Space Flight Mechanics
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Lecture – 23
Kepler's Equation (Contd.,)

Welcome to the 23rd lecture so we have been discussing about the Kepler's equation using graphical method we will continue with that.

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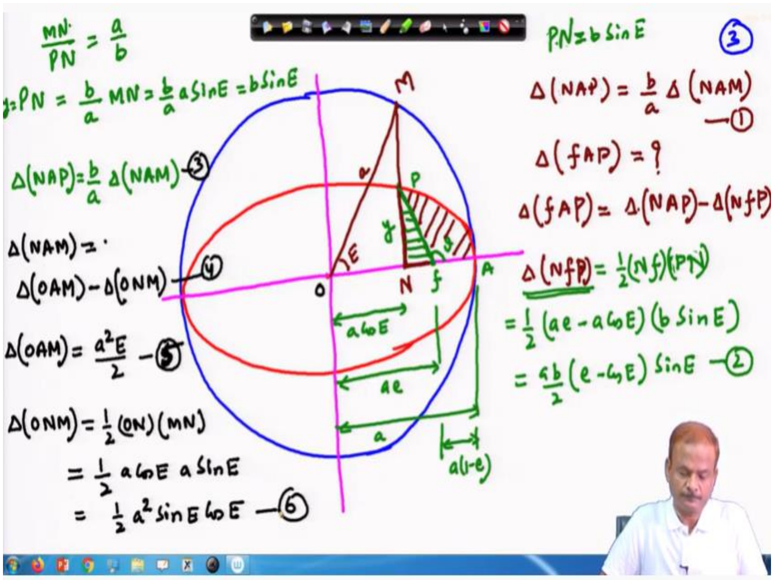

So as we have looked into this figure so we derived we have this 6 equations 1, 2, 3, 4, 5 and 6 using this we have to work out what we are looking for this Δ fAP and once we know that well be able to work out the time taken from point A to point P. So, this we have already done this part this part if you have to calculate Δ NAP and for calculating NAP, we require Δ NAM okay and the NAM is given here by this part and out of this two are given here. So, we have to combine all of these results on the next page first I write Δ NAM.

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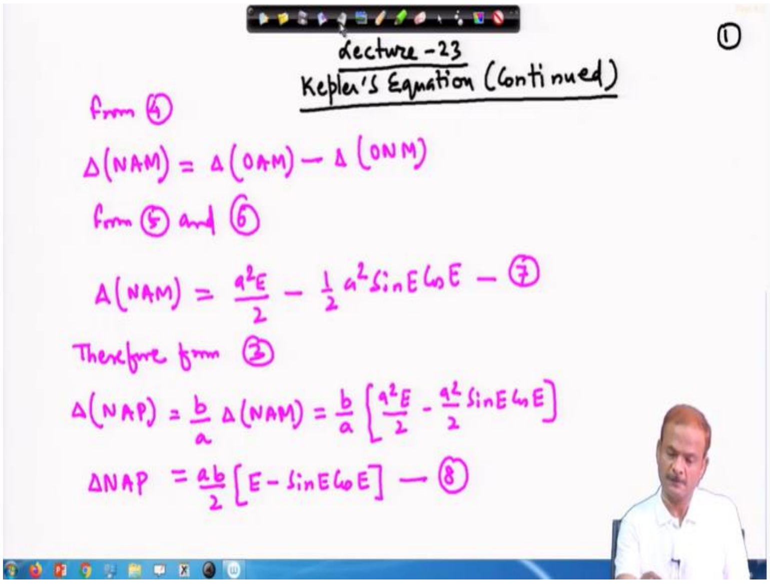

So, the

$$\Delta\,(NAM) \;=\; \Delta\,(OAM) - \Delta\,(ONM)$$

and this is from 4, from 5 and 6 so we can write here Δ NAM, OM is given here a^2 E divided by 2 and this is ONM is 1/2 a^2 sin E cos E this is our equation number 7. Now once we have got this ONM and OM basically we are getting here NAM. So, now NAM is here in this place okay and from there we can get this Δ NAP.

So, from 3 get therefore from 3 we get here

$$\Delta\,(NAP) \;= \frac{b}{a}\,\Delta\,(NAM) = \frac{b}{a}\left(\frac{a^2E}{2} - \frac{a^2}{2}\,sin\,E\,cos\,E\right)$$

and this gets reduced to a can be taken outside

$$\Delta\,(NAP) = \frac{ab}{2}\,(E - sin\,E\,cos\,E)$$

this is our equation number (8). Now from 3 then we go to this we write as here equation this is (3), (2) already we have used this equation we write as 1A.

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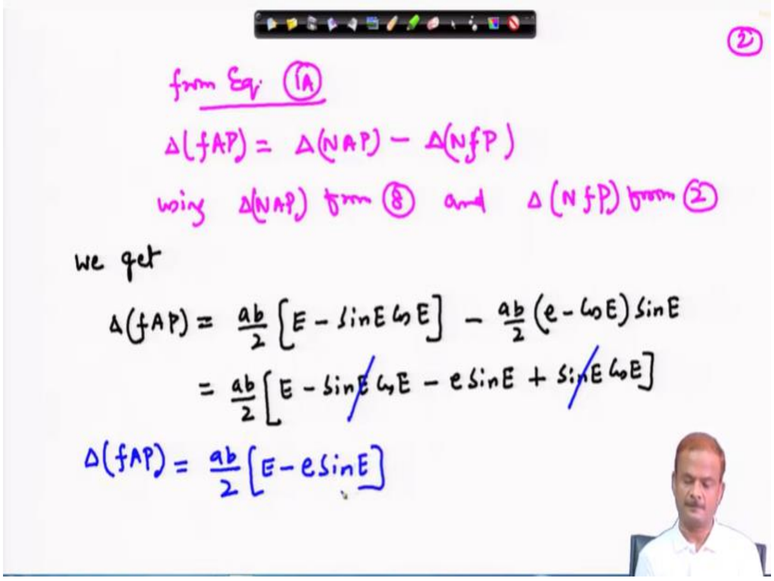

From equation (1A).

$$\Delta\,(fAP)\;=\;\Delta\,(NAP) - \Delta\,(NfP)$$

so using Δ NAP from NAP we have derived here in (8); from (8) and Δ NfP from NfP we have derived here from (2) we get Δ fAP = Δ NAP so Δ NAP we have ab/2 (E – sin E cos E) and from there then we have to subtract the equation 2 which is ab/2 ab divided by 2 (E – cos E) times sin E and then we expand E sin a times cos E – e sin E + sin E times cos E . So, what we can observe that this term and this term they cancel out so this leaves us with

$$\Delta\,(fAP) = \frac{ab}{2}\,(E - e\,sin\,E)$$

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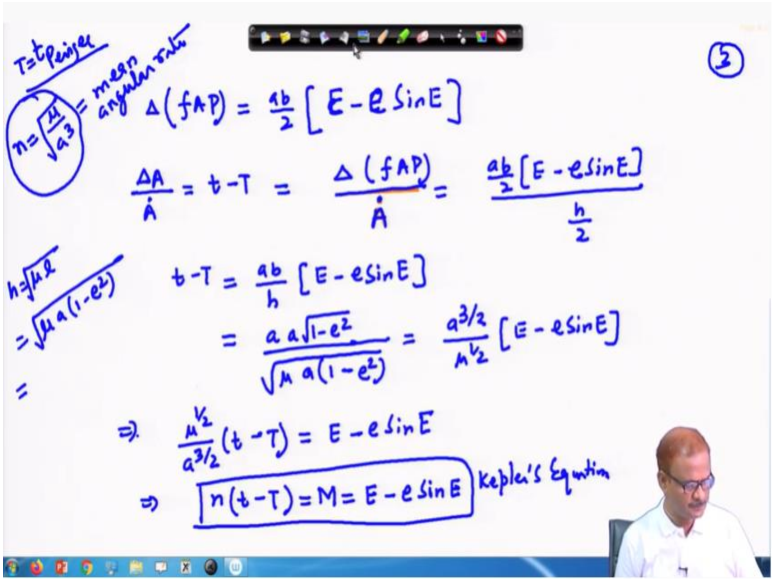

Okay this is what we have got there now we apply the formula ΔA divided by $\dot{A}$ equal to t –

T or where T = t position at the perigee the time at the perigee so here in this case Δ A is nothing but your area fAP. I hope now I can draw the line this is not visible this is from here to here then this place to this place and this so this is the area the hashed area we are looking for fAP so fAP divided by $\dot{A}$.

So, this gets reduced to then ab/2 (E – e sin E) and $\dot{A}$ is nothing but your h/2. Therefore, t – T this gets reduced to ab/h (E – e sin E) and we take it on the left-hand side some of the terms or we can first work out here itself and then take it on the left-hand side. So, what we want to do we want to write in terms of n which is μ/a^3under root mean angular rate we have derived earlier also angular rate.

So we do that here a times b is nothing but a times; a $\sqrt{(1 - e^2)}$ and h is μ times l under root so h is μ times l under root and this gives μ times $a(1 - e^2)$ for an ellipse and therefore if we insert it here $1 - e^2$ under root this becomes $a^{3/2} (1 - e^2)$ this cancels out $\mu^{\frac{1}{2}}$ × E – e sin E and this implies if we bring it on the left hand side 3/2.

And as we have already written here this quantity is nothing but in this part as n (t –T) and this quantity has the mean anomaly which is just a mathematical quantity no physical representation unlike eccentric anomaly and true anomaly so this is your Kepler's equation using graphical method. So, this has been various small thing but doing and assembling from one place to another place it takes time.

But it is a not a very long one once you know that just you have to scale the area and the scale the distance these are the two basic concepts used in this and the rest are simple geometry so this are the simple geometry which we have used here. Now thereafter the question arises how do we solve this?

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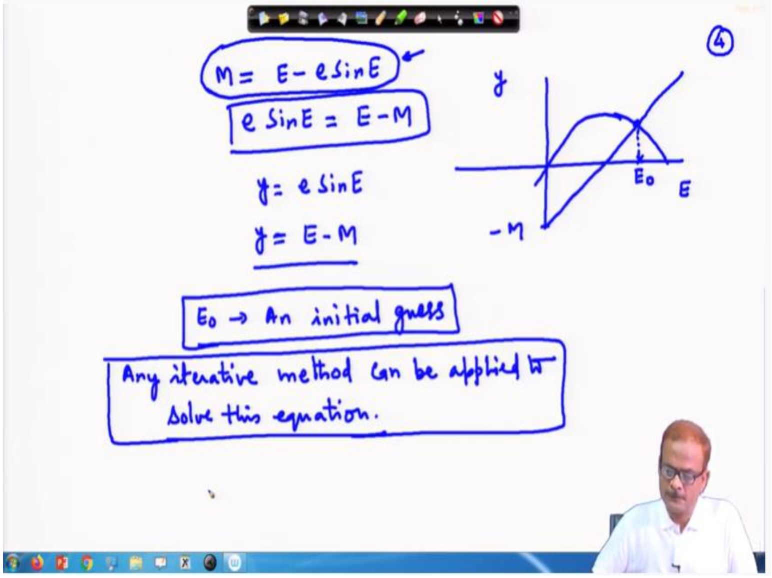

So, what we have got here

$$M = E - e \sin E$$

if we rearrange and write it and then let us write

$$y = e \sin E$$

and also

$$y = E - M$$

and plot it plot both of them here E and y on this axis. So this is e sin E so it will pass through the centre and it will look like a sin curve okay while y = E – M while E= 0

$$y = - M$$

so somewhere it cuts the y axis here okay and thereafter it has a positive slope of 1 so it goes like this.

So wherever they intercept so this you can take as a starting point which we will write as E_0 so this is your E_0 an initial guess and then there are various methods for estimating the value of e because this equation solving it, it is not as straight forward and therefore we have to do it by iterative method. So, any iterative method.

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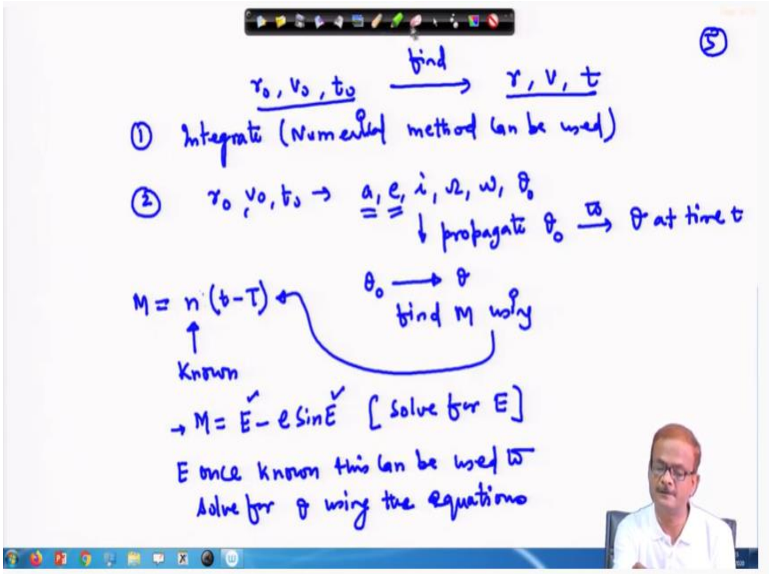

As we have discussed earlier now there are two problems related to what we turn this as Kepler's problem so given r_0 , v_0 and t_0 find r, v, and t. So, one-way I have already stated this integrate as the numerical method can be used. Then another way the other way I have also stated r_0, v_0 , $t_0$this is converted first into a, e, i, Ω ω and then θ the true anomaly which I will write it as t0 θ 0 and then propagate θ 0 to θ so at time t so how to propagate till time t.

So, for that what we need

$$M = n\,(t - T)$$

so if you know the orbit a is known to you and μ is known okay. So therefore this quantity is known to us therefore to propagate to θ 0 to θ what we need to do find out M find M using this expression once you have got M then M can be solved or E - e sin E using M then you can solve for capital E because e is small e is already known.

So from here solve for E once you solve for E now here once known this can be used to solve for θ using the equation or the equation derived earlier which I will again write due course of time equation derived earlier that means θ is a function of e and therefore once you have got this e you can also get θ. So, this way you will solve for θ.

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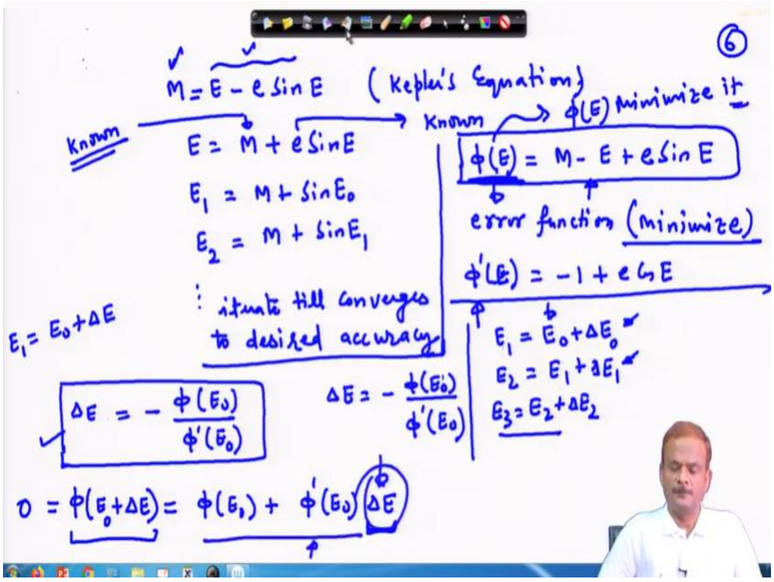

So what we have got here the initial guess as e_0 so this we can take as starting point and solve this equation M = E – e sin E this is our Kepler's equation so one very primitive way of doing this is write E = M = e sin E and then start with E_0 E_1 becomes because M is known this quantity is known this we do not have to change and this quantity is also known this is also not going to change.

So, we will start with sin E_0 we will get E_1 thereafter whatever the E_1 we get we use here in this and so on okay so iterate till converges to desired accuracy. So, this is a primitive way of doing it another way of doing it can be using the Newtons method so that we need to write it ϕ E = M – E + e sin E. So here what is happening that if M the left-hand side and the right turn side both are equal okay so this difference will be 0 okay if they are not equal.

So this difference will not be 0. So therefore, this is acting as a as an error function and we want to minimize it. So, using the Newtons method then we can work it out okay and this is pretty straight forward ϕ' prime E we write this as – 1 + e cos E okay and thereafter the basic equation for the newtons we use and work out this problem. I am just going to write it here Δ E_0 = - ϕ E_0 divided by ϕ' E_0 this is the correction that we get.

So we can give it to E_0 this correction so E_1 = E_0 + Δ E_0or Δ E itself you can write here Δ E so this is the desired correction and how do we get it this ϕ E + Δ E this we can write as E_0 + Δ E ϕ E_0 + ϕ' E_0 times Δ E and if this is minimized if this quantity is minimized means going from ϕ0 to ϕ E_0 + Δ E so if this is exactly the minimum so the left hand side must be 0.

So, from here we can write Δ E = - ϕ E 0 divided by ϕ' E_0. Once we have got this so you just accept this part. So, here we have ϕ this is E_1 = E_0 + Δ E E_0 already we have worked out using this method or using the graphical method and then similarly E_2 will be E_1 + okay if we tag this Δ E_0 so it will be easy for us to do then this will be Δ E_1.

So next step you calculate this ΔE so this goes here in this place okay similarly E_3 will be E_2 + Δ E_2 and this way we will be able to update it. Any other method use this you can apply even it is Markov or whatever is available there are many methods of optimization so what we have to optimize we have to minimize this ϕ minimize it okay. So, we complete this lecture here and we will continue with this lecture later on. Thank you very much.