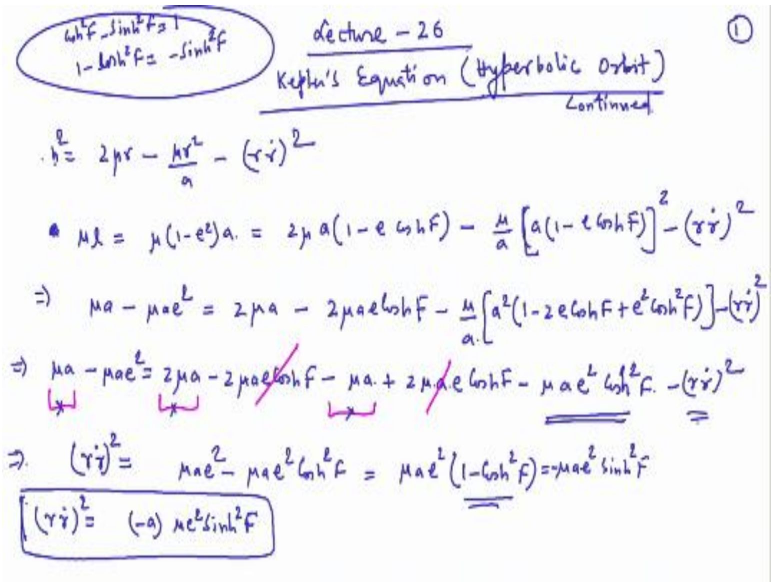
Space Flight Mechanics
Prof. Manoranjan Sinha
Department of Aerospace Engineering
Indian Institute of Technology – Kharagpur

Lecture – 26
Kepler's Equation (Contd.)

(Refer Slide Time: 00:14)



Welcome to 26 Lecture. We were discussing about the Kepler's Equation for hyperbolic orbit and in that context we derived one equation. We worked out till this point

$$h^2 = 2\mu r - \frac{\mu r^2}{a} - (r\dot{r})^2$$

So this part we are going to replace here this part we are going to replace. This we have a × this is $\mu l = \mu \times (1 - e^2)a = 2\mu$ a now the value for the r we insert here. The r is given to 1 -e cosh F.

We write here this way. One more step we need here break the bracket here - $\mu a + 2\mu ae \times \cosh F - \mu ae^2 \cosh^2 F - (r\dot{r})^2$. Now whatever the terms can be cancelled out we will cancel it. So this term and this term this cancels out and you can see that this part, this part and this part together it will cancel out. So this implies.

And thereafter we take this part to the left or maybe we can take it to the right hand side and this part we can bring to the left hand side. So this became $(r\dot{r})^2 = \mu ae^2$ so this will cancel out this together this will cancel out. So μae^2 - μae^2 cosh F. This is r$\dot{r}$ and so if we take the square root and what this quantity is? This quantity we have to write it here.

This is the cosh F- sinh^2 F; this = 1 so 1 - cosh^2 F = - sinh^2 F. So this quantity here is μae^2 sinh^2 F and with a minus sign. Now look on to both the sides this is (– a) we will write it separately μe^2 sinh^2 F.

$$(r\dot{r})^2 = (-a)\,\mu\, e^2 \sinh^2 F$$

Thereafter, let us say straight forward to work out this problem. Now we need to insert here the value for the r and $\dot{r}$ and the problem will get solved out.

(Refer Slide Time: 07:57)

2

$$\left[a(1-e\cosh F)(-a\,e\sinh F\,\dot{F})\right]^2 = (-a)\,\mu e^2 \sinh^2 F$$

$$-a^2 e(1-e\cosh F)\sinh F\,\dot{F} = \pm\sqrt{-a\mu}\; e\sinh F$$

$$(1-e\cosh F)\,\dot{F} = \mp\sqrt{\frac{-\mu}{a^3}}$$

$$\int_0^F (1-e\cosh F)\,dF = \int_T^t \mp\left(\frac{-\mu}{a^3}\right)^{1/2} dt$$

$$(F - e\sinh F) = \mp\sqrt{\frac{\mu}{-a^3}}\,(t-T) = M_h(t-T)$$

$$\boxed{M_h(t-T) = \pm(F - e\sinh F) = \mp(e\sinh F - F)}$$

$$\boxed{M_h(t-T) = (e\sinh F - F)}$$ Kepler's Equation for hyperbolic orbit.

So we will have to move to the next page so r we have written as a × (1 - e cosh F) and $\dot{r}$ we have written as - a e sinh F $\dot{F}$. This is r$\dot{r}$. So this square this quantity = - a × μ e^2 sinh, μe^2 sinh^2 F take the square root. So this get reduce to - a^2 e × (1 - e cosh F) × sinh F × $\dot{F}$ this equal to - a under root μ also let us take each side e sinh F. We have taken the square root.

And also we can put a plus minus sign here because we are taking a square root so plus minus sign we can insert here in this place. So, what we can see here if sinh F this quantity is non zero so we can cancel out from both this sides and therefore we will get here (1 - e cosh F) × $\dot{F}$ minus plus μ/a^3 under root with minus sign and e cancels out this is what we are getting here.

Let us verify all the signs here whatever we have done. We got till this point and from here we started this part we have written μ × a (1- e^2)and on the right hand side we have expanded it –

μ square - $(r\dot{r})^2$and then we further expanded this is - μa this is μ^2 - $\dot{r}$ this is fine then we have taken it on the right hand side we have written it this way r$\dot{r}$ we have taken it on the left hand side.

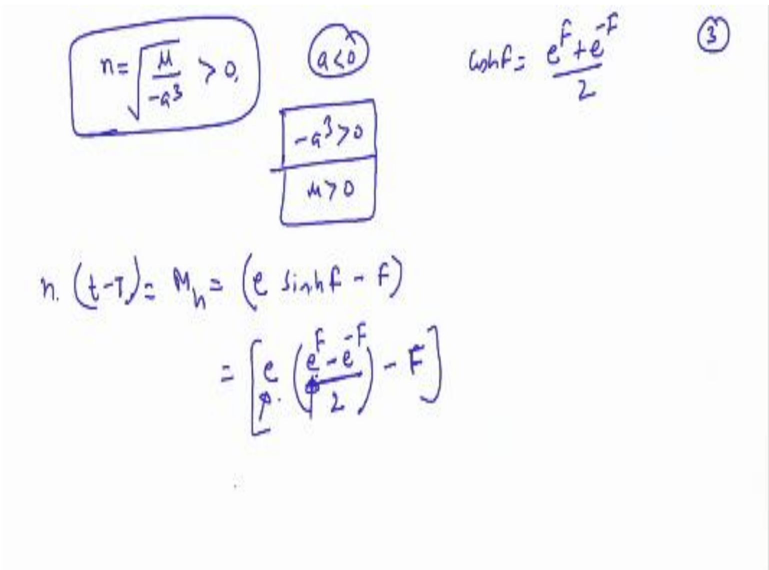
So from there this part again we have put inserted r and $\dot{r}$ here and then taken the square root so for that plus minus sign we have introduced here, left hand side we have this way - a^2 e then we simplified it further by cancelling out e and we got this format. Now from this place we can see that this quantity is - e cosh F – (μ/a^3) to the power 1/2 dt, and if we integrate it between 0 to E or we can integrate from E_0 to E it is all the same.

So instead of writing here e we write capital F so this become F - e now this will become sinh F this quantity we write as N the whole thing we are writing as M (t – T). So this is the mean angular rate in the case of h orbit.

$$M_h(t - T) = e\ sinh\ F - F$$

So we ignore the negative sign here and write it as so this is our final Kepler's Equation for hyperbolic orbit.

(Refer Slide Time: 16:38)



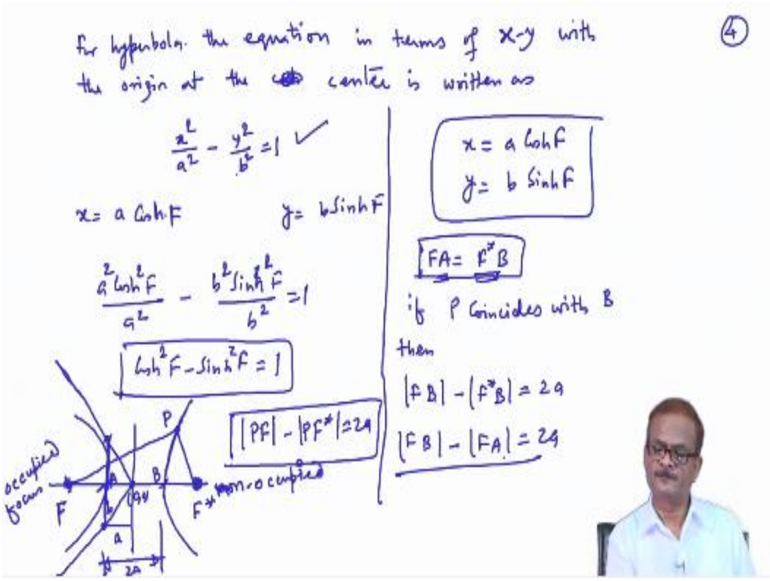
So this was the equation we were looking for and remember in the n equation what we are getting here - a^3 under root so here a is less than 0 and therefore the quantity under the square root sign is always positive and this will be greater than 0. So - a^3 this is greater than 0 and μ is also greater than 0 so there is no inconsistency in the result and obviously in the result where we have written

$$(t\ -\ T\ ) \times\ n\ =\ M_h\ e\ sinh\ F - F$$

So you can see that this is eccentricity okay and here this is your base this is not eccentricity.

So $(e^{F} - e^{-F}) / 2$ this eccentricity and this is different these are not the same it should not be confused with where the cosh F we have written as $(e^{F} + e^{-F}) / 2$

(Refer Slide Time: 18:50)



So we have covered this part and it is a geometrical interpretation also can be looked into, but before that we will just go through mathematics and let us look other way. So for hyperbola the equation in terms of x, y with the origin at the center is written as then we can if we put here x = a or x = a cosh say θ or cosh let us say cosh F itself and y = b sinh F and insert into this.

So we get a

$$\frac{a^2\cosh^2 F}{a^2} - b^2\frac{\sinh^2 F}{b^2} = 1$$
$$\cosh^2 F - \sinh^2 F = 1$$

So we can see that this is satisfied, this is the identity which we are getting which is true. Hence, for the hyperbola the equation once written in this form this is satisfied by x = a cosh F and y = b sinh F and this can we utilized for solving the problem.

Now we look into this hyperbola so your center is located here which is your origin this are the focus one may be occupied focus another may be the vacant focus. So let us say this is the occupied focus F and here we have F*. So this distance from this place to this place this is b and this is a and property of hyperbola is that if we take any point say P on this hyperbola so

$$|PF| - |PF^*| = 2a$$

this quantity is fixed.

And also we have say if we write this point as A and this point we write as B so we will have

FA = F*B

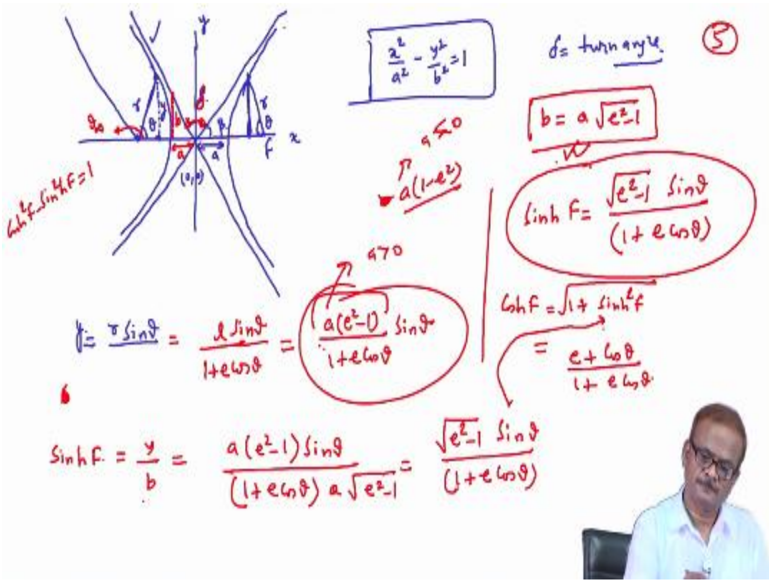
because this are the perigee position. One is occupied focus another one this is the vacant focus and this is the occupied focus, this is vacant no or non-occupied. Therefore, if your P coincides with B then FB – F*B this should also be equal to 2a. Now if we look from this place F * B is nothing, but FA.

So this becomes FB – FA mod = 2a or

$$|FB| - |FA| = 2a$$

If you look into this equation so this is nothing, but this distance from here to here. So this distance is your 2a magnitude and if we draw a asymptote here like this if we draw a asymptote like this. So the point where from the perigee if we draw a vertical or draw a straight line perpendicular line up. So the point where it cuts so stop there where it is cutting the asymptote I will show it in the next page. So distance from this point to this point this is your B.

(Refer Slide Time: 25:03)



So, I will show it on the next page. So we are drawing an asymptote say here it is going here in this direction so we need to draw an asymptote, an asymptote will pass through the center.

So this is asymptote of the hyperbola. Similarly, here on this side asymptote can be drawn. This angle we show as beta now if we draw a line perpendicular to this and wherever it cuts so this red one is b and magnitude by distance from this place to this place this is your a.

On the right hand side, we have another part of the hyperbola which we have not shown here, we can complete it maybe. So this distance also this is a this is not to the scale not a very good figure, but it is okay. In this direction x and y and origin lies here in 0, 0 and therefore for this we are getting

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

as the equation of the hyperbola.

So we will discuss about the hyperbolic eccentric anomaly later on. First, we look here into this part so we have $y = r \sin\theta$ divided $y = r \sin\theta$ that means we have any point here taken this angle is θ and this is r this will become y. Similarly, on the right hand side we can show suppose the focus is here if this is your focus and this is r and you are showing angle this as θ so $y = r \sin\theta$ it can be written.

So $y = r \sin\theta$ this angle we show it as the turn angle delta this is the turn angle. A line parallel to this asymptote. So this angle will be shown here by this particular one and this is θ infinity we write as so this is the asymptote angle. From here to here this is your delta, this is the turn angle. You can look into any standard book about the hyperbola. So $r = l \times l$ divided by $(1 + e \cos\theta) \sin\theta$.

So this is another way of working here say if I write here $a \times (e^2 - 1)$ so what we are assuming here that $a > 0$ so we are taking the positive value equally we could have written here $- a \times (1 - e^2)$or $l = a \times (1 - e^2)$we could have written in that case a will be < 0. So these are the ways you will find in different books they have their own ways of describing different authors they work in different ways.

So I am introducing both the ways of working out the problem and also we will prove that $b = a \times (e^2 - 1)$ under root $a \times$ under root $(e^2 - 1)$. So this is a simple result which can be derived later on for the time being we just take it for granted. Now we have sinh F equal to already we have looked into here in this equation y/b. Okay so from here this is y/b.

And if we insert this so this becomes from this place y = r sin θ equal to this quantity so we insert it here $(e^2 – 1)$ sin θ divided by 1 + e cos θ and then we have b here in this place so for this b also we introduced the quantity which is a × $(e^2 – 1)$ and if we simplify this so this gets reduced to $(e^2 – 1)$ under root sin θ divided by (1 + e cos θ).

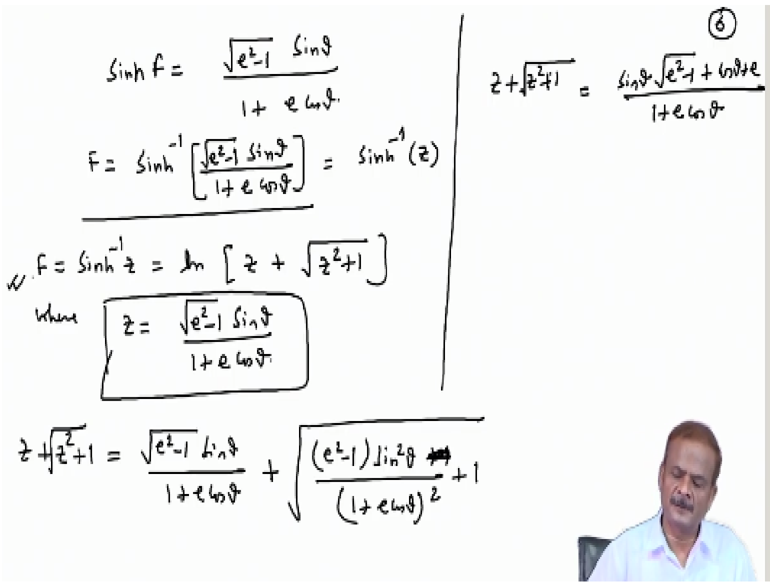
So this gives you the result sinh F this is

$$\sinh \mathrm{F} = \frac{\mathrm{y}}{\mathrm{b}} = \frac{\left(\sqrt{\mathrm{e}^2 - 1}\ \sin\theta\right)}{1\ +\ \mathrm{e}\cos\theta}$$

This is for hyperbolic orbit you are getting the result and if you use this equation cosh F = 1 + here cosh F – sinh F $\sin^2$h F this equal to 1. So from this place cosh F can be obtained so this becomes 1 + $\sinh^2$ F square root.

And if you insert this here in this place this value so this will get reduced to this step I am skipping you can check it yourself e + cos θ / (1+e cos θ). So these are some of the standard results which are used in solving the problems and both way the things can be worked out either whatever see using the method worked here we have done this particular part this also can be used or either we can utilize the way we are working here to solve the Kepler’s problem.

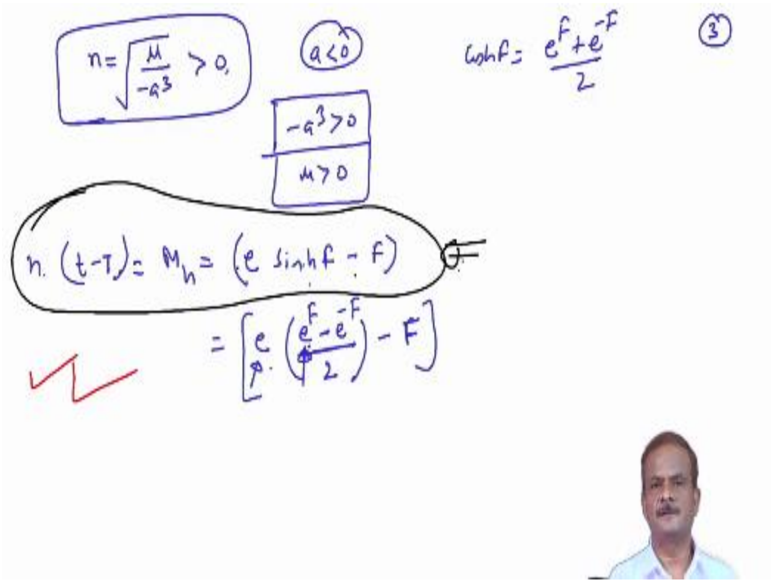
(Refer Slide Time: 33:57)



So we have got sinh F = $\sqrt{(e^2 – 1)}$ sin θ divided by (1 + e cos θ) and let us write this quantity. Now it is a property of the hyperbolic function this will be equal to and these are given in standard textbook where and if we insert all these things so your z + $\sqrt{(z^2 + 1)}$. This quantity once you insert the value here $\sqrt{(e^2 - 1)} \sin \theta$ and this can be simplified.

So this step you can carry out I will skip this step here z + $\sqrt{(z^2 + 1)}$ this get reduced to sin θ cos θ + e divided by (1 + e cos θ) and we need to further work on this to complete it. So from there if we complete it so we will get for the expression for F which we are going to do it in the next lecture and then we will put it into the hyperbolic orbit the equation we have derived.

(Refer Slide Time: 36:53)



And from there we will see that this is exactly the same as the equation we have derived here in this place except it for the sign change means here instead of minus it will be plus and here instead of plus it will be minus. So, we will do it in the next class. Thank you very much.