Space Flight Mechanics
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Lecture – 24
Kepler's Equation (Contd.)

Welcome to lecture number 24 so we have discussed about the Kepler's problem for parabola and ellipse are the parabolic an elliptical orbit and the hyperbolic orbit is remaining but before that we do one problem and we have some other resource that I want to discuss so let us start with the problem first.

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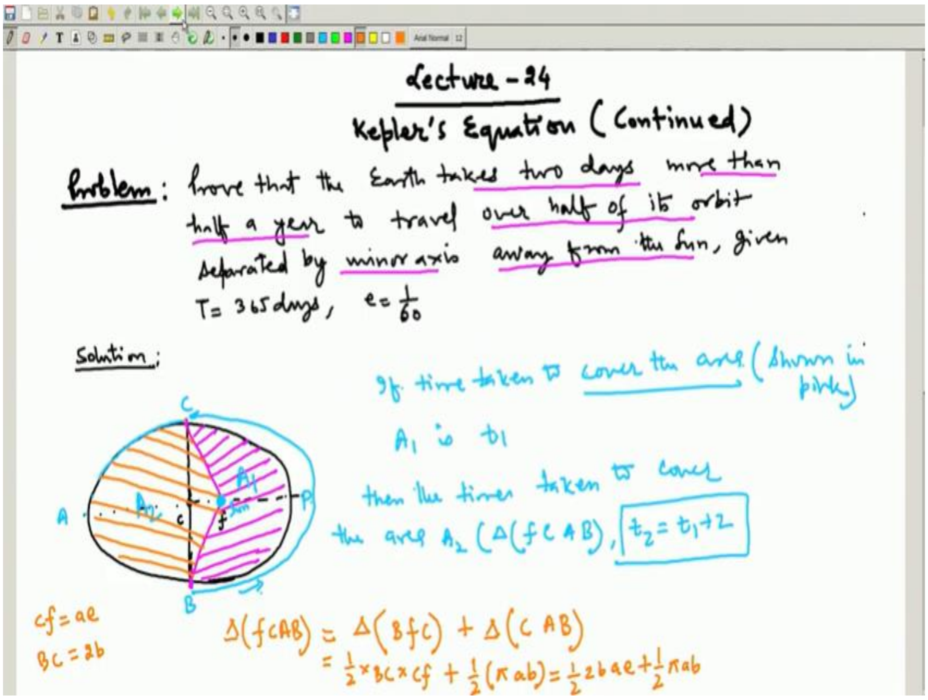

So, here so the problem is given that prove that a earth takes two days more than half a year to travel over half of its orbit separated by minor axis away from the sun so what is the problem that this is the centre C focus is somewhere say here in this point f and this is your minor axis. So, but it is telling that prove that earth takes 2 days more than half a year to travel over half of its orbit separated by minor axis away from the sun.

So, what it is telling in effect that this is half of the orbit this is the separated by my minor axis so away from the sun and the sun is located here. So, you have the sun here in this point this is sun okay and earth going around in the orbit. So away from the sun means we are talking about this area so covering let us say this is A this is P and we can write this as A, B and P we will write here f and C we will name this as C and if time taken to cover the area or shift the area means the suppose the satellite starts from this place and it goes all the way up to here okay?

So, starting from B and going to C so let us say that cover the area this area shown in pink so it takes time, time taken to cover the area and we will note this area is A_1. So cover the area A_1 is t_1 then the time taken to cover the area A_2 this we will write as A_2 area A_2 which is nothing but we will show this as fCAB cover the area A_2 is then the time taken is

$$t_2 = t_1 + 2$$

this is what has to be proved means 2 days more.

So now I am going to hash this area A_2 so you can see that in the orange I am hashing this area and covering this area it takes 2 more days than the pink one that means the sweeping this area by the radius vector it will take 2 more days and this is what we need to prove. So, Δ fCAB this area can be written as area BfC which is triangle + area CAB okay and this area the triangle area BfC this will be 1/2 this will be = 1/2 times this base, base length BC times this distance Cf times Cf plus here Cf is nothing but ae.

As we know for from our discussion about the ellipse and BC is nothing but equal to 2b and area CAB. So this is half of the ellipse area. So, 1/2 times πab, πab is the area of the ellipse. So, all the things are available here so we insert so this becomes 1/2 BC = 2b and Cf = ae + 1/2 πab.

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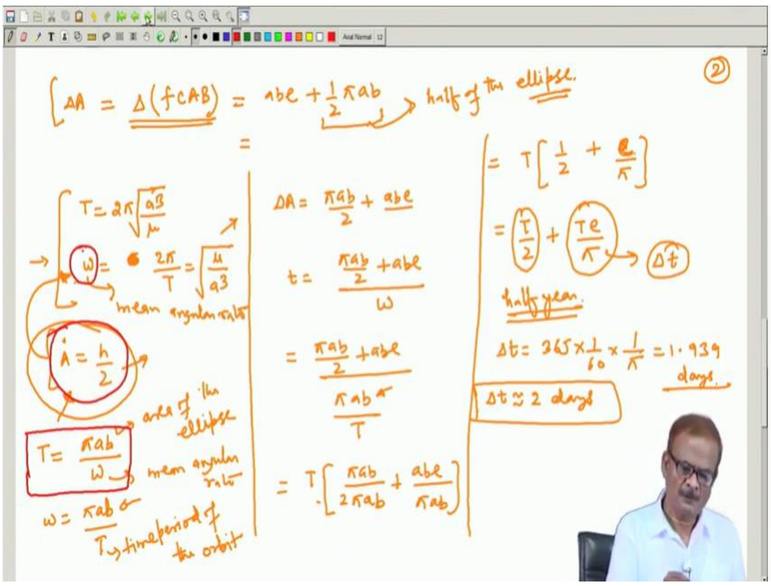

So, we have the area fCAB and this becomes 1/2, 2 cancels out so that is a factor of 2 so we will remove that and simply reuse this see here we are in this place 2, 2 cancels out so abe + 1/2 πab. So now we have to compute this is the area we have done just know okay and let us write this as ΔA, ΔA = this is the quantity so and what is this area? This is half of the ellipse already we have written.

So, this area is known to us now time period equation already we know this is 2π divided by 2π and square root a^3/μ under root and from there we have got the mean angular rate ω to be 2π /T = μ/a^3 under root this is what we have got so once we get this means angular rate we will be able to solve the problem. Now we also know that A dot the rate of sweep of area this is nothing but h/2.

So, this information are available to us and we have to solve this problem. So, we have ΔA = πab/2 + abe; this is the quantity okay so divided by now this ω is the mean angular rate. So, we can also use this information but already because based on this we have derived this ω so directly we can use this ω is a constant it is a mean angular rate and rate of sweep of area is a constant. So, either we directly use this or either we use these results will be the same.

So therefore, this Δt this will be πab divided by 2 + abe divided by ω and then we can insert the value here π/2 ab and ω is μ/a^3 under root so this goes as a^3/μ under root. Also, see one thing also we have to note that this time period is nothing but π ab divided by ω this is the whole area of the ellipse area of the ellipse and this is the mean angular rate.

So, ω also can be written as πab divided by T so where T becomes now the time period this is the time period of the orbit. So, either we can go through this let us go through this route because this will be a little simpler to work with if you can do it fast. So, πab divided by 2 + abe and ω we directly put from here you can see the advantage that ab is already here present in this place here.

So, cancellation of the some of the terms takes place and that makes our life easy working on this problem. So, this becomes π/2 + ab, ab cancels out e/ π T, π, π also cancels out we divide it by let us do one more step we take here t and then this is πab/2πab + abe divided by πab. So, there is one more step I have added so t times πab cancels out so 1/2 + ab cancels out e/ π.

So, this is

$$t = \frac{T}{2} + \frac{Te}{\pi}$$

so this is your half year okay T/2; half of the year T/2; T is the total period of the orbit then what this quantity will be we have to just insert the value so this is extra ΔT so the time taken so here instead of writing ΔT let us read this as T. So, therefore this quantity where it appears this is the extra time ΔT. So, this extra time Δt then we can write as t = 365 v e =1/60 divided by 1/ π.

And this turns to be around 1.939 and therefore

$$\Delta t \ \approx 2\ days$$

means 2 days extra it is going to take over half the year to cover the area which has been orange here in this place which I have shown by A_2 which includes again I will draw this area I will mark it by red boundary this area you can see this area from here to here to this place. So, this is your A_2 and time taken to this cover this this is t_2 and t_2 is nothing but T/2 + 2 extra days approximately the rest of the time it will go over this orbit from this place to this place.

So this was a very small problem but it gives you the concept for working we have taken this part here ω = 2π /T and also this is the corresponding equation but equally we could have used this also and directly we could have worked and result will be the same.

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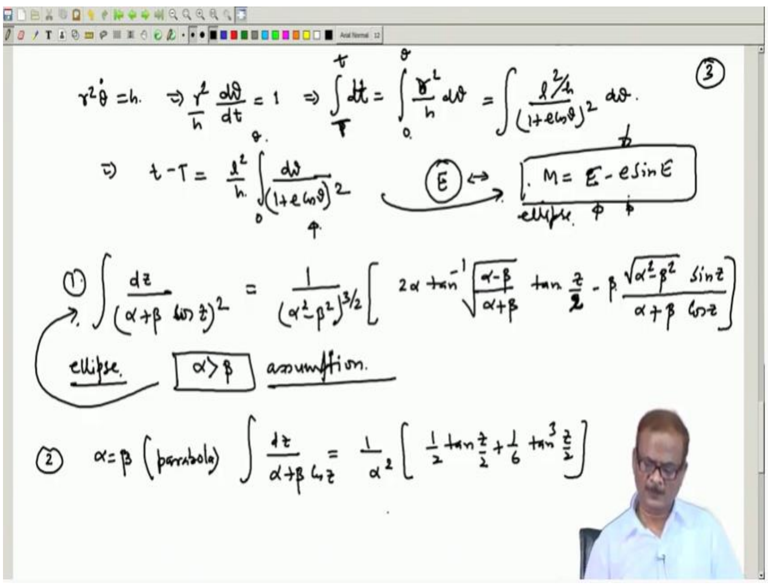

once we have done this now, we will get back to the elliptic problem again and there are some standard integrals available in the books. So those you can utilize to solve the problem for the ellipse hyperbola and parabola. So, what we had the problem that is a given

$$r^2 \dot{\theta} = h$$

and this we wrote as dθ /dt r^2 this can be written like this and this implies dt = r^2 divided by h dθ and this we integrate it from 0 to t and this side from 0 to θ or maybe θ_0 to θ both the integration limit you can put.

So, for doing this then we replaced l r/(l + e cos θ) then so this whole square this square and therefore we have written this as if we start from t here say the capital T we have written perhaps earlier. So, this gets reduced to l^2 divided by h dθ/ (1 + e cos θ) whole square and the issue was integrating this so we did in terms of eccentric anomaly and the benefit was that it got reduced into a simpler format which we call this as the Kepler's equation.

So

$$M = E - e \sin E$$

from there by integrating we got this equation for the case of ellipse. Now integrating this there are some standard equations available in textbook on integration. So, I will write that and you can use that also but this one was more elegant and that it showed finally because our objective was also getting the Kepler's equation in this format whatever we have written here okay?

So the process we followed it was intentional now the other way of doing the same problem will be we use this integrations say this is dz ($\alpha + \beta$ cos z) whole square the integration of this quantity this will be $(\alpha^2 - \beta^2)^{3/2}$ 2 α $\tan^{-1}$ (α + β) under root here do you need to remember this equation? Okay that is the difficulty therefore I have not taken that in the beginning this is z/2. tan (z/2) – β ($\alpha^2 - \beta^2$)under root sin z β cos z.

So, this is for ellipse this can be used for ellipse. So, what is the assumption here, assumption is that α is greater than β provided this is the assumption we will write here under this assumption this is valid. Similarly, if you have α = β so in that case so this case will be applicable to parabola and in this case the simple integration is α + β cos z. that was the case of parabola now we have the case of hyperbola.

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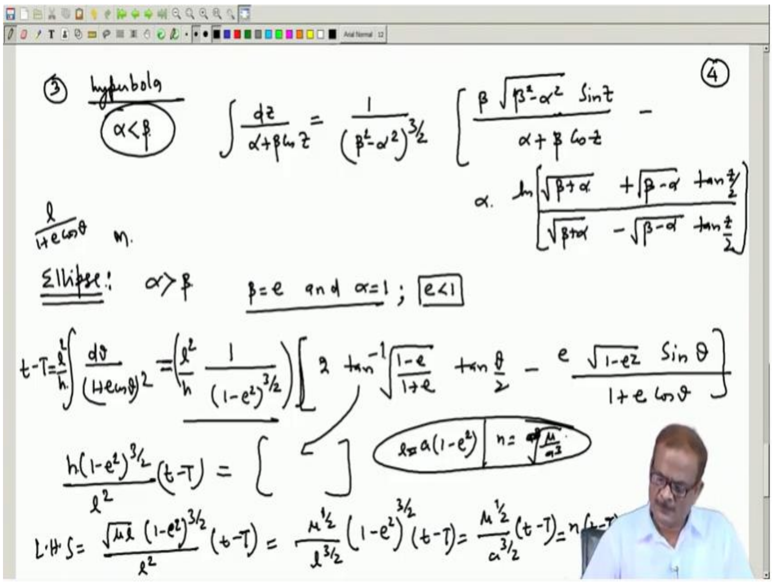

Now we have the case of hyperbola, in the case of hyperbola the following your equation will be applicable and here α is less than β so for this case you can look into a standard textbook I am not going to work at a disintegration particularly here we lack a lot of we have a very short time to finish this course okay. So, these are the 3 cases of integration depending on how α and β they are different from each other okay. So, considering the first equation this one and from here also we can work out.

So here in this case we assume α is greater than β. So, the case of ellipse now we are going to just work out α is greater than β. So, if we take α outside okay and basically what we are interested in we have to express it in this format if we are able to do this our job is done okay. So, we write here $\beta = e$ and $\alpha = 1$ where $e < 1$ okay so therefore this is satisfied less than α.

So, if we do this so our equation gets reduced to dθ /1 + e cos θ whole square in this particular equation β equal to we are writing e and $\alpha = 1$. So, this is $1 - e^2$ so here we get $(1 - e^2)^{3/2}$ and rest of the things we have to write here 2 $\tan^{-1}$ $\alpha = 1$; $\beta = e$ here in this place we have to insert all those values. So, $\sqrt{(1 - e / 1 + e)}$ tan (θ/2), z is replaced by θ in this case so that becomes θ/2 – e; this β is replaced by e.

So, e $(1 - e^2)$ z is replaced by θ so we get here sin θ and 1 + e cos θ and of course this quantity is t – T and we have here also we have the quantity l^2/h here presenting this case l^2 divided by h. So, on this side we have to multiply it by l^2 divided by h. So, we rearrange this part just this part we have to rearrange to work it out. So, ultimately, we have to express it in terms of we bring this quantity on the left-hand side and this will be h times $(1 – e^2)^{3/2}$ divided by l^2 times t – T and on the right-hand side.

We have this whole quantity to be copied and this part is simple h = μ times l under root $(1 – e^2\ )^{3/2}$ divided by l^2 (t – T) this is LHS. $\mu^{\frac{1}{2}}$ divided by $l^{3/2}$ $(1 – e^2\ )^{3/2}$ and this quantity as you can see this is nothing but $\mu^{\frac{1}{2}}$ $a^{3/2}$ which we can write as and obviously one term is missing.

So, that term also we add t – T = μ to the power 1/2 divided by and we know this quantity is nothing but n (t – T) okay here what information we are using we have sued 1 = a $(1 – e^2\ )$and n = μ/a^3 under root as we have used earlier. So, this information we are utilising. So, left hand side then here in this equation gets reduced to M.

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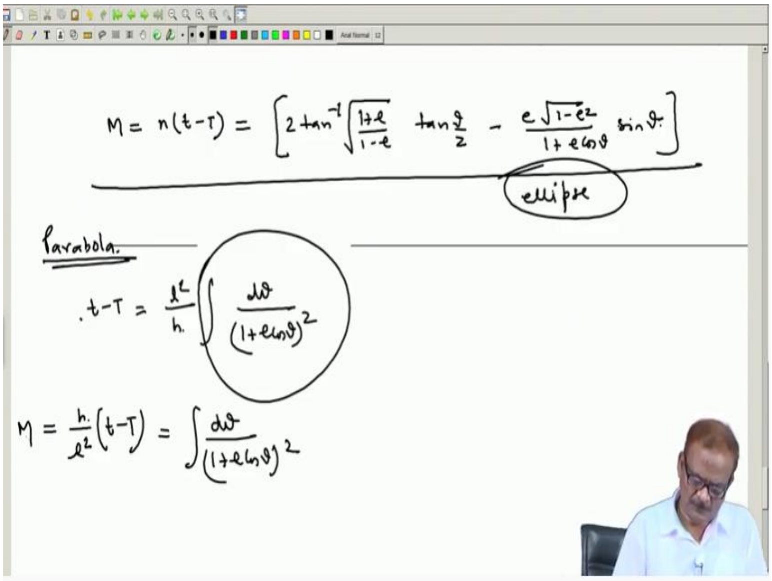

So therefore, you get here this

$$M = n\ (t - T)$$

and on the right-hand side then the whole thing we have to copy from this place which is

$$= 2\tan^{-1}\sqrt{\frac{1+e}{1-e}}\ tan\frac{\theta}{2} - \left(\frac{e\sqrt{1-\ e^2}}{1+e\cos\theta}\right)\ sin\ \theta$$

. So, this expression is for ellipse the same way the second equation if we use here this particular one so we get the equation for the parabola. So, if we take α = β = 1 so in this case β = e = 1 for the case of parabola this case here itself we can write it in short.

So, you can see there and z we replace it with θ so we can see this here the square is missing this is square we have to put it here square is missing for that case of here also the square is missing we have to put the square okay so if we utilize this so you can see that this gets reduced to

$$t - T = \frac{l^2}{h}\int \frac{1}{(1 + \cos\theta)^2} d\theta$$
$$= \frac{1}{2} tan\frac{\theta}{2} + \frac{1}{6}\tan^3\frac{\theta}{2}$$

So we know once this quantity can be written like this so it becomes easy therefore we can take the case of parabola.

So, t - T = l^2/h as we have done here l^2 divided by h and the rest of the things they appear as 1+ e cos θ whole square and this quantity is known to us okay. So, you can utilize it and if this part we take it on the left-hand side so that gets reduced to h/l^2 this gets reduced to t – T equal to dθ /(1 + e cos θ) whole square and we know this quantity is nothing but.

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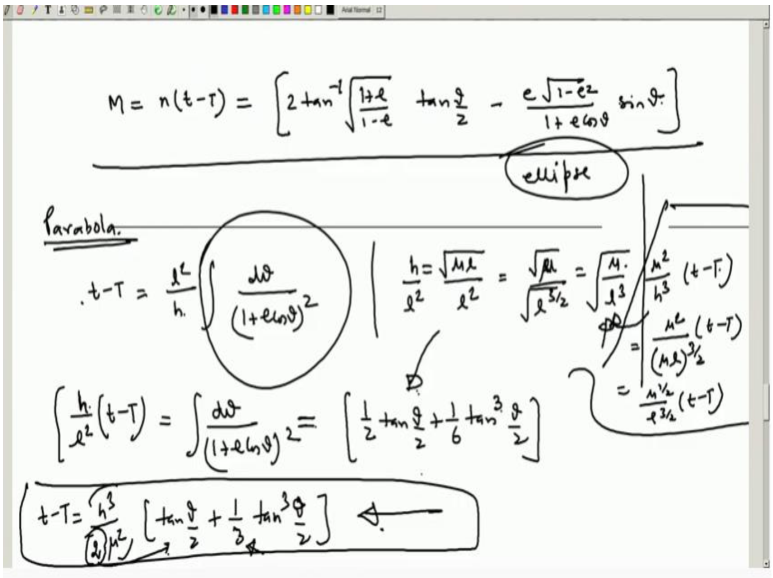

One more part here this part we do not write as M because 1 – e^2 was also there. So, we have to take care of that we will simply write it this way and on the other hand side and then we put the whole thing. So, right hand side here in this case will be 1/2 × tan (θ/2) + 1/6 × $\tan^3$ (θ/2). So, this way we will be able to work out the whole problem.

And obviously the way we want to express h = μ times l under root and therefore this divided by l^2 this will be divided by l^2 here so this becomes μ under root divided by l to the power and this case this comes as and perhaps we have obtained the same equation using the normal integration process earlier I will just check once okay earlier we have written the equation this I will repeat there we have written

$$t - T = \frac{h^3}{2\mu^2}\left(tan\frac{\theta}{2} + \frac{1}{3}\tan^3\frac{\theta}{2}\right)$$

So, these two we are now taking it inside so this becomes ½ and this will become 1/6 this is 2 here. So, only part remaining is this thing h^3 and μ^2. So, if you take it on the left-hand side so that becomes μ^2/h^3 times t – T this is other part. So, if we try to reduce it so here in this case, we have written in terms of see we have in this part μ/l^3 as we can express h/l^2 whatever the way we want to express.

So, say we eliminate here in this case let us say we eliminate h so it should get reduced to this format okay so this μ^2 and h = μ times l under root. So this becomes 3/2 t - T and therefore this becomes $\mu^{\frac{1}{2}}$ and $l^{\frac{3}{2}}$ times t - T so we get the same term. So, this is by usual integration process we got and this we have got by using the standard equation given in the text.

So, there is no difference between this either this or that both ways are the same but in one case you have to remember the equation another case you can derive if you know a little bit of the integration process okay. So, this way the complete this lecture here and I turn back to the same issue that is we are going to write the equation for the ellipse again because the hyperbola I will take at the end not right now.

So, once we finish this part, we are willing to take the case of the hyperbola. So already we have written the equation for the hyperbola using the standard equation from the your integration textbook. So, thank you very much for listening and we will continue in the next lecture.