Space Flight Mechanics
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Lecture – 21
Kepler's Problem (Contd.,)

Welcome to lecture number 21. So we have been discussing about the Kepler's problem, so we continue with that.

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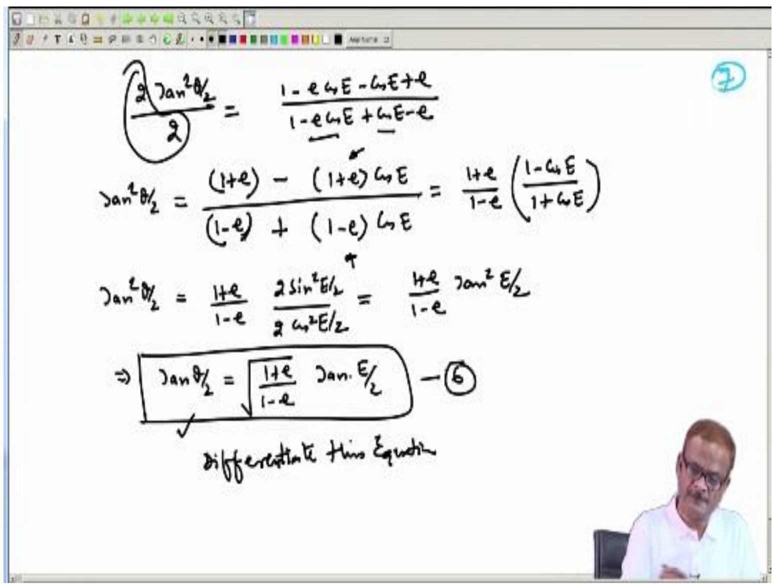

So going back to our; the equation that we have derived tan (θ/2) = 1+e divided by 1-e tan (E/2). So we will continue with this; now from here we can extract this.

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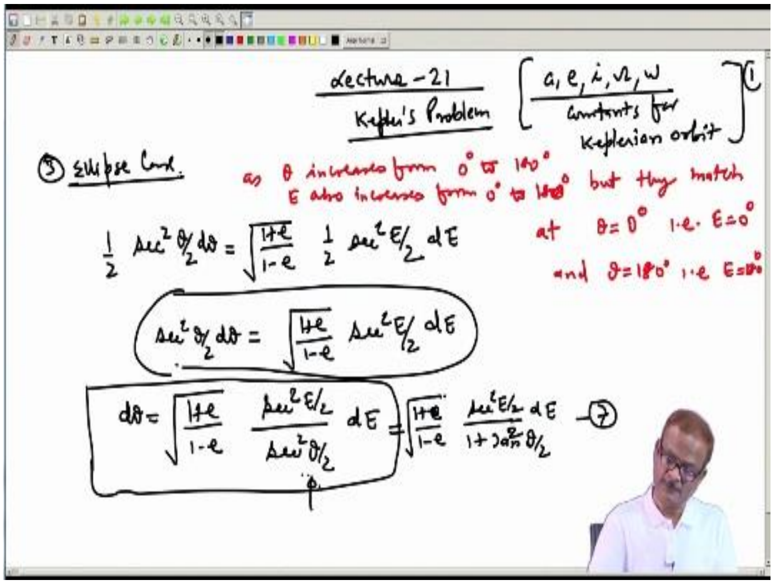

If we differentiate this basically $\sec^2$ (θ/2) suppose this equation is differentiated so differentiate this equation. This will (5);

$$tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+e}{1-e}}\tan\left(\frac{E}{2}\right)$$

this we will write as equation number (6), $\sec^2$ (θ/2) 1/2 and on the right hand side then we have, as we know these are constant for Keplerian orbit, so already I have stated many times a, e, i, Ω, ω, these are constants for Keplerian orbit, okay therefore there is no point in differentiating this.

And then we have to take the differentiation of this so this becomes 1/2 $\sec^2$ (E/2). And on this side we also have to write the dθ term, so here dθ and here we have write d. So this gets reduced to $\sec^2$ (θ/2) dθ (E/2) dE, and we are interested in getting the dθ, so we write it in this format. We need to further work on this and put it because here (θ/2) is appearing so we need eliminate this in order to get the proper result.

$$d\theta = \sqrt{\frac{1+e}{1-e}}\,\frac{(\sec^2\left(\frac{E}{2}\right)}{1+\tan^2\frac{\theta}{2}}\,dE$$

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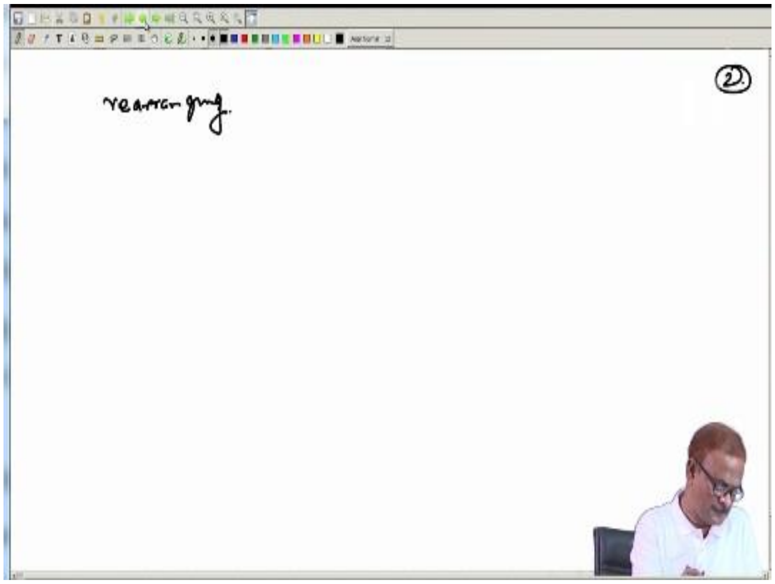

So rearranging; so we need to rearrange it. Okay one more thing we need to observe before we go into the other part. If we look into this equation:

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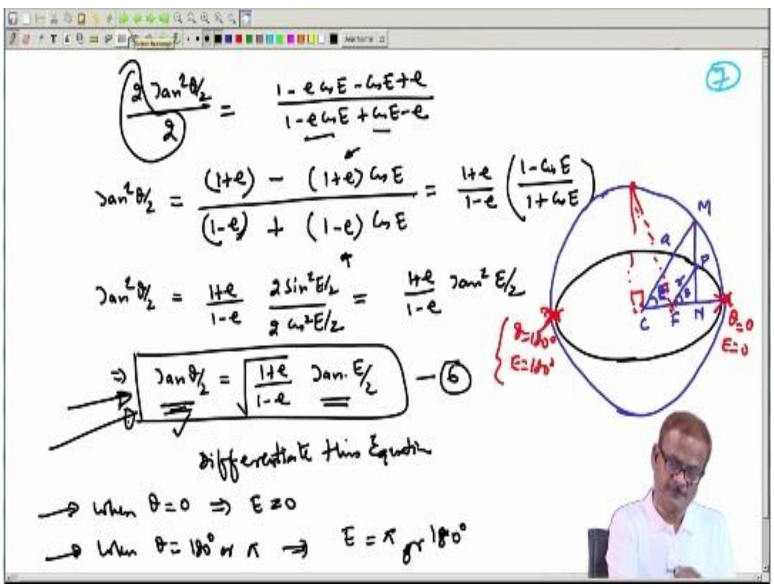

Or either better here in this place, if we look here in this equation, so what we can observe that when θ equal to 0, this implies e is also equal to 0, okay and when θ equal to 180° or π in terms of radians, so this will imply, so this side the left hand side becomes infinity or similarly the right hand side then will be infinity, so this implies E is also θ equal to π/2, so this also implies E equal to π or180°.

So these are the two points where θ and E they match, the other points they are not matching. So this is simply visual from, if you look into the ellipse and the auxiliary circle; this figure is not very good but still it will do. This is the focus here. This is R. This is θ. And here this is your a, this point we have taken as M, this we have taken as P and this we have taken as F and this as center, okay.

So you can see that this angle as E. So what we can see that E=0 so that means M is lying here in this please and at that time θ also lies here in this place, that is very obvious if you drag P here the M will also get dragged here in this place. And if M comes to this place that means you are extending it such that the R vector you are extending from this place to this place and then dragging one vertical from this point to this point, so at that time whatever the θ value will be E will be equal to 90° but θ will not match, because θ, this will be an inclined distance from here to here, so this angle and this angle this θ angle not going to match.

But once you come to this point again so you can see that at this point θ will be also 180° and E will be also 180°, so at this point and this point only they match, so here θ equal to 0 and E is also equal to 0 at this point θ equal to 180° and E is also 180°. So these are the two points at which they match, the other points they will not coincide.

So this we can state as θ increases from 0 to 180° E also increases from 0° to 180°, but they match at θ equal to 0° andθ equal to 180° that is in this case θ E will be equal to 0° and here this E will be equal to 180°, so these are the only two matching points, other places they will not match.

So this equation again we rewrite here, $\tan^2$ (E/2) dE. And then we insert these values, tan (E/2) is; tan (θ/2) is known to us so we can insert here, sorry this is (θ/2) so here this should be (θ/2), so here this is $\tan^2$ (θ/2). So $\tan^2$ (θ/2) is available from tan (θ/2) is available here, so we can insert that and rearrange that equation.

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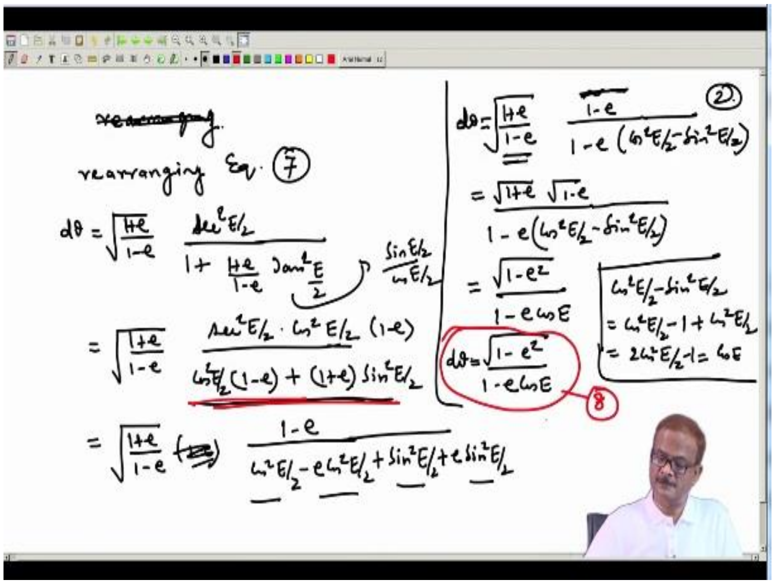

So rearranging equation (7) that gives us dθ equal to 1+e divided by 1-e times sec^2 (E/2) divided by 1+ ; and now tan (θ/2) whatever the value we have, so that value we need to insert, so tan (θ/2) we have just written as √(1+e/1-e) tan^2 (E/2). Okay. So if we rearrange it, so this comes here in this format and here one thing we have to take care this is tan (θ/2) is, if you look here in this place tan (θ/2) is under root of this 1+e.

So here we have the term, we have replaced this tan; sec^2 θ by 1+tan^2 θ, so we need to square also. So this squaring we have square d this term but this term is not square d, so we have to square it. So if we square this under root will go and with this we have then 1-e, we simplify it little bit 1+e and this will break up in terms of sin (E/2) divided by cos (E/2), so this will be sin ^{2} (E/2) and here we will have cos ^{2} (E/2).

And of course then we will have here sec^2 (E/2) times cos ^{2} (E/2) and times 1-e. So after rearranging we get this. So one little bit more steps are required. Sec ^{2} (E/2) times cos ^{2} (E/2) is 1, so this we write as 1 or we can write here 1-e and this term we will rearrange, okay. So we have to rearrange this particular term. So for doing this we write here bracket here and then this is cos ^{2} (E/2) – e. 1+e divided by 1-e and this term, this term that gives us 1 and here this two terms we have to combine them.

So this we can write as –e times $\cos^2$ (E/2) – $\sin^2$ (E/2). Okay, so final state we have to write for, so this becomes 1+e under root, this one and this one we will combine together by dividing so this will be 1-e because here this is in under root, so therefore we get this term and in the denominator we have 1-e $\cos^2$ (E/2) – $\sin^2$ (E/2). So this is 1-e ^{2} under root divided by 1-e cos E.

And this is following from this particular principle, $\cos^2$ (E/2) – $\sin^2$ (E/2), this is the trigonometry principle you must be aware of, I just took a shortcut. This will be 1-cos ^{2} E, so another this $\cos^2$ (E/2) so this is 2 $\cos^2$ (E/2) – 1 and this is nothing but equal to cos E. So this is what we have utilized in this place. So dθ finally we get as

$$d\theta = \frac{\sqrt{1 - e^2}}{1 - e\,cos\,E}\;dE$$

So reason for doing this all these exercise, it is a simplification of the whole process, so (7) and then this equation is now (8).

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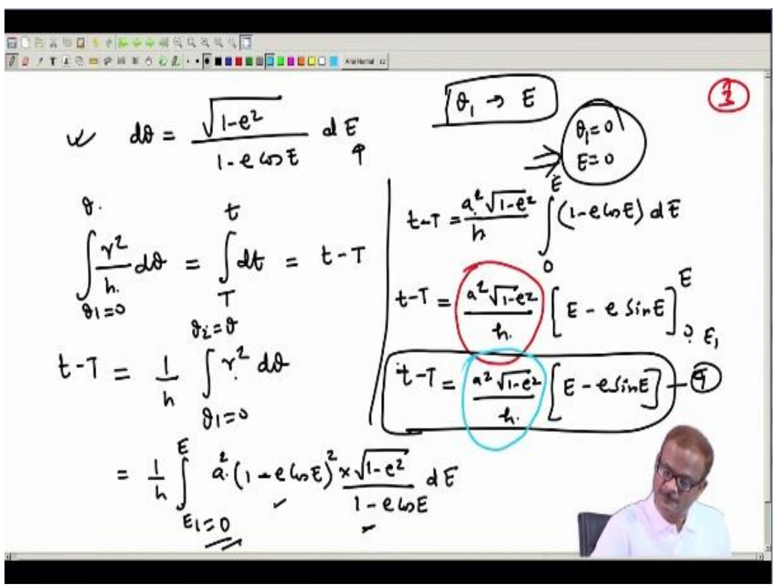

Now finally we can work with our equation where dθ, we have got as

$$d\theta = \frac{\sqrt{1 - e^2}}{1 - e\,cos\,E}\;dE$$

, this is available to us so we utilize it in this equation. Now we need to put the value for the *r*; whatever we have derived. So h we will take out first, θ_1 to θ_2 and as soon as we replace dθ in terms of E and *r* also in terms of E so this θ_1 and θ_2 they also, they will get replaced in terms of the corresponding E variable.

So let us first work in terms of; leave it like this and later on we will change it. So we have here r^2 dθ. And going back and looking for the; this r value, so this r equation we have got as r equal to a times 1-e cos E this was the equation we got, so this got in this square format and then dθ from this place is $1-e^2$ under root divided by 1-e cos E, dE. h, I have already taken outside.

Now this θ_1 instead of once we have converted into θ_1 so θ_1 is corresponding to certain value of E, we have to compute it, okay using the relationship we have developed for E and θ, so using those relation we can work it out. So we will do this problem later on in the trajectory transfer especially this is very much required, so at that time we will do this problem. So this θ_1 then we replace by E_1 and this is θ_2 so θ_2 we write as θ generally, the upper one this we write as E.

And if we start with θ_1 = 0 so at that time E will also be equal to 0, so this is a simplification of this case. So here we can write in terms of rather than writing in terms of E_1, we can write in terms of this, here this can be replaced by 0 if this is the case as indicated here. So θ_1 equal to, we can insert it to be 0 like this, so if we do this; this is a pretty simplified place then. So finally we have t – T this equal to 1/h, A^2 is also A constant.

So A^2 we can fetch it outside E is also a constant so that also we can take it outside, this gets stressed into a^2 times $1-e^2$ and inside the bracket we have 0 integral sign, okay. This becomes, this and this term; one term will cancel here so this is 1-e cos E and this is dE. So you can see that now we have got into; got a form which can be integrated very easily and this was the reason of doing this exercise, nothing else.

So this is $1-e^2$ divided by h and here we have E – e sin E and integrating between 0 to E. So this term we will work little later. Let us finish this part first, so this is t – T then gets reduced to a^2 times $1-e^2$d divided by h and if we break this limit, so this is E – e sin E. So this is the situation. So we are measuring, remember that we are measuring from this E_1 equal to 0. If we do not do this, so here instead of 0 we will have E_1. So, one more term will simply get into this place.

$$t - T = \frac{a^2\sqrt{1-e^2}}{h}\ (E - e\sin E)$$

So this is equation number (9). Okay now we will simplify the term, we simplify this term, okay and put it in a proper format.

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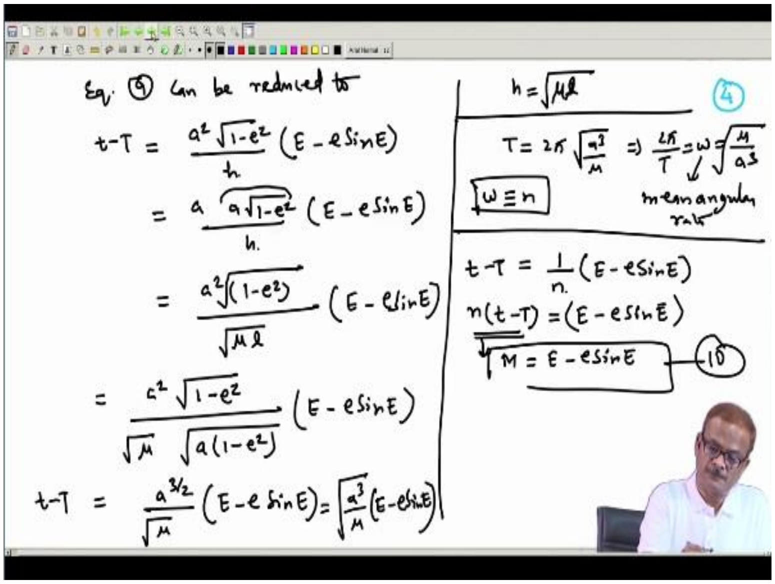

So equation (9) can be reduced to t – T equal to; first we will copy this equation there, a^2 times $1-e^2$ divided by h and then E – e sin E. And this we can write as a times $1-e^2$ divided by h. Okay. So this is nothing but, this quantity from here to here, this quantity is b; ab/h (E – e sinE) or little bit of simplification because h also we need to eliminate, so we can do some simplification here in this place, what we will do that h we will replace in terms of, so I will rub out this step.

Let us continue with this and later on we will come to this part, h^2 equal to μ times l under root. So h we are going to replace, h equal to μ times l under root. So this is μ times l under root. E – e sin E so μ under root, l equal to a times $1 – e^2$ so this is; this is the way it is coming. So E – e sin E, and you can see that $1-e^2$ will cancel out and we get here a to the power cube by 3 divided by μ under root E – e sin E.

So this is t – T. And we use another relationship: we know that time period is $2\pi \sqrt{(a^3/\mu)}$ We have already derived it, okay. So rewriting it $2\pi/T$ equal to ω, this becomes μ/a^3 under root. So in ω this is nothing but here mean angular rate. And it is a customary to write ω as n. So this symbol is used, okay and therefore if we use this, so this gets reduced to t – T = a^3 by; so here in this case we have taken inside the bracket.

So I have simply written as a^3 by μ under root E – e sin E, this equation we are using, so a^3 by μ from this place as you can see this μ by a^3 under root is 1/ω. So this is 1 by ω or 1/n we can write it, 1/n, E – e sin E, okay. And putting it here in this format, and the left and side this quantity we denoted by M.

$$M = E - e \sin E$$

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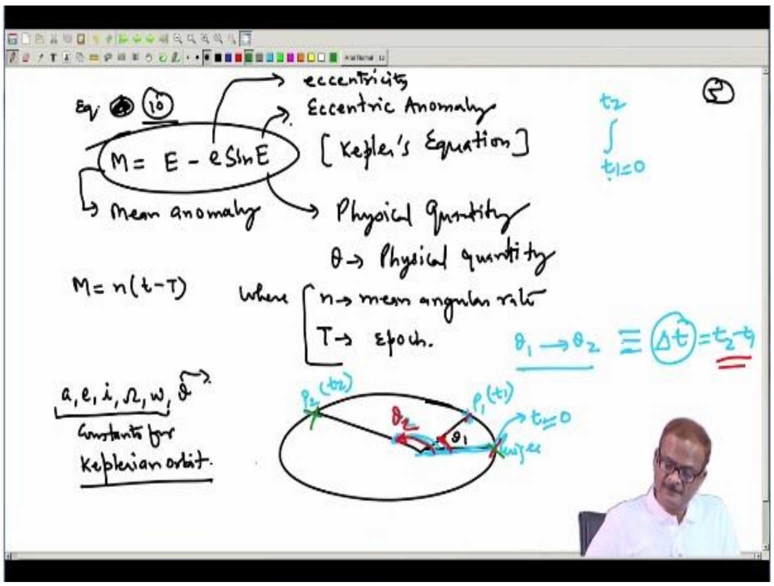

So the equation (9); equation (10) here, equation (10) which we have written as

M = E – e sin E

this we call as the **Kepler's equation**. Left hand side as I have told you earlier this is called Mean Anomaly and while the capital E this we have written as the eccentric anomaly. And this is written as the eccentricity. Okay so now, if you remember I told once during the last lecture or maybe before that, that eccentricity for this eccentric anomaly this is the physical quantity.

Also true anomaly θ this is a physical quantity. Physical quantity means you can visualize it on the figure. If you go back and look here in this place, so θ is visual here and also the eccentric anomaly E is visible here this is E and there is a θ both are visible, so on the figure you can visualize it but M you cannot visualize. The reason is very simple; this is because of the relationship we have got here. It is a purely mathematical quantity, okay.

But it has got a lot importance in solving varies problems, especially in trajectory transfer in rendezvous problem so on, okay. So we will discuss about this further. So what we have written here M = n times t – T where n is the mean angular rate and capital T which is the beginning time also we called the Epoch. So now we are going into advancing little bit further. Okay. So the next question will be, next question is how to solve this problem.

See the situation is that we are given a, e, i, Ω, ω andθ so this quantity is already, these are constants for Keplerian orbit. These are constant for Keplerian orbit. Only θ is a variable. So question can arise that if this is θ_1 and the satellite has to go another position which is say from this place to this place this is θ_2, okay so this is position P_1 and this is position P_2 in the orbit, this is an elliptical orbit obviously and this is the perigee position or the periabscess, okay.

So how much time it will take to go from this place to this place? This maybe one question that means from θ_1 to θ_2 how much time is involved. What is the corresponding; this is equivalent to certain value of delta t which equal to $t_2 - t_1$, how much it is going to take? So we will reckon time from this time, we will reckon time, we will put here t=0 so at this place this will correspond to t_1 and this will correspond to t_2, okay. Unlike the previous one where we writing here the limits as t_1 and t_2, so this t_1 here it can be this point also; if you try to work it so this can be this point and this point also.

But it is a convenient that we put it to t_1=0 so this goes to the perigee position. So perigee position to this one and then from this to this, that means perigee to θ_2 and perigee to θ_1; we will use another color, this angle and this angle. This is your θ_1 and this is θ_2, so going from θ_1 to θ_2 that is equivalent to $t_2 - t_1$. So how we can calculate? You find out the time from this place to this place and then again find time from this place to this place. So this is done. So if the angles are given, so you find out the corresponding time, how much time it going to take to come from this point to this point, from here to this point.

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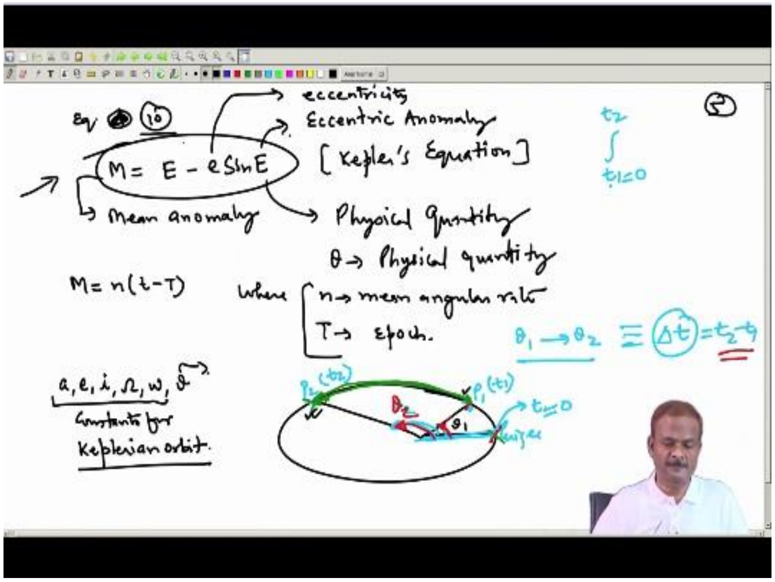

So to solve this, the technique we use, so from t first you get m, because already you know that

$$n\,(t-T)\;=\;M$$

so here t will be set to 0, so nt this becomes equal to M, so therefore n times t_1 becomes equal to M_1, okay. So from t we get m and from M then we can solve for E. and once we have solve for E then we can get to the position r and θ. Because already r and θ both of them are described in terms of E, so solve for this. So we have M_1 here, you get E_1 so the corresponding r_1 and θ_1 is available.

Similarly, corresponding to t_2 you have M_2 here, you can get this M_2 value and then E_2 and from here we will have r_2 and θ_2. So this way the orbit is propagated. So what we have done here in essence, we are finding out if the; if you are looking for the angular position; so what was our desire that the initial θ is known and the final θ. I am given, so how much time it is required? Other way it maybe the time is given so how much θ it will cover? So these are the two problems.

So we have two problems here. So two problems involved, given Δθ find Δt, or given Δt find Δθ, so these are the way; two things we can do. So what we will do that we will stop here and continue in the next lecture. So we will start from the same place where we have derived the Kepler's equation and thereafter we will proceed with this particular one. So how to solve this some; that we need to get certain value for this, so how to guess that and then solve it, so we will discuss about that problem. Obviously on computer you can always do the optimization and work out, but by hand; using hands also, using simple calculator you can work it. Thank you very much.