Space Flight Mechanics
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Lecture – 25
Kepler's Equation (Contd.,)

Welcome to lecture number 25. So, we have been discussing about the Kepler's Equation and we used the standard integration to solve this problem also and in normal process also by doing various steps we have also worked out. Today, again we are going to look into the ellipse problem and it is a different way of looking at the same problem and if you are aware of this different ways then it will help you in solving various problems.

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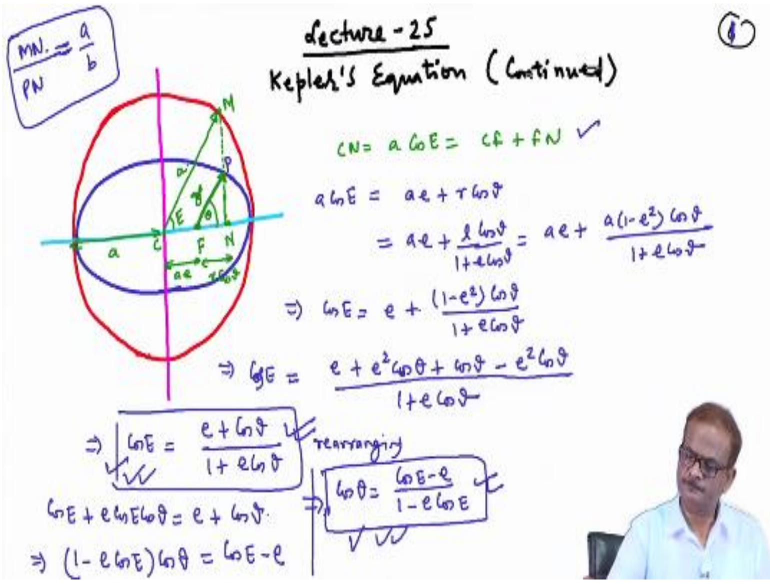

So, let us look into this ellipse. So we have an ellipse here and then there is circumscribing circle which we have named as auxiliary circle. We take the focus here in this place, this is your r and this angle is θ and this is r to this place. From here we draw up a perpendicular and extend it upward wherever it cuts. So this point we have named as M, this point we have named as N, this is your focus.

And then we connected the center of the ellipse and this angle we have written as E which is eccentric anomaly and this point we write as C so CN this becomes and this is obviously a the radius of the circle which is nothing, but this quantity this is also your a semi-major axis so

$$CN = a \cos E$$

and this quantity will be equal to from here to here this distance is ae and this distance is r cos θ.

So,

$$CN = a \cos E = CF + FN$$

and then we can write CF as ae + r cos θ this equal to a cos E and this we replace this by l /(1 + e cos θ)and this is l cos θ one more step we need to do here. These relations we are aware of so we utilize them and this implies cos E will be equal to e + a cancels out from both the side $1 - e^2 \cos\theta$ divided by $1 + e\cos\theta$ and therefore cos E this can be reduced to $e + e^2\cos\theta + \cos\theta - e^2\cos\theta$ divided by $1 + e\cos\theta$.

And this implies

$$\cos E = \frac{e + \cos\theta}{1 + e\cos\theta}$$

So, we have derived this relationship earlier also, but we have done it in a different way and if we rearrange this rearranging this gives us cos E + e cos E times cos θ e + cos θ. Cos θ if we bring it on one side so 1 – e cos E times cos θ cos E – e and this implies

$$cos\,\theta = \frac{cos\,E - e}{1 - e\,cos\,E}$$

So here this is your cos E and this is cos θ which we have worked out earlier also, but there we used this information that

$$\frac{MN}{PN} = \frac{a}{b}$$

it is in this ratio and based on that then we worked out all the relations, but here also you can proceed in this way it is a little longer, but you get the same result. So, we have got here cos E and cos θ and therefore sin θ and sin E can also be computed within these two relations.

(Refer Slide Time: 07:12)

2

$$\sin E = \sqrt{1-\cos^2 E}$$

$$\sin E = \sqrt{1-\left(\frac{e+\cos\theta}{1+e\cos\theta}\right)^2} = \frac{\sin\theta\sqrt{1-e^2}}{1+e\cos\theta}$$

$$\boxed{\sin E = \frac{\sin\theta\sqrt{1-e^2}}{1+e\cos\theta}}$$

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\left(\frac{\cos E-e}{1-e\cos E}\right)^2}$$

$$\boxed{\sin\theta = \frac{\sqrt{1-e^2}\,\sin E}{e\cos E-1}}$$

Sin E can be written as 1 – $\cos^2$ E and this is the reason I have not taken it earlier in the conic section because discussing at a time when we needed it is a much better. If you simplify it you are going to get the equation sin θ this step I am leaving to you and therefore

$$\sin E \; = \frac{\sin\theta\ \sqrt{1-\, e^2}}{1+e\cos\theta}$$

In case of; this is one equation, this is another equation here we have the sin E and the same way cos E already we have used.

You see here also we write it properly this is $\cos^2$ E okay cos θ is also given so in the same way sin θ can be obtained. So sin θ this = 1 – $\cos^2$ θ under root and insert the value for the cos θ cos θ is cos E – e cos capital E – e and then 1 – e cos E this whole square under root and if you simplify it again I am writing the final result here to save sometime.

$$\sin\theta \; = \frac{\sin E\ \sqrt{1-\, e^2}}{e\cos E-1}$$

So, this way you can proceed to solve the problem of the ellipse. Once you know all these things so rest of the procedures we have carried out it remains the same the rest of the procedures remain the same. So we do not need to elaborate that, but what I was trying to show here that how sin E and cos E sin θ cos θ they can be retained in terms of each other.

Therefore, based on this you can write the relationship for r also which I am not going to do because it will be a mere repetition what we have done earlier. So, we skip that step here in this place.

(Refer Slide Time: 10:55)

Hyperbolic Orbit (Kepler's Equation) ③

$$\int_{\theta_0}^{\theta} \frac{d\theta}{(1+e\cos\theta)^2} = \frac{1}{(e^2-1)^{3/2}}\left[\frac{e\sqrt{e^2-1}\,\sin\theta}{1+e\cos\theta} - \ln\left(\frac{\sqrt{e+1}+\sqrt{e-1}\tan\theta/2}{\sqrt{e+1}-\sqrt{e-1}\tan\theta/2}\right)\right]_{\theta_0}^{\theta}$$

$$(t-T)\frac{h}{l^2} = \int_0^{\theta} \frac{d\theta}{(1+e\cos\theta)^2} = \frac{1}{(e^2-1)^{3/2}}\Big[\qquad\Big]$$

$$(t-T)\frac{h}{l^2}(e^2-1)^{3/2} = \left[\frac{e\sqrt{e^2-1}\,\sin\theta}{1+e\cos\theta} - \ln\left(\frac{\sqrt{e+1}+\sqrt{e-1}\tan\theta/2}{\sqrt{e+1}-\sqrt{e-1}\tan\theta/2}\right)\right]$$

$$(t-T)\frac{\sqrt{\mu l}}{l^2}(e^2-1)^{3/2} = (t-T)\sqrt{\frac{\mu}{a^3}} = \left[\frac{e\sqrt{e^2-1}\,\sin\theta}{1+e\cos\theta} - \ln\left(\frac{\sqrt{e+1}+\sqrt{e-1}\tan\theta/2}{\sqrt{e+1}-\sqrt{e-1}\tan\theta/2}\right)\right] \quad a>0$$

$a>0$

Next, we have the case of hyperbola, hyperbolic orbit. For this we can to derive Kepler's Equation so what form this will be so obviously we have written on standard integration. So using that we can work it out which we have written here in this place this is your standard these integration for hyperbola. So, if we utilize it we can work out the problem. So we do this.

And thereafter we will do it by some other method which will be quite interesting to look into. So we will finish this part and then go into the next step. So for the case of hyperbola we have dθ this quantity will be now we are in this case what we are doing. If we go back here in this equation, we are using this equation. So β we are writing as e and alpha we will write as 1.

So, therefore here in this case you can see that alpha is less than β so that means $1 < e$ so this information we are going to utilize. So, looking into this part so only particular part we are writing currently so this gets reduced to e times replacing everywhere β by e and alpha by 1 and these equations will be very useful later on while doing the trajectory transfer problem.

So, now we have t – T this quantity times h^2 h divided by l^2 this equal to d θ / (1 + e cos θ) whole square and this is between 0 to θ we have written either we can write from θ_0 to θ it is not a problem. So, in that case only the boundary will change we will write here θ 0 upside we will write θ this is the only change then we will do here.

So, here we can write this as θ_0 and θ. So this term is available to us so $1 /(e^2 - 1)^{3/2}$ and rest of the terms which will get copied from this place to this place. The left hand side then gets reduced to and the right hand side again we copy the whole thing here. If we reduce the left hand side so we are interested in eliminating h here so we write here μ/l under root l^2 $(e^2 – 1)^{3/2}$ and this quantity will be (t – T) μ times a^3 under root and this equal to on the right hand side this whole thing copied here.

I will copy here let us say e times $\sqrt{(e^2 – 1)}$ sin θ. So μ/a^3 here a remember a here in this case which is appearing a > 0 this is greater than 0. In this place a is appearing so this quantity a > 0.

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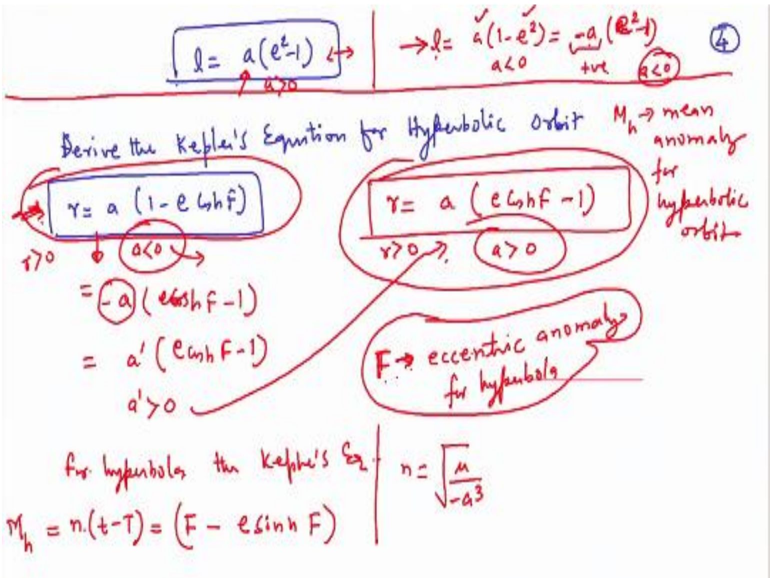

Here, we have used the information l = a times or l times

$$l = a\,(e^2 - 1)$$

. So this way this problem is also solved and here we write M this is the mean anomaly and on the right hand side this whole thing is to be copied. So, better we write here in this place itself M equal to rather than copying one more step and wasting time here so we skip that step, so this is your solution. Your solution is here M = M this equal to finally this quantity here.

Now I will take you to some different way of doing the problem and that too in the Kepler's Equation form we will write and see the most important thing that I am going to write here that either you write this way or write this way. Both are valid for hyperbola this is your semi-latus rectum and this is semi-major axis and this is eccentricity both are valid, but here in this case a < 0 here in this case a > 0.

So say this if I write it as – a (e^2 – 1). So you can see that this quantity l is a positive quantity so here – a this should be a positive quantity that means a < 0. Here in this part a appears as less than 0, but here a > 0. So, these are the two ways of writing the same thing. So now what our objective is to derive the Kepler's equation for hyperbola hyperbolic orbit where it is given that

$$r = a\,(1 - e\cosh F)$$

Again here I would like to point out and especially I will write it in red that I can also write this as e cosh F – 1. This is another way of writing r is always greater than 0. Here in this case a > 0 here in this case a < 0 so this is the difference of representation and both are valid. As you can see here in this place if I rewrite this equation as e cosh F – 1 minus sign if I put it here then you can see if I write this as '.

So e cos h F – 1 so this is not this a' I can write this as greater than 0 was being here because a < 0 so this get reduced into this format you can see that both are the same then. So, we are going to use this equation and what we have to prove that for the hyperbola the Kepler's Equation this can be written as

$$M_h = n(t - T) = F - e\sinh F$$

where F is here just like we have eccentric anomaly for ellipse.

The same way this is eccentric anomaly for hyperbola and what this quantity is right now do not worry. We will take up this issue later on and here the quantity n is – a^3 – μ/ a^3 under root because T becomes equal to this is μ/- a^3. So, on the left hand side then we write this as the mean anomaly for hyperbolic orbit M_h is mean anomaly for hyperbolic orbit. So we are not considering then this equation we will work with this. You can also work with that there is no problem. There are actually so many issues while discussing all these problems.

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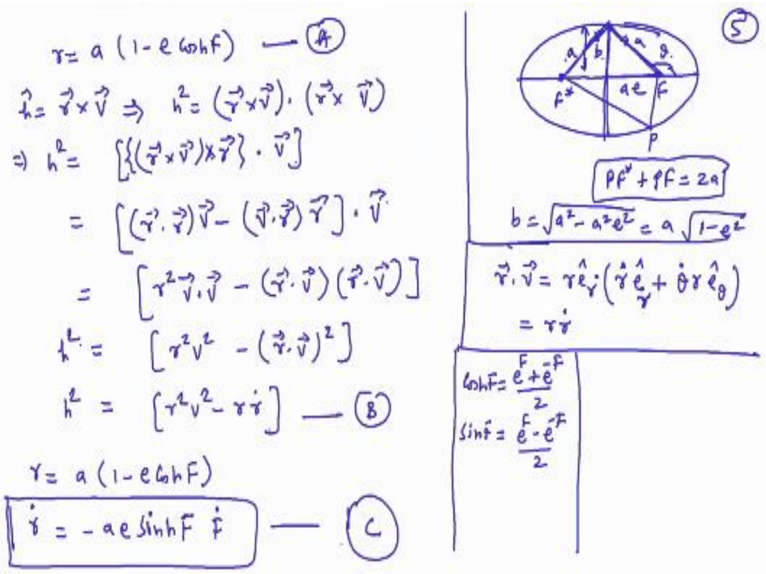

Like in the case of ellipse if you remember that for deriving this b. This is your b this particular part from here to here semi-major axis we took this angle and then we worked in a particular way it was a long way of working obviously, but that could have been done in a very short way also. Purposefully, I followed that path because that was more generalized. So here in this case we always know that in the case of ellipse this distance from here to here plus here to here or to any other place from this place to this place.

This is always 2a that means if I write this as P so

$$PF* + PF = 2a$$

which we have discussed in the case of conic section. So, once we take the minor axis so in that place this becomes a and this becomes a and this distance we know this distance is ae. So, therefore b then in that case becomes $\sqrt{(a^2 - a^2 e^2)}$ so this is $a\sqrt{(1 - e^2)}$.

So this is so straightforward, but we took a generalized way of doing in conic section while discussing the conic section that was good because it was independent of this information that we are using here that this length and this length they are equal to a we did not use that there. So, let us forget this and return back to what we are interested in doing here. So, our objective is to use this equation and prove that the mean anomaly for the hyperbolic orbit which is Mh.

It is given by this particular equation. So we start with

$$r = a(1 - e \cosh F)$$

This is the angular momentum per unit mass. This we have done earlier so I have written here quickly $\vec{r}$dot $\vec{v}$ already we have also worked out again writing here the symbols have their usual meaning this unit vector $\widehat{e_r}$ unit vector in the direction of the radius vector and $\widehat{e_\theta}$ is perpendicular to that $\widehat{e_r}$.

So this gets reduced to and there is a dot product here so this is r times $\dot{r}$. So, therefore this equation gets reduced to now we take

$$r \ = \ a\,(1- \ e\,cos\,h\,F)$$

So $\dot{r}$ will be equal to

$$\dot{r} = - \ ae\,sinh\,F\,\dot{F}$$

Here, this hyperbolic function you may be aware of I presume that you are aware of all this things

$$cosh\,F \ = \frac{e^F \ + \ e^{-F}}{2}$$

And sinh F – ; so if you differentiate cosh F you will get sinh F. So we have here $\dot{r}$ and together with $\dot{F}$ will come here. Now we insert this let us say this is equation (A) and this is equation this is equation (A), this is equation (B) and this is equation (C). So, we insert equation (C) into equation (B).

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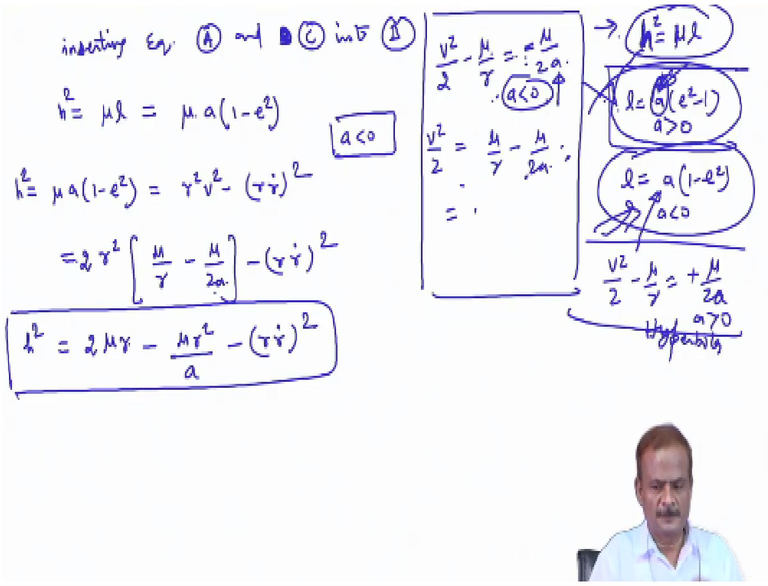

So (A) and (B) they will be inserted in equation (B) h^2 this is nothing, but μ times l. Remember, that from the conic section we got this expression h^2 = μ times l and while working out the expression for the conic section we did not assume anywhere that whether it is hyperbolic, it is parabolic or it is elliptic irrespective of that this is valid. So, therefore we replace h^2 / μ times l and this μ times l, l we write as $1 - e^2$.

So again remember we are not using this expression l = a (e^2 –1). If I use this means I am assuming here a > 0 if I write l = a (1 – e^2). So here we assume a < 0 so I am using this part not this part. So using this then we can write μ a times (1 – e^2) this = r^2 v^2 – r $\dot{r}^2$. Now v^2 also in the central force motion what we have seen that the equation we drive for the energy we derive this equation.

So, if you take a is positive this is for ellipse. If a is negative then it becomes hyperbola if a is infinity it becomes parabola, but it is not necessary. See anywhere if I write like here in this case if I write like this. So here we know that a is negative for the hyperbola the semi-major axis, but if we are writing here in this fashion so we are just putting the magnitude of a not the sign. While here in this case we insert along with the sign okay this is the difference here.

So, we utilize this equation here in this place so from here what we see that $v^2/2$ + μ/2a this will be equal to μ/r. So what we need to replace here this is the quantity v^2 not the other way what we have written here. We need to replace this v^2 so we write it here $v^2/2$ = μ/r – μ/2a so here r^2 times μ/r – μ/2a. So again see I am not putting here like earlier I have told you that for the hyperbola $v^2/2$ – μ / r = + μ/2a.

So there we assume that a > 0 this is for hyperbola, but if we go with this minus sign so in this place we will assume that a < 0 only then this is valid otherwise this will become invalid. So, we continue with this expression so here what we are assuming that a < 0 under this circumstances this condition we are working here and this is $v^2/2$. So this will get multiplied by 2; here in this place – r $\dot{r}^2$.

And of course once we break the bracket so this becomes μ r 2, μ r and here minus 2, 2 cancels out and this is μ r^2 divided by a – r$\dot{r}^2$ this is your h^2.

$$h^2 = 2\mu r - \frac{\mu r^2}{a} - (r\dot{r})^2$$

So we will complete it in the next lecture. Thank you very much. So we have achieved till a point where now it is easy to work out this problem again we will continue in the next lecture. Thank you.