Space Flight Mechanics
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Lecture - 27
Kepler's Equation (Contd.)

Welcome to lecture number 27. So we have been discussing about Kepler’s equation for hyperbola. So already we did by one method and in the last lecture we have started with deriving with another method. So the same method we will continue and derive and thereafter, we will look into various terminologies, we have used and various equations used in this derivation.

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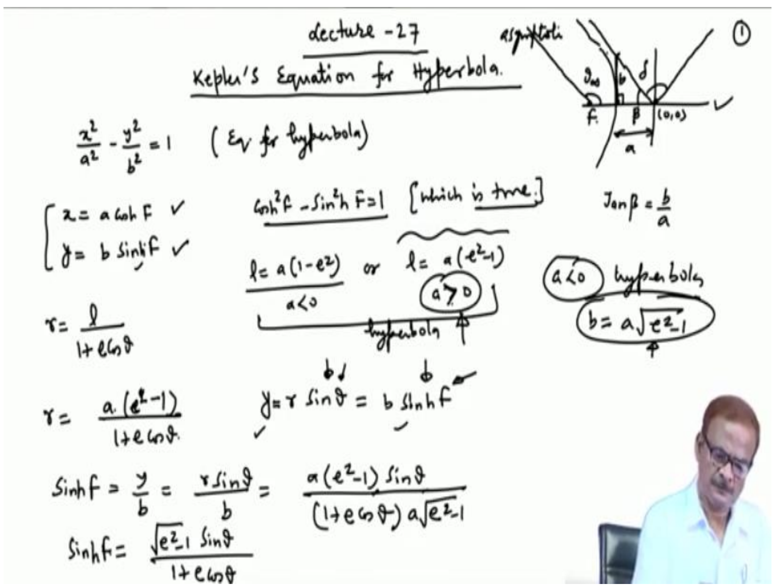

So if you remember that we have written equation for hyperbola as

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

in Cartesian coordinates and then we stated that this is satisfied by a cosh and y = b sinh F. If we insert this, so we get here

$$\cosh^2 F - \sinh^2 F = 1$$

which is true. Hence x and y described by a cosh F and y = b sinh F, they lie on the hyperbola.

And we have the hyperbola, we have shown as. This is the center of the hyperbola not directrix and this is the hyperbola is this one, this distance we have magnitude wise, we have shown it to be a and if we draw asymptote for this, so this is the asymptote for this hyperbola, so on this side from here from the perigee, if we raise a perpendicular line, so it cuts here in somewhere. So this b and this is a. Similarly, on this side also, in the last lecture I have shown that figure.

So I am not drawing the whole thing. So this is the turn angle, this is the asymptote and somewhere this is your focus and this is the asymptote line, parallel to asymptote and this we have written as θ_∞ and this angle, we have shown by β. So from this place, if you look magnitude wise, this is tan β = b/a. It will come to this particular figure again. Right now, that is not required and thereafter using this, we have started writing by equation for the conic section.

This is

$$r = \frac{l}{1 + e\cos\theta}$$

and remember that l can be written as $1 - a^2$ and here this can be represented or it can be written as $l = a(e^2 - 1)$. In this case, a is greater than 0. In this case, a is less than 0 and we are restricting it for hyperbola. Hyperbola in basic definition, for hyperbola a is taken as less than 0, but if you write the equation in a particular way like this, so at that time you have to describe the semi-major axis as a greater than 0.

So then using this equation, we have written this as $a(e^2 - 1)$ divided by $(1 + e\cos\theta)$ and then we rearrange it. Also $y = r\sin\theta$ and y from here this is available, so this is b sinh F and this implies $r = b\sin\theta$ or you just write in the next step here. So what we are going to do? What we have done? We have expressed sinh in terms of θ angle or F in terms of θ angle. So with the same effort, we are putting here.

So sinh F this equal to y/b. From this place, we can see, this is y/ b equal to r sin θ divided by b and r is known from this place, so $a(e^2 - 1)$ divided by 1 + e cos θ and sin θ as we put here in this place. So till this extent, we have done this and b is missing here. So b again, which we have not proved for the hyperbola yet, $b = a(e^2 - 1)$. This will be necessarily written in this way $(e^2 - 1)$, not $(1 - e^2)$.

So here we have to also write a $\sqrt{(e^2 - 1)}$. So this will get reduced to a cancels out and here we have $\sqrt{(e^2 - 1)}$ sin θ divided by (1 + e cos θ). So this is sinh equation, we have got.

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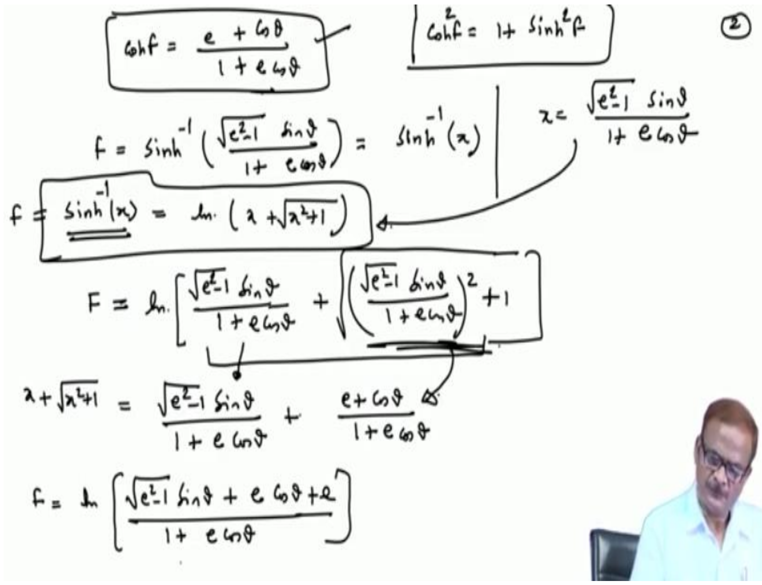

Along the same line, cosh F this can be written as

$$\cos F = \frac{(e \; + cos\ \theta)}{1 + e\ cos\ \theta}$$

This you can check yourself. This is left as an exercise, where we have used the fact that $\cosh^2$ F this equal to 1 + $\sinh^2$ F. So you can use this equation and you can get to this result. Thereafter, we wrote F = sinh inverse and whatever we have written the terms here.

This equation we have used it $\sqrt{(e^2 - 1)}$ sin θ divided by 1 + e cos θ and this can also be written as this. So there is one standard equation from text you will get this sinh inverse x this can be written as. This we will prove later. For the time being, we take it for granted that this is written like this. So this implies that here x is $\sqrt{(e^2 - 1)}$ sin θ divided by 1 + e cos θ and then we insert here in this equation.

So from there by inserting what we are getting here, this is nothing but F. F = $\sinh^{-1}$ x, so therefore we get F = ln and in bracket here the whole thing has to be copied $\sqrt{(e^2 - 1)}$ sin θ + 1 e cos θ and then x^2. Of course, $\sqrt{(e^2 - 1)}$ sin θ divided by 1 + e cos θ this whole square and then +1 and square root of this whole thing.

$$F \ = \ \sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$$

Now this quantity can be expanded and particularly this inside square root sign, this can be expanded and it can be written in a particular way. So in that case, you x = √ (x^2 + 1) This gets reduced to, if you expand it and you can check. Few steps I am skipping here again because there is constraint on time. The term under the square root sign is here and this term is copied here in this place.

So therefore,

$$F \ = \ln\left(\frac{\sqrt{e^2 - 1}\, sin\, \theta \ + \ e\, cos\, \theta \ \ + \ e}{1 \ + \ e\, cost\, \theta}\right)$$

So this is the equation we are getting. Now this has to be simplified. This needs to be simplified and for that, we need to do certain substitution.

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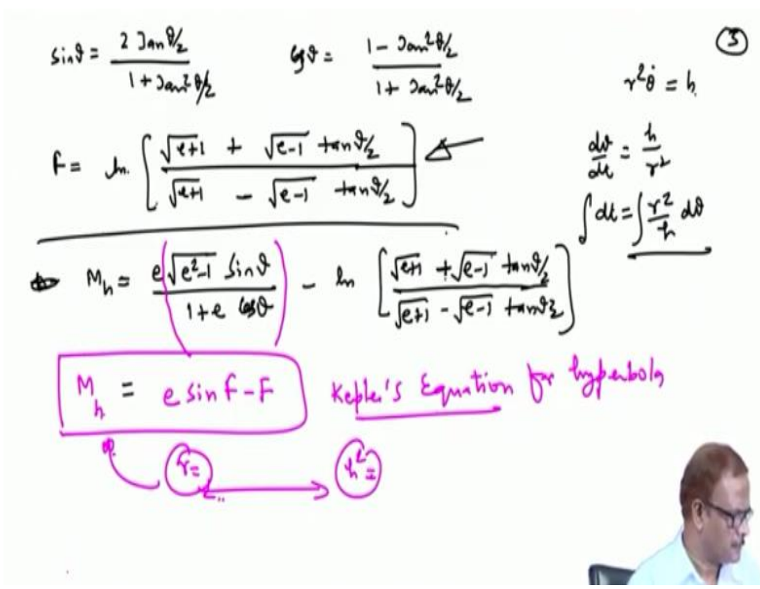

So we will use this substitution

$$sin\, \theta \ = \frac{2\, tan\frac{\theta}{2}}{1 \ + \ \tan^2\frac{\theta}{2}}$$

and

$$\cos\theta = \frac{1 - \tan^2\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}$$

. This step again you can carry out and if you insert this, this will get reduced to ln. It is not difficult to derive this particular equation. Now we are going to the equation we have got, this M_h hyperbolic, this equation we have written earlier as, if you remember that by integrating this equation $r^2 \dot{\theta}$.

We re-wrote it and thereafter we rearranged it to write it as and integrated it. Once we integrated it for hyperbola, we got this equation. This we have written that this is a standard integration. This is an integration found in some standard textbook. So this quantity, this is cos θ – e + 1 under root tan (θ/2). This will be plus here and e + 1 under root – e – 1 under root tan (θ/2). So if you check now, the quantity here written this is nothing but F.

So this side becomes F and what this quantity is, if you look here in this part, so this part we have written earlier, sinh F, this is your this part. So here we have

$$M_h = e \sinh F - F$$

we write here and this is nothing but your Kepler's equation for hyperbola, which we have also often using, right in the beginning, we have often using purely the mathematical method by differentiating, where we were given this r and we used the equation for h^2.

And then we differentiated this r and we had the equation for h^2. So combining these two, we obtain this Mh. Therefore, we are getting the same equation by different methods and it also verifies that whatever we are doing, it is correct. There are few more things that we need to work out. Particularly, we have few relationship that we should look into

$$\tan\left(\frac{E}{2}\right) = \sqrt{\frac{e - 1}{e + 1}}\ \tan\frac{\theta}{2}$$

and the same way, it can also be written as

$$\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{e + 1}{e - 1}}\ \tan\frac{E}{2}$$

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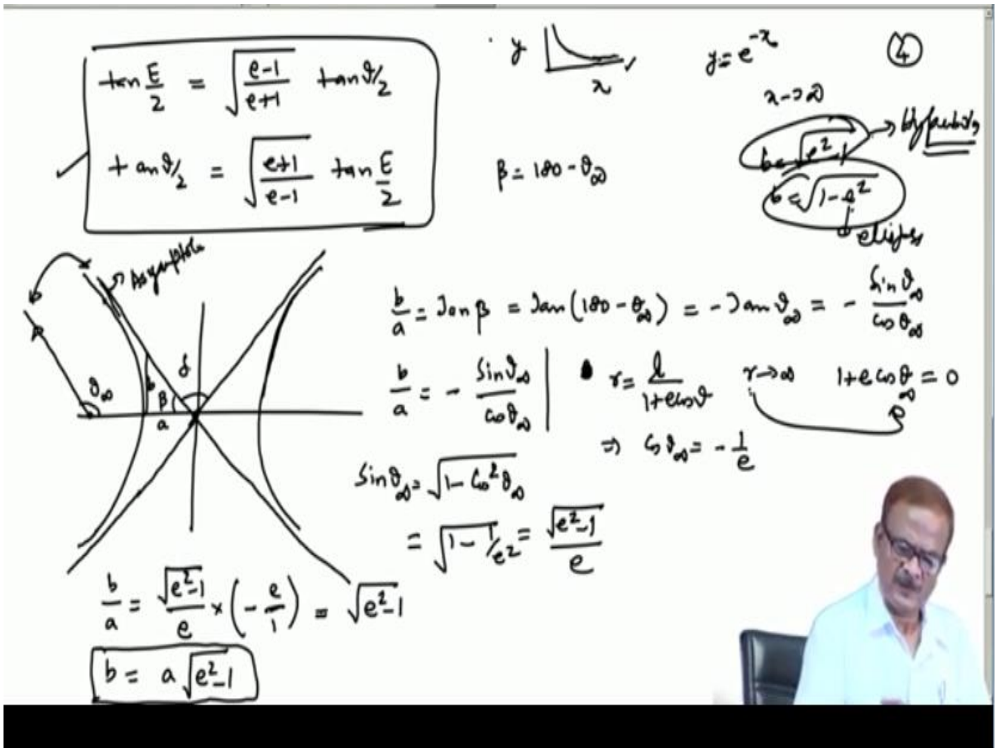

So we will work on this also, but before that we need to do few more exercises. Now let us look into the properties of the hyperbola. We have to get these equations. Already, we have drawn this earlier. Asymptote is here. Asymptote is the line to which say if we have an x_1 initial function, so you can see that e^{-x} if I plot it, y = e^{-x}. So it will converge like this. So the x axis is, let us say, asymptote.

This is the asymptotic value, means as x tends to infinity, we will see that y tends to 0. So this is an asymptote to which this line becomes converges, once r tends to infinity. So already, we have shown this distance to be b, this distance to be a and this angle as δ, this part was remaining in our discussion, so I am taking it up at this stage, θ_∞. This angle, we write as β. So we can see that β = 180 – θ_∞, because this is parallel to this asymptote.

This is asymptote, so this is parallel, these two are parallel. So we have b/a = tan β = tan 180 – θ_∞ = - tan θ_∞. This we can write as – sin θ_∞ / cos θ_∞. For hyperbola, we write r = 1 by r for the conic section, e cos θ. In this equation, as r tends to infinity, 1 + e cos θ then can be written as θ_∞. When r tends to infinity, this θ is referring to that. This is referring to this part.

So this becomes 0 and this implies cos θ_∞, this equal to – 1/e. Therefore,
sin θ_∞ this becomes

$$\sin\theta_\infty = \sqrt{1 - \cos^2\theta_\infty}$$

1 – 1/e^2 b by a can be written as sin θ, which is √(e^2 – 1) divided by e and then cos θ_∞, which is minus and this gets reduced to √(e^2 -1)

$$b = a\sqrt{e^2 - 1}$$

So you have to also note that b will be always written in terms of b = √(e^2 – 1). You cannot write it like this √(1 – e^2). This is only in the case of ellipse and this part is for the hyperbola. For hyperbola, we always write it like this. So we have got the expression for b here. These were few of the things which we have left in the conic section.

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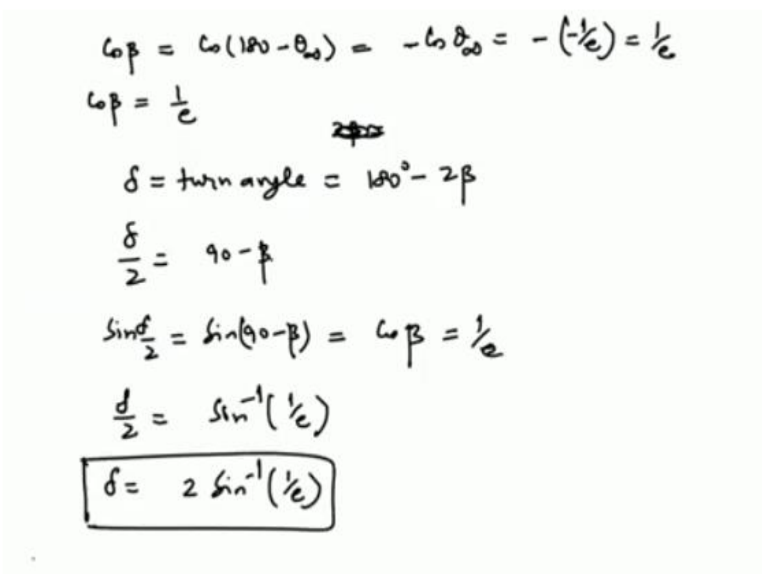

And I told that we will cover while we discuss about the hyperbolic orbit. So cos β = cos 180 – θ_∞ = - cos θ_∞. Now going back into this figure, this angle is β, because this is these two lines are asymptotes, this is the right part of the hyperbola and this is left part and this tends from here to here, this is r-perigee and from this place to this place, this is r-apogee.

So cos β is 1/ e and also we can see that 2β equal to or we can write δ the turn angle. This equal to 180 – 2β; δ/2 = 90 – β, cos β = 1/e as derived above and therefore δ/ 2, this is sin inverse -1/ e or δ equal to 2 sin inverse.

$$\delta \ = 2\sin^{-1}\left(\frac{1}{e}\right)$$

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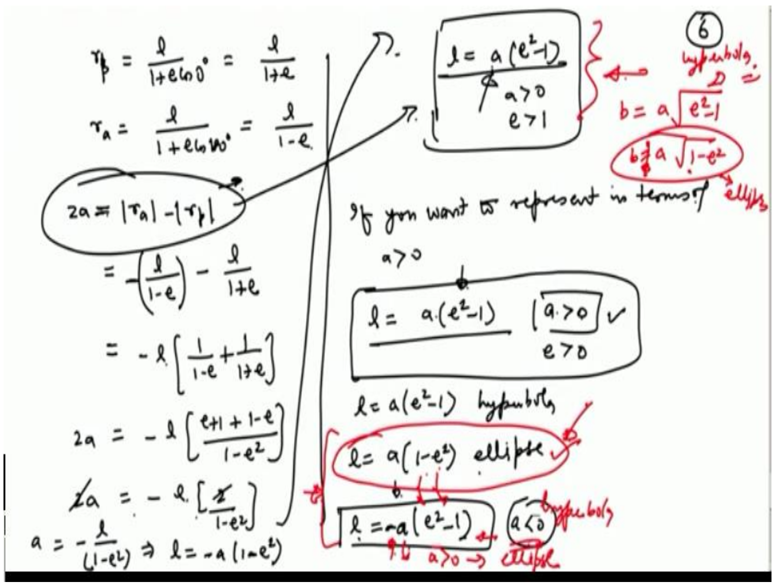

So these are the expression for r-perigee and r-apogee. You can notice the difference in the position perigee location in the ellipse. Here in this case, this is perigee here as shown by this red dot, while the apogee is shown by this blue dot here. So these are on the two sides of the center and center is lying here which is shown by the yellow point, center is lying in this point, whose coordinate we have written as (0,0).

So there is a difference between the ellipse and hyperbola and accordingly some of the things change and also if you notice, we have shown that distance from the focus to any point on the hyperbola, say we take a point here on this and then we join the other focus, so this is f and this is f*. We will show it by f and if we write this as p, so here

$$|Pf| \ - \ |Pf^{*}| \ = \ 2a$$

This is the basic result. This is always constant.

So we will utilize this information. In that case, if we look here at this point, this is r-apogee and this point is r-perigee. On the blue dot, I am putting a black dot and in the red dot I am putting here a black dot. I hope it is visible. So this is your apogee and this is perigee. So your p point coincides with this, it becomes easier to work out. So let us look into this. So at that time r-perigee and r-apogee, they are defined.

So 2a we will write as $r_a - r_p$. This is distance wise in that case. So we have here r_a = l/(1 – e), but here remember that e for the hyperbola is greater than 1 and therefore we will put a minus sign, because this l/(1 – a), this becomes a negative quantity and therefore we put a minus sign here and this is already a positive quantity, so we write it in this way and if you rearrange it, so l can be taken outside.

So this is the semi-latus rectum, but in this case then a is less than 0, but if you want to represent in terms of a greater than 0, so the same thing you should write as, here in this case this will be representing a greater than 0 and e also greater than 0. Here e is greater than 0. Therefore, this quantity, this becomes negative. If this part is little easier to handle, because you can remember that latus rectum, this will be given by

$$l = a\ (e^2 - 1)$$

, this is for hyperbola.

And for parabola you are using

$$l = a\ (1 - e^2)$$

parabola. In the case of this parabolic equation, if we want to change, sorry this is for ellipse and if we want to change it, so we have written as

$$l = - a\ (e^2 - 1)$$

Here we need to correct in this part, otherwise it will be confusing. We will follow this scheme. Here in this case, this a is, if we write it like this a is less than 0, in this part. If we write here in this way, so a is greater than 0. This is the expression for hyperbola.

This is also the expression for hyperbola, but here in this case a is less than 0. So this thing we have to take care of while writing, otherwise it may be confusing. So this part we have rearranged and this 2, 2 cancels out and $a = - l/(1 - e^2)$ This implies $l = - a \times (1 - e^2)$and once we rearrange, so this $l = a \times 1 (e^2 - 1)$. So here in this case, a is greater than 0 and also e is greater than 1.

The format in which we are working, we are writing here, because this is the correction I have done. This should be consistent with this. So we have written in this particular format and because of this, the result we got it here this way $l = a \times (e^2 - 1)$. Otherwise, we follow a standard form, which is the most standard one is this one and based on this, in the beginning we have derived the Kepler's equation also $l = a \times (1 - e^2)$ this is valid for ellipse.

And if it is a hyperbola, so e will be greater than 1. So in that case, you will shift the sign outside. Therefore, there is exchange of e^2 and 1. So it gets into this form and minus sign comes outside. So in this case, we write a is less than 0. So if we write a is less than 0, so this becomes hyperbola and if we write a greater than 0, so here in this case if we write a greater than 0, so this case will be an ellipse.

This is the most standard form, but many of the books, they bring it to this form for their convenience maybe, but if we go by our basic concept, I will say that you follow this step and it will be the most convenient, only thing that you have to keep clear that b will be always defined in terms of $b = a \times (e^2 - 1)$. We never write it as $b = a \times (1 - e^2)$ this is not correct. This is only for the case of ellipse.

This will be valid only for the case of ellipse and this is for hyperbola. So this difference you have to notice.

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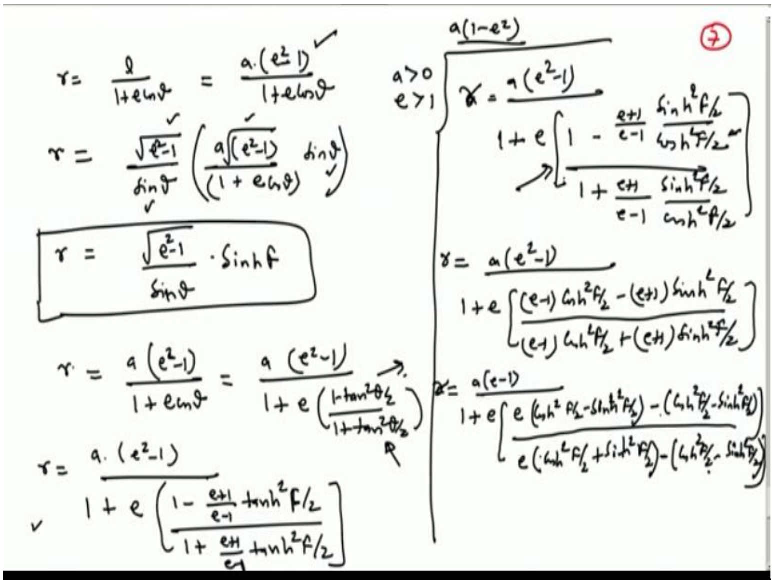

Some of the basic things we have already covered and expressions also we have already written. Now let us wind up this first

$$r = \frac{l}{(1 + e\cos\theta)}$$

We have used this earlier. This we are writing as a ×(e^2 – 1). Again this is known standard format I am writing here. Otherwise, you can write it in the same way 1 – e^2, but there will be a minus sign outside in that case.

Specifically, you should write a greater than 0, and e greater than 1, if you are expressing like this. Thereafter, we break this term (e^2 – 1) into two terms here, square root and square root and the denominator we have to write sin θ and the numerator also we write sin θ. If we write it in this way, so this can be expressed as (e^2 – 1) sin θ and this part if you remember this is nothing but sinh F as derived earlier.

$$r = \frac{\sqrt{e^2 - 1}\sinh F}{\sin\theta}$$

We do one substitution here as we have done earlier also e × 1 – $\tan^2$ (θ/2) 1 + $\tan^2$ (θ/2) and then we need to rearrange it. So now $\tan^2$ (θ/2) if you remember, I have written the equation, I will also prove it here, because this is required, but let us use that information whatever we have written earlier. So this is the information I have used earlier and this I will prove here (F/2) (e -1) $\tanh^2$ (F/2).

This can be rearranged and if we rearrange it

$$r = \frac{a\ (e^2 - 1)}{1 + e\cos\theta}$$

This part needs to be rearranged and if we rearrange it this can be simplified. First simplify this part $\cosh^2$ (F/2) – e + 1 $\sinh^2$ (F/2). The denominator will cancel out from the numerator, here this part will cancel out. So I am skipping one step and writing here and we need to rearrange this term again e ×, e is one outside and inside also.

This is e × cosh h^2 (F/2)– sinh h^2 (F/2)and the other term comes with here only cosh. So this is cosh h^2 F divided by 2 – sinh h^2 (F/2)and in the denominator we have the same way e cosh h^2 (F/2)+ sinh h^2 (F/2)or we write even in terms of minus here, - cosh h^2 (F/2)and – sinh h^2 (F/2). Here this is minus and cosh – sin it derives us here, so this becomes plus. This would come with a plus sign.

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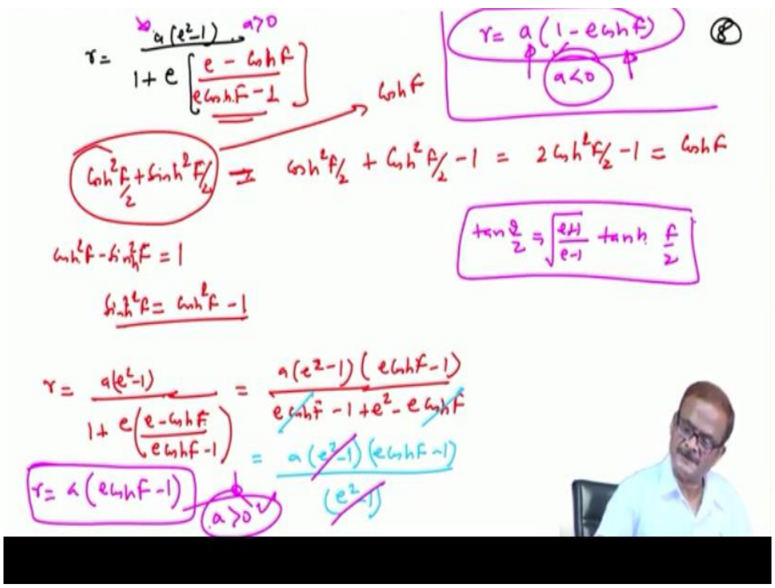

So going to the next step, so r becomes

$$r = \frac{a(e^2 - 1)}{1 + e\left(\frac{e - \cosh F}{e\cosh F - 1}\right)}$$

This term needs to be simplified. You can see that this particular term here, this equal to 1

$$\cosh^2 F - \sinh^2 F = 1$$

. So this we write as 1, so that becomes e, the first one becomes e. So here you have e and thereafter this particular term is there cosh (F/2)+ sinh (F/2). So this term we need to work out.

Here also we have this particular term and this term equal to 1. This term, the right hand side is 1 and we need to work out. So we will show that this is cosh F and here cosh F multiplied by e. So we have here the last term we have written $\cosh^2$ F $\sinh^2$ (F/2). So this term we have to rewrite it. So we know that $\cosh^2$ F – $\sinh^2$ F this equal to 1 or $\sinh^2$ F.

This will be cosh h^2 F – 1. So we can utilize this information here in this place. So this term then becomes cosh h^2 (F/2)+ cosh h^2 (F/2)– 1. That means this is 2 cosh h^2 (F/2)– 1, which is nothing but cosh F and which we have used here in this place. So this term we are replacing by cosh F. So on the previous page, this is your quantity, cosh F and this is cosh F.

But here with this, e is multiplied and therefore we have in this format and this is the minus sign here in this place. Once we have done this, rest of the steps is not much difficult to work out. This this cancels out and we get here a × (e^2 – 1) and cosh F – 1 divided by (e^2 – 1) and then this term and this term will cancel out leaving us

$$r = a\,(e \cosh F - 1)$$

So you can notice the difference that earlier we have got here the expression a × (1 – e cosh F) for r, but there then a was taken as less than 0. Here a is greater than 0. This is the difference. This is the way we have expressed here. Here a is greater than 0 and therefore in this expression a is greater than 0 and once we have proved our this Kepler's equation for hyperbola, at that time you remember that I have taken r equal to this.

$$r = a\,(e \cosh F - 1)$$

So this difference always you should notice, because the different authors, they will express in their own way and I have presented here in both the ways it can be done. So this is the expression for r. Now only thing we are left with proving this

$$tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{e\ +1}{e-\ 1}}\ tanh\left(\frac{F}{2}\right)$$

and this is similar to what we have got for the ellipse. In that case, here instead of F, it was replaced by a. So this particular part, we will work out and before winding up the things. Then what we will do, we stop here and continue in the next lecture. Thank you very much.