Space Flight Mechanics
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Lecture – 22
Kepler's Equation

Welcome to the 22nd lecture we have been discussing about the Kepler's equation and already we have done it using the theoretical method. So, I told you that we are discussing it using a graphical method. So, I am going to discuss the graphical method today.

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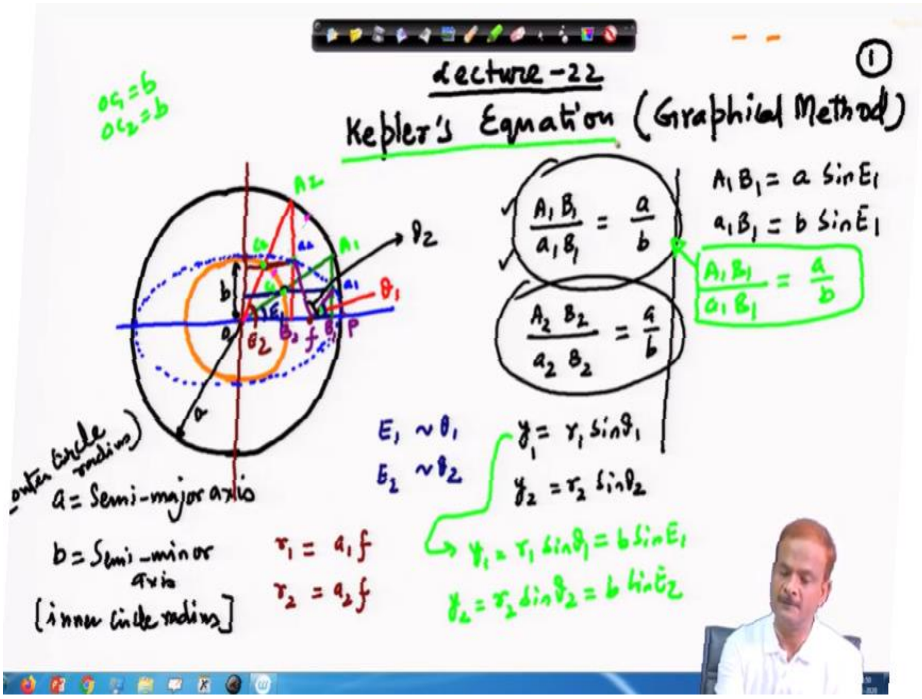

So, we have two circles, two concentric circles the centre here. So, already we have observed that what is our eccentric anomaly. This is the centre of the circle let us write this as o and of course we are aware that if I draw two lines like this maybe another line little at distance. So, from this line we drop a perpendicular this is the line so I drop a perpendicular on the horizontal axis.

Similarly, we drop a perpendicular from this point on the horizontal axis. Now wherever this cutting here suppose in this case it is a so I extend it like this okay. Similarly this circle is being cut here in this point I have shown it by dot there I will try it by some other line so, I extend it this way and take to this point so you can see that I have got points here in this place another point here in this place.

So if I draw of various like that and let us name this say this is A_1 this is A_2 okay so corresponding Point 1, we are getting here in this place this is the one of the point and the next point we are getting here in this place which I am showing by dots. So, if this way I construct this point and let us name this point as small a_2 and this is small a_1. So, I can construct this point and then connect it this way.

So, if I connect it this way so this ellipse is constructed so the same way it is on the left-hand side this will extend toward the left and from this side also you can construct ellipse like this. So somewhere let us assume that focus is located here in this point. So, from another colour I have to use so I joined this point okay another point I joined here in this place. So a_1 and a_2 and this is your f and this point I can write this as P , and this point, I will write this as B_2 and this point I will write as B_1.

So, corresponding angles this is θ_1 now this is your θ_1 angle and similarly we have angle from this place to this place to the horizontal line and this is θ_2. So now you compare with the what earlier we have discussed so we have

$$\frac{A_1B_1}{a_1B_1} = \frac{a}{b}$$

Similarly,

$$\frac{A_2B_2}{a_2B_2} = \frac{a}{b}$$

so this ratio is constant.

Why this is constant? Let us assume that this is the thing I wanted to discuss with you this is angle E is the eccentric anomaly by definition this is eccentric anomaly and corresponding true anomaly is appearing here in display this is the angle here. So, this I will write us E_1 so corresponding to E_1 we have the true anomaly θ_1 and corresponding to E_2. Similarly, the true anomaly will be θ_2 so what is the where is the E_2 angle, so E_2 angle lies from this place to this place so this is your E_2.

So, from the graph then you can see that $y \sin \theta_1$ because we are measuring distances from the let us write r_1 = a_1 f and r_2 = a_2 f. So, therefore y becomes equal to or y_1 becomes

$$y_1 = r_1 \sin \theta_1$$

and y_2 is $r_2 \sin \theta_2$. So, other question is getting this part okay. So, we have A_1 B_1 this is nothing but this is what we are looking for so A_1 B_1 is nothing but a sin E_1 where a is the radius of the circle this is *a* so this becomes the semi-major axis of the ellipse and *b* is the minor axis which is the distance from this point to this point inner circle radius.

So, semi-minor axis this is the inner circle radius and this one is the outer circle radius. So A_1 and B_1 a sin E_1 and similarly small a_1 B_1 this gets reduced to as you can see from this place b sin E_1. So, here we are taking on this graph this point we have taken as o this is a point o. So, the point here and this point we will denote by some light colour this is the point okay where I am putting the cursor okay.

So, this point let us name this as C_1 this is on the circle and similarly this point we will name as C_2. So, to C_1 this = b and OC_2 this is also = b because it is lying on the inner circle. Now if you divide it so A_1 B_1 divided by small a_1 B_1 this gets reduced to a/b. So, you can see that whatever the ratio you were using earlier so this has got recovered here. So here once we are writing in this part $y_1 = r_1 \sin \theta_1$ so this quantity is also equal to b × sin E_1.

Similarly, y_2 will also be = $r_2 \sin \theta_2$ = b × sin E_2. So, this technique we are going to utilize to construct to derive the Kepler's equation using the graphical method but the same way we can derive this equation also okay exactly in the same way. Now we go on the next page.

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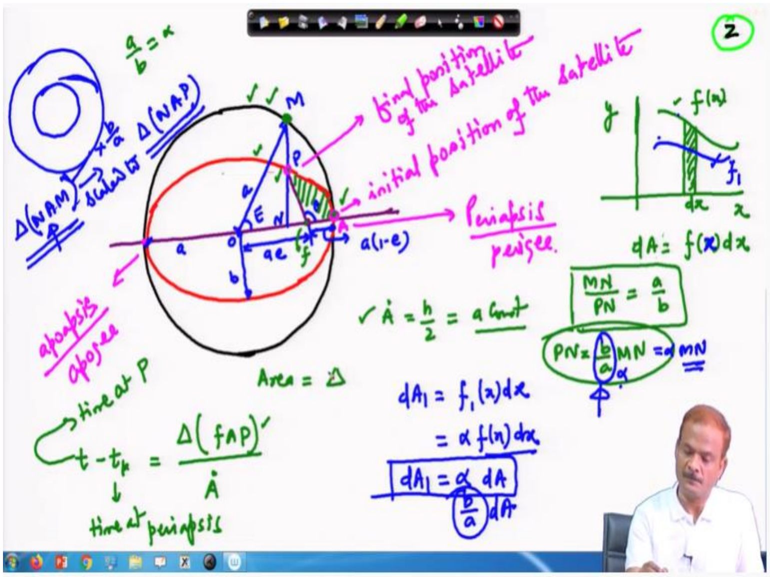

So, what we are going to do we are going to find out the Kepler's equation using the graphical method. So, this represents a and b. I am not showing here b you can show here in this place and this is b so this is your half of the semi-major axis from this point to this point this is a this distance is as usual a (1- e) in the case of ellipse and distance from this point to this point and this is ae.

This angle becomes E from the previous figure okay and this we have written as M and earlier we have used the symbol P for this place and this point we have written as N. So, if the satellite is suppose located here in this place which is showing by the pink so this is the initial position of the satellite secondly and thereafter satellite reaches this position. So, this is the final position of the satellite.

So going from and we will name this as just point as a better we should have named it as P because this is the periapsis this point is your periapsis and this point is apoapsis or we have written also as apogee and this is as perigee okay so the question is how much time it takes to go from this position to this position. So that will depend on the area swept as we have observed for the central force motion including the inverse the square gravitational field that

$$\dot{A} = \frac{h}{2}$$

this is a constant.

So, the top slip of the area this is constant so if we are able to find and this point let us name this point as f this is your focus f this point is focus and area we are going to denote by Δ. So what we are interested in finding out Δ fAP and if we know this okay so this is the area and if I divide it $\dot{A}$ so I will get that time to move from perigee to this is the time at perigee time at periapsis and then this becomes time at p.

So, because the rate of sweep of area is constant so this total area which is fAP divided by the rate of sweep of area that gives me that time this a very simple and this is the technique I am going to use it. But for bring this we need another observation say I have this curve okay so area under this curve this is f (x) we are plotting x this is y. So, area under this curve becomes you can write this as dA = f (x) dx this is a small element dx here. Now suppose I scale this curve by certain amount.

So as you know this circle has been scaled to this ellipse and what ratio in the ratio a/b okay and if I write

$$\frac{a}{b} = \alpha$$

so that means this is a factor by which these are being scaled that means if this is the radius of this is a at any point say this is the point here shown by this green point and this is the pink point P. So, m and p and so what we have written that

$$\frac{MN}{PN} = \frac{a}{b}$$

So this is true for all the points on the ellipse whichever point which choose on the ellipse and a construct like what is shown in this figure so we are going to get this ratio so that means the PN = b/ a × MN. So, this circle is getting mapped onto the ellipse basically all the points on the circle we are reducing it to the ellipse the same way I can do here that if i take it to the particular ratio suppose this is α so this is α × MN.

So if I use here scaling α so this may get reduced like something like this suppose so why is this curve or the distance is changing the distance is changing because your MN is changing so if the MN changes so your PN also it is a changing okay it is not parallel to that circle okay we are not getting after reduction a circle like this rather we are getting an ellipse like this. This is very obvious you want to apply a little bit of your sense you will get this.

So, here this new curve let us write this as f maybe f_1 dA_1 then we can write us f_1 (x) dx and because this is a skilled curve so f_1 is α × f (x) dx. So, what does this mean that this area dA = dA_1. Similarly, here in this case the area which I will show by Δ MAN or NAM. I have taken it in the position so I am write in that sense only NAM this gets scaled to NAP in area NAP so this is the scaling involved.

So, if somehow or rather we can find this area so we know that if I multiply this by b/a okay because this is the scaling here okay you can see because of this, this α is nothing but b/a and this multiplied by dA so the area NAM this will get scaled to area NAP and this is the technique I am going to use to solve this graphically okay so for this we go to the next page and again I construct the same figure and I start working.

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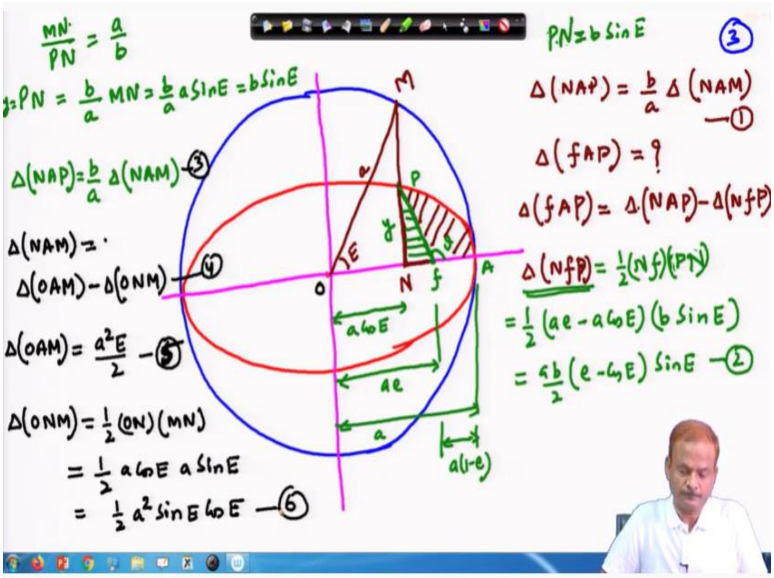

Okay so the same figure I am duplicating there so this is your a and we are interested in finding this particular area. So, the area NAM this can be written as b/a × area the opposite we will write area NAP as we have done here and this is NAP getting mapped to this your NAM is getting mapped to NAP. So area NAP is

$$\Delta\, NAP \;= \frac{b}{a}\,\Delta\,(NAM)$$

So this is the first equation. We are going to use; there after we are interested in what is the area fAP this we have to find out.

Okay so for finding area fAP what we can observe that

$$\Delta\, fAP \;= \Delta\, NAP - \Delta\, NfP$$

Now the things are simple and straightforward to work out so a few of the things. I will write here. So Δ first let us write Δ NfP area of the this is a triangle N to f and to P and this is the green line here this line okay so this is the area first we are working out so

NfP = 1/2 × Nf × NP

So, this is 1/2 and what is the quantity Nf we have to decide distance from here to here this is a cos E and to the focus the distance from here to here this is ae and this distance from here to here and this is a and distance from this point to this point this is a (1 – e). So, this relation we are going to use here. So, therefore Nf is ae – a cos e and NP as we have observed on the last page this is nothing but b sin see because this is the quantity NP is quantity your y.

So, the relation we have written on the earlier page. we are going to use b sin E_1 b sin E_2 so wherever we are working with the same sign is to be used. So, this is b sin. So, therefore this becomes

$$\Delta\, NfP \;= \frac{ab}{2}(e - \cos E) \times \sin E$$

this is our equation number (2) here we have used NP = b sin E and also from our ratio because we have MN/PN or NP we have taken in the this let us write this NP as PN so this is PN here we will remove this and write here as PN.

So, MN/PN we know that this quantity is a/b and therefore

$$PN \;= \frac{b}{a}\; PN$$

which is nothing but your y = b/a MN hence b/a × MN is a sin E = b sin E we have used this relationship earlier. So, this is the quantity we have got here Δ NfP. Now area NAP so we have to get area NAP and we know that this will be the scaled area b/a times area NAM. So, therefore this gets reduced to b/a area NAM okay we have to work out area NAM also.

So first let us work that we have to work out area NAP and for that we need area NAM. So,

$$\Delta\, NAM \;=\; \Delta\, OAM - \Delta\, ONM$$

so we need to work out this also. Now OAM this will be equal to a^2 E/2 because this is a sector of a circle. So, this we are naming as (3) or we can name this as (3) and this equation as (4) and this as (5). Now area ONM this is a triangle so this will be

$$\Delta\, ONM = \;1/2\; \times\; ON\; \times\; MN$$

Therefore, this can be written as ON is a cos E and MN is a sin E this is equation (6).

So, we have worked out the basic things needed and thereafter we will be able to derive the Kepler's relation using graphical method. So, we will continue in the next lecture with the same figure. Thank you very much.