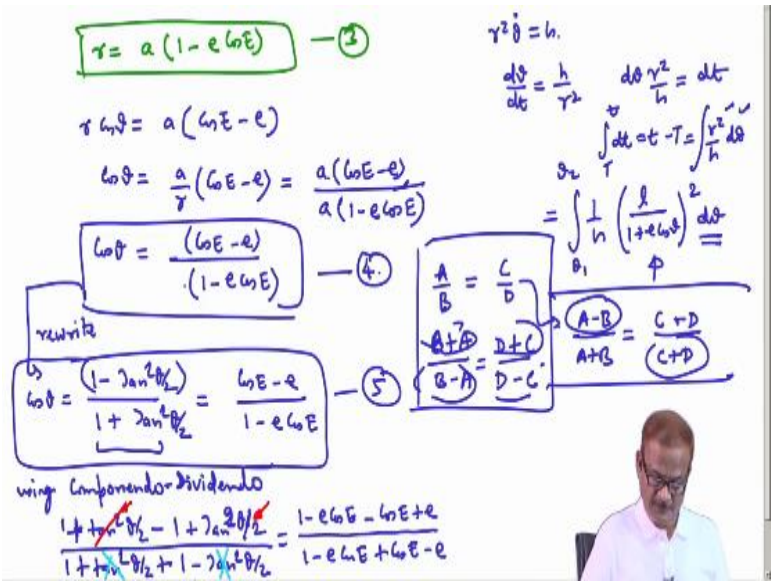

So, therefore r becomes equal to because r is always a positive quantity so this becomes

$$r = a\,(1 - e\cos E)$$

. So this we will write equation number (3). Now we have to get the θ also. So what we are trying to do as I have told you in the last lecture that while we were working with this equation $r^2\,\dot{\theta} = h$

so this we then wrote as

$$\frac{d\theta}{dt} = \frac{h}{r^2}$$

and dθ r^2 / h = dt.

And this we integrated to get t – T = r^2 / h dθ. So here r and θ we are trying to because it is a not comfortable to work with the terms like l / 1 + e cos θ because

$$r = \frac{l}{1 + e\,cos\,\theta}$$

and then here we have 1/h and θ_1, θ_2 integrating between θ and θ_2 this is dθ. So this was not comfortable situation so we are converting into a form where we can easily integrate it.

But such a long process of integration can be avoided if we use little geometric property and graphical method. So that I will show you later on, but this is also very useful while working with many problems so therefore we are discussing this. So, next we have to determine this cos θ so cos θ already the equations are known to us the x value we are aware of so from there we can extract.

So r cos θ here cos θ is given by this a cos E – e r cos θ = a (cos E – e) and therefore cos θ that becomes

$$cos\,\theta = \frac{a}{r}(cos\,E - e)$$

Cos E – e a cancels out so we will eliminate a from this place, but still we have a form with which it is difficult to work we have to eliminate here so the r if we replace in terms of this and then cos θ we also require here dθ and we require it in a proper format.

So for that we need to work little bit more so this cos θ this equation is rewritten as rewrite this as 1 – $\tan^2$ (θ/2) divided by 1 + $\tan^2$ (θ/2). So we are using this trigonometric identity and this equal to cos E – e / 1 – e cos E. Now, using Componendo-dividendo so we write this as 1 minus simply we have to add the denominator into the numerator and so on. On the right hand side there are various ways of doing this.

So the first term we added 1 / $\tan^2$ (θ/2) you might be aware of this there is nothing in this. If you are not aware of just look into the algebra by Hall and Knight. $\tan^2$ (θ/2) and then we subtract this term also – 1 – $\tan^2$ (θ/2). So this subtracting this gets plus equally what you can do that first we instead of adding here this we can do the subtraction and this place we can see what I mean most of you might be aware of this principle there is nothing great in this.

So the principle it goes like this. If we have

$$\frac{A}{B} = \frac{C}{D}$$

So we can write as say the

$$\frac{B + A}{B - A} = \frac{D + C}{D - C}$$

and equally we can also write as the same thing can also be written as

$$\frac{A - B}{A + B} = \frac{C - D}{C + D}$$

So, there is nothing peculiar in this, this is very simple relationship which is used quite often. So we will see that this is just exchange of numerator and denominator this is A – B.

If you bring it on this side so this is A – B times C + D so A – B and C + D is there so here the sign reversal will take place so both are the same thing. So the same principle we are using here and if we do that so we get the result 1 + $\tan^2$ (θ/2) it is a matter of convenience which one either we can use this or either we can use both are same so it is nothing different.

So, here we subtract here in this case so that if we subtract if we take this first and subtract from this particular term so tan will add in the numerator. So this way we will have here $\tan^2$ (θ/2) $\tan^2$ (θ/2) divided by 1 + $\tan^2$ (θ/2) + $\tan^2$ (θ/2). So once we have subtracted in the numerator in the denominator we will add so this becomes then 1 – $\tan^2$ (θ/2).

Similarly, on the right hand side 1 – cos E 1 – e cos E – cos E + e and divided by 1 – e cos E + cos E – e. So here on the left hand side what we see that this two cancel out this two add up. Okay this and this term they add up and here this two terms they cancel out.

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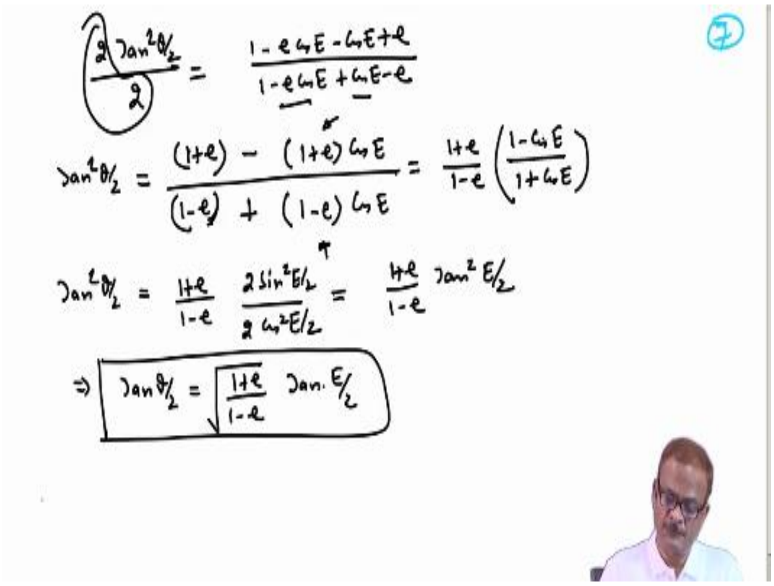

So we write it on the next page. So this becomes 2 $\tan^2$ (θ/2)and from this position so 2 $\tan^2$ (θ/2) divided by 2 on the left hand side and on the right hand side we have as we can see from this place this is 1 – e cos E. We will write the whole term there 1 – e cos E and thereafter we have – cos E + e and the other term 1 – e cos E and then cos E – e. So this is the term so this two will cancel out will drop out and that gives us $\tan^2$ (θ/2)and here we have to rearrange it in a proper way to solve it.

So this will become 1 + e – cos E we will take it outside so this becomes 1 + e cos E and similarly here in this place this is 1 – e and in this place we get it as 1 + e and this is 1 – e this term and this term we are taking. So once we take the we will rearrange it because depending on minus or plus sign outside the sign will depend so 1 – e cos E we are taking if we write it with a plus sign. So the 1 can be here in this place and then we will get this – e cos E.

If we write here a minus sign so this thing will change, but we want to put it here in this the same format. So as it appears from here this becomes 1 + e divided by 1 – e, 1 – cos E divided by 1 + cos E. So $\tan^2$ (θ/2)and this becomes 2 $\sin^2$ (E/2) using this trigonometric identity $\cos^2$ (E/2) and this implies

$$tan\ (\theta/2) = \sqrt{\frac{1\ +\ e}{1 -\ e}}\ tan\ \left(\frac{E}{2}\right)$$

So we have taken only the positive sign. So we stop here and so we will continue with the same point in the next lecture.