Space Flight Mechanics
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Lecture – 28
Kepler's Equation (Contd.)

Welcome to lecture 28. So we continue with whatever we have been doing last time.

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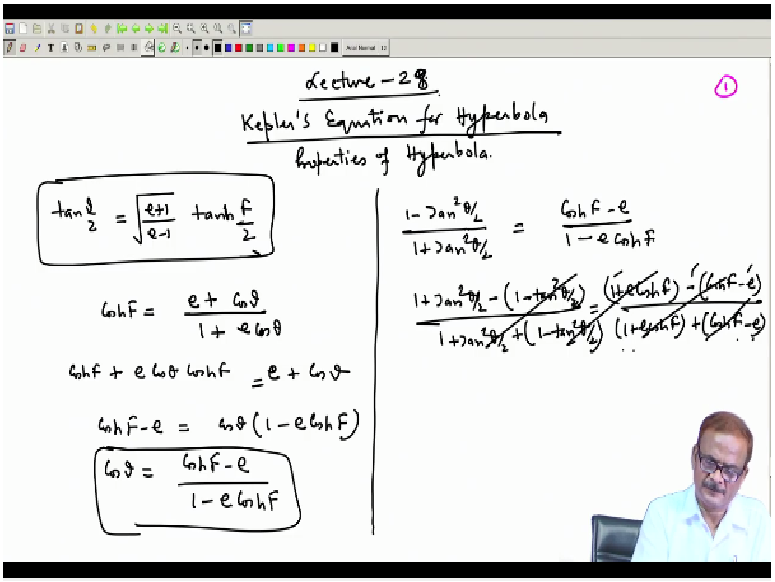

So we were interested in deriving certain relation and that relation was

$$tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{e\ +1}{e-\ 1}}\ tanh\left(\frac{F}{2}\right)$$

So this we have to derive. So to derive this what we need to do that we go back and look into this equation, we have written

$$\cosh F = \frac{(e + \ cos\ \theta\ )}{1 + e\ cost\ \theta}$$

this we have derived in the last lecture. So rearrange this equation. This becomes cosh F + e cos θ times cosh F, this equal to e + cos θ, cos θ term we combine together (1 – e cosh F).

So from here we get cos θ equal to

$$\cos\theta = \frac{cosh\ F - e}{1 + e\ cosh\ F}$$

Rewrite in the left hand side and the right hand side the same way cosh F – e divided by; Use Componendo, Dividendo 1+tan^2 (θ/2) – tan^2 (θ/2); this will be equal to; this rewriting the whole thing; and here –cosh and this is guess 2 minus minus plus. Here we have to check this particular part.

This is (1+e).1 here in this place and this is minus, minus that gets plus, so this is (1+e) and thereafter we have cosh F, we have taken outside, so this is minus, this minus, minus, plus, so part is okay then and here also need to change (1-e) and then plus (1 + e). Something wrong here. We will go on the next page.

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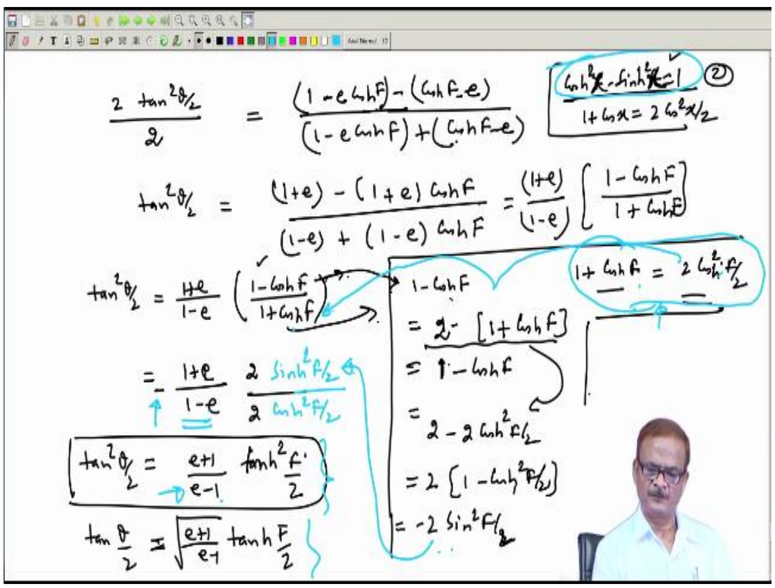

So left hand side is 2 tan^2 (θ/2) divided by 2 and on the right hand side we have 1-e; this is 1-e here we are doing the mistake, this is (1-e), here this is (1-e); I will strike it off this particular part, with this the things will get modified, so we write it on the next page 1 – e cosh F and then –(cosh F – e); cosh F + cosh F – e. This is the right thing now. tan^2 (θ/2), 1+e and then here minus sign we take it outside, so this is (1 – e); 1 + e cosh F.

You can check, and thereafter this becomes (1 – e) and here in this place plus (1 – e cosh F), (1 + e) can be taken outside and here (1 – e) can be taken outside, so this get reduced to

$$\tan^2\left(\frac{\theta}{2}\right)=\frac{1+\mathrm{e}}{1-\mathrm{e}}\left(\frac{1-\ cosh\ F}{1\ +\ cosh\ F}\right)$$

So $\tan^2$ (θ/2), this gets reduced to (1 + e). So we need to rearrange this part. This quantity is 2 times $\cosh^2$ (F/2)and similarly this part we can express here 1 + cosh F, this we can rearrange as; you can rewrite as – cosh F and say here we write as 1 – cosh F.

So here if we bring the bracket so this will gets reduced to -1 + cosh F and if we write here 2, so this will reduced into this format. So then this becomes 2 – $\cosh^2$ (F/2); so 2 times 1 – $\cosh^2$ (F/2) and this is nothing but -2 times $\sinh^2$ (F/2). Because we are using this expression $\sinh^2$ (F/2)= 1 or say this is x or any quantity here, $\cosh^2$ x – $\sinh^2$ x = 1, so following that notation we get this expression.

So here we can write with a minus sign 2, $\sinh^2$ so this gets reduced to

$$\tan^2(\theta/2)=\left(\frac{e+1}{e-1}\right)\tan^2\left(\frac{F}{2}\right)$$

In this part we need to correct here also. This should come in the format here; we have done one mistake here. Okay, so this will ultimately come here in this format tan (θ/2), which we can write as

$$tan\ (\theta/2)\ =\sqrt{\frac{e+1}{e-1}}\ tanh\ \left(\frac{F}{2}\right)$$

So the remaining part we have to work out 1 – cosh F and 1 + cosh F. Okay, so; the logic is the same that means if you remember from our trigonometry that 1 + cos x is nothing but 2 times $\cos^2$ x/2, so the same analogy also applies here in this place. Okay. So if we follow that, so here in this place this is your 1 + cosh F=2 $\cos^2$, $\cosh^2$ F/2, okay and then this part we can rewrite as 1-; so we want to first express it to this way. So cosh we have; this part we are trying to express here.

So here we write 2- (1 + cosh F) and then see that this gets reduced to this format 1 – cosh F and this quantity as you can see, this quantity here this can be written as 2- using this 2 $\cosh^2$ (F/2)so 2 times (1 – $\cosh^2$ (F/2)) and this is 2 times; this will come with a minus sign here so – Sinh (F/2)as we have; we have not written here. Okay anyway, this part is directly coming from this place; from this equation, you can rearrange and check it.

So we have to keep care of that identity we are using it is almost similar to what we have the trigonometry identities in the case of a unit circle. So in the case of hyperbola it follows the almost the same rule, so 1 + cosh F = 2 $\cosh^2$ (F/2) so we utilize it and insert it here, in that case you can see that 1 + cosh F this becomes from this place, this goes as 2 $\cosh^2$ (F/2) which is written here, we have written the opposite way here we correct, $\cosh^2$ (F/2).

And this part, this goes here in this place with a minus sign, so minus sign we have observed here and thereafter we reverse the sign of this, so (1-e) this becomes (e-1) and finally we get this equation, so this is what we were looking for. So this is proved in the case of hyperbola, this result we were looking for. Now one more thing I would like to mention here before we further proceed, that while we derive this equation.

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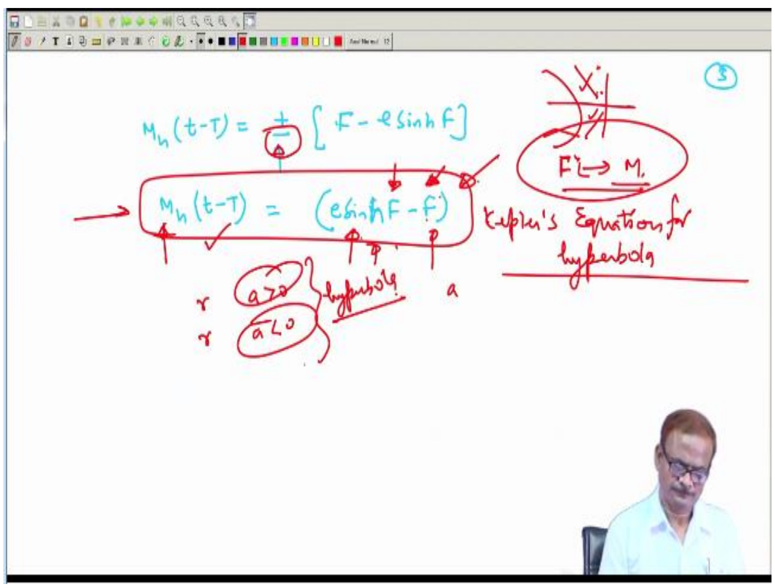

t – T times M_h, so at that time we got + -, F – e sinh F. And then we selected the minus sign. And thereafter we wrote this as

$$M_h(t - T) = e \sinh F - F$$

. So minus sign was accepted only and plus sign was rejected. The reason for is, this is the consistency in the result, okay. The F, actually this F is defined in terms of area which is my next topic to discuss, okay. So this way the area below this is negative and the area upside that will be positive and accordingly your mean hyperbolic; this hyperbolic mean anomaly this will be positive and negative either corresponding to; sorry this; if the quantity F that we are writing here.

So this is defined in terms of area or the certain area which is projected on the hyperbola which I am telling that I am going to discuss. So this is the consistency of; we are looking for the consistency of result, so if F is positive, so corresponding m we are looking for m should also be positive. So for making that we take this negative sign, you can check it numerically, this negative sign is selected and we put it here in this format which is our Kepler's equation for hyperbola.

So for this part you should remember also while we have written, we have written use this format. And this result is independent of whether we are using the format for

$$l = a\,(1 - e^2)$$

or the other way. Either we are using the expression for the r where $a > 0$ and the expression for r where $a < 0$, both for the case of hyperbola. So independent of that we are getting this expression here and also we have looked earlier that, in the case of hyperbola using the, one integral standard integral we used and in that we inserted the equations we derived so thereafter we got this expression.

So both way we get the same expression, which we have done in the last lecture. So now after winding up this part let us go into and look into the; what is this F, which is pending for a long time.

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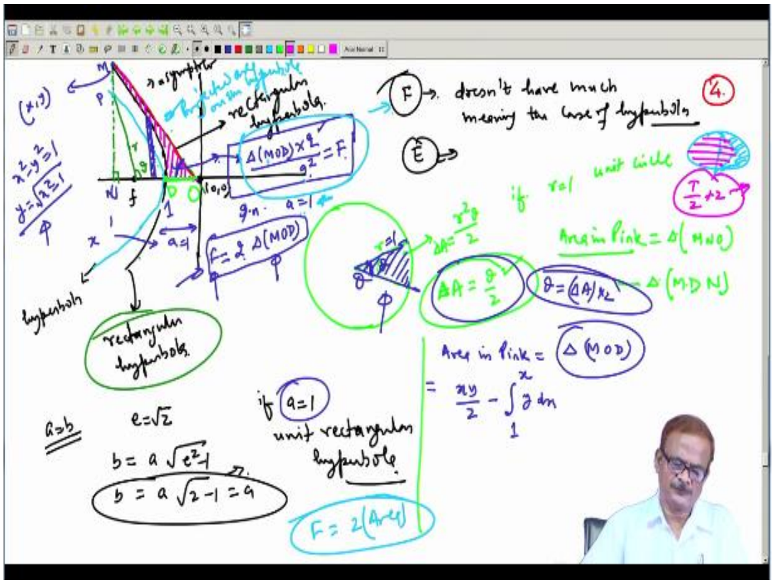

In case of here it has not passed through the; this line is little curve, but this is passing through this origin. Here this is the focus. The F here does not have much meaning in the case of hyperbola just as the mean anomaly. So this is hyperbola and this is a rectangular hyperbola. Rectangular hyperbola is the case where a=b and that corresponds to e equal to root 2, so you can look into this expression

$$b = a\,(e^2 - 1)$$

, so if you put e equal to √2 so this gets reduced to (2-1) a so

$$b = a$$

so this is the case of the rectangular hyperbola and this is asymptote to rectangular hyperbola.

Now why this F is not defined as in the case of e is defined in the case of ellipse that centric anomaly. The reason is very simple, that in the case of ellipse there was a circumscribing circle which in this case is not present, okay and therefore instead of defining this in terms of circle we utilize a rectangular hyperbola, so this is your rectangular hyperbola and if a=1, so this is called the Unit rectangular hyperbola.

So this concept is used for finding out what we are looking for the F, how it should described, the inner circle we will show it by some other color like this hyperbola hyperbola. And this hyperbola also both of them they touch here at the perigee. Now from this place draw this r, this is your r and then draw a perpendicular and pull it up also, this is your θ and this one is already we have written this one rectangular hyperbola. So rectangular it goes and cuts here in this point.

Now to this join here, this is not a very good figure, okay. Rectangular hyperbola again I will redraw it, it will look something like this, okay. So asymptote is different. So I will remove the asymptote. This asymptote line I am removing. So in the case we; for the ellipse we wrote this as m and upper one perhaps written as; upper one was written as m and lower was n. Now this area which I am hashing here; this area bounded between the; this red line and this black line for the rectangular hyperbola and this access, okay which is shown by green line here.

This is used to define F. See analogy is given for the case of a circle, so in the case of a circle the; if this angle is θ so area of this sector will be given by; and if this is r, so r^2 θ divided by 2, this is the area of this sector. And if r=1 that means the unit circle, so in that case this area ΔA; if we write this as ΔA, so ΔA becomes equal to (θ/2), means this area is direct representation of this θ.

$$\Delta A = \frac{\theta}{2}$$

Similarly, this analogy is made here and whatever the area I have shown by this hashed pink line, okay that area is used to represent F. So as such, just like the centric anomaly we do not have the concept here of a particular angle here in this case, okay. So you can see from this place that for the case of the rectangle hyperbola this hashed area, area in pink, this can be written as, if we write this point as O, so area MNO which is a triangle minus area and this point we write as; this is the perigee point so let us make this as say, we write here as d.

So M, D and N. so this is referring to rectangular hyperbola where we are trying to find area for this rectangular hyperbola, and this rectangular hyperbola if it is unit hyperbola means a=1 so the things gets more simplified. So coordinate of this point, here in this point and this is xy. So what will be the area of the shaded region? So area in pink which is MOD this gets equal to, this coordinate is xy.

So therefore you can see that this xy divided by 2 and minus the area inside the hyperbola means we have to draw a area like this elementary area and integrate it and this point if we write it as 1 in the case f rectangular hyperbola, so this is you are a, a=1 so from this point to this point then we have to integrate where this is the point where I putting x. Okay. So then this area becomes; and we know that this hyperbola for rectangular if it is, so a becomes, a=1.

So in that case it will be written like this and therefore y becomes

$$y = \sqrt{x^2 - 1}$$

So upper curve can be written like this, similarly for the lower curve also, but that will come with a minus sign. So this becomes then y dx to be integrated from 1 to x.

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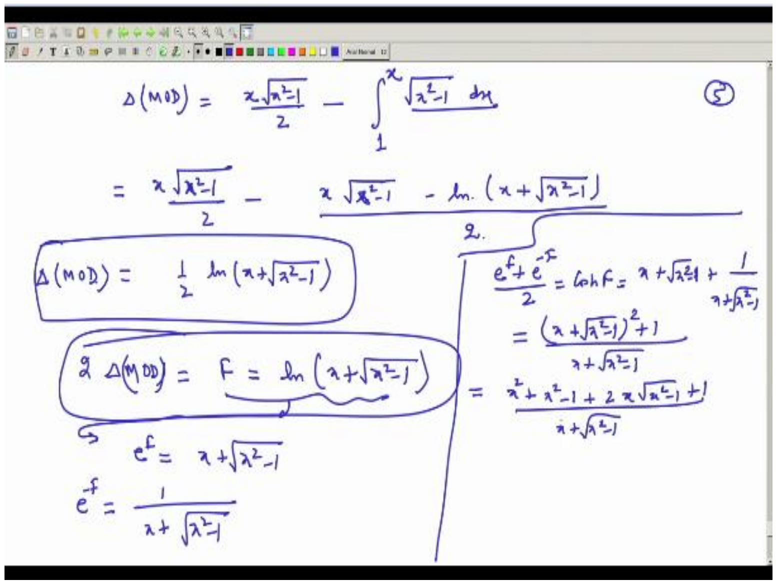

Therefore, area (MOD) x and y equal to; already we have written here x^2-1, so we replace here $\sqrt{(x^2-1)}$ divided by 2 dx, why dx? Now this area the area shaded in pink, okay. This quantity if the area say we have already written here Δ (MOD), this is the area times 2 divided by a^2, this is defined, F is defined like this. So this is the area here, pink area multiplied by 2 divided by a^2 equal to F.

In the case a=1, so we have

$$F = 2\ \Delta\,(MOD)$$

. So this angle is, F is defined as twice the area which is shown here in pink. Okay, so this definition is not as, you can corrugate something from here that this area is numerically equivalent in the case of unit circle r=1 this area is equivalent to (θ/2), means this angle θ is nothing but ΔA times 2. Similarly, here in this case this angle F is 2 times Δ (MOD).

But here in this case see that, directly this is the angle here which is represented. This is your θ angle. But here in this case we are not telling that this angle is the F angle. So let us go further then we will look what the exactly the things are happening here. So x (x^2 – 1) divided by 2 and this part it is a integration, it is a standard integration given in many of the books so you can use it on the integral, so x (x^2 – 1).

And also on Wikipedia you will find some of this material, you can refer to that also. And once we rearrange it we get it as

$$F \;=\; 2\;\Delta\,(MOD) = \frac{1}{2}\;ln\left(x \;+\; \sqrt{}(x^2 - 1)\right))$$

This is the area and twice the area Δ (MOD), this we are defining as F, so F becomes then equal to ln(x + √(x^2- 1). So this is a very abstract representation here in this case, but it helps in solving out certain problems.

Now we know that this quantity is nothing but e to the power f and therefore e to the power -f will be ln(x + √(x^2- 1) . And if we try to rearrange this and write it like this (e^f + e^{-f})divided by 2 so from here we will get this as the; this is nothing but the cosh function. You can see that this gets reduced to;

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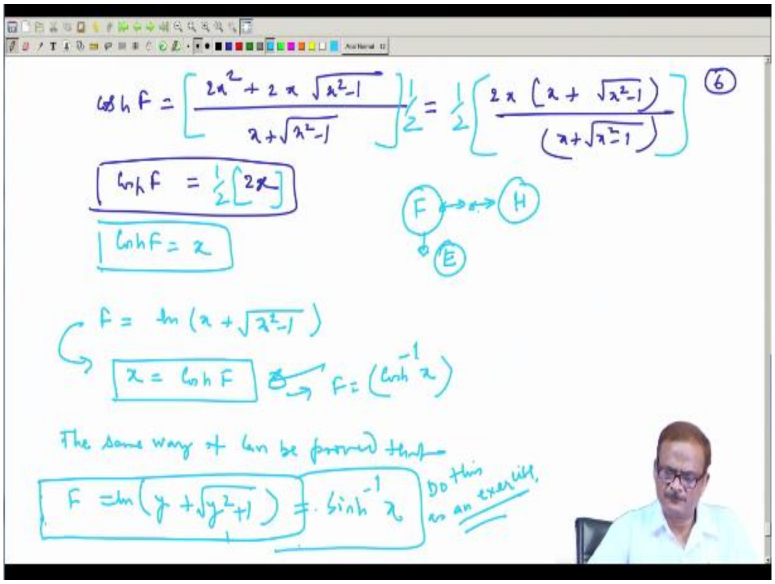

So we have from this place cosh F, this equal to $2x^2$, 1 ,1 cancel out, $2x^2 + 2x$ times $\sqrt{(x^2-1)}$ divided by $x + \sqrt{(x^2-1)}$. x taking out this is a common and yet the factor 2 we have taken, so first let us remove this factor 2 here, so this is only this first we have written here, so $e^f + e^{-f}$, okay; no first; if we write divided by 2 only then this cosh F we can write then in that case we need to write here 1/2 and take it inside the bracket the whole thing.

So we have to take it inside the bracket divided by 2, this is 1/2. So here also we need to put 1/2 then 1/2 and then 1/2, so this gets reduced to finally

$$\cosh F = x$$

So this means what we have proved that we have started with $F = \ln(x + \sqrt{(x^2- 1)})$ and then from there we proved that $x = \cosh F$.

Okay, similarly the same way you can also prove another expression, so let us say, we write in terms of sinh F this equal to or we write in a different way little; $\ln(x + \sqrt{(x^2- 1)})$, this quantity we can write as F or you can use a different notation maybe. You can write in terms of y. And the way we have proved here, here in this place the same way it can be written as.

So here F then becomes $\cosh^{-1} x$. So in this place then F becomes $\sinh^{-1} x$ and this is left to you as an exercise. So thus, what we have concluded that the quantity F we are using here, in some of the text it may be represented by H, okay. So this quantity F this does not have a physical meaning as the eccentric anomaly E. This is basically a mathematical quantity and there is a more of; it is a represented in terms of the area which is shown here in the pink and this part also you remember.

This is two times the area divided by a^2 and only in the case of a rectangular hyperbola where a=1, okay, F becomes equal to 2 times area. So, case of a hyperbola is turned out to be little different from what we have done for the ellipse. But if you go in a mathematical way it is not much difficult. In the case of the geometric representation it becomes little complex because there is no auxiliary circle and we have to use this hyperbola so this is the; this pink area is the projected area; projected area sometimes called on the hyperbola.

Okay so we will continue to discuss about maybe one problem we can take up later on the next lecture. So we will conclude this lecture here and next time we will start with the 3 body problem and we will also consider taking one problem related to hyperbola, already we have done one problem and regarding ellipse also we have taken one problem related to the sun-earth; we have proved that moving in the ellipse and the time taken to cover this blank area.

It is this the; again this pink area here, this will be $\frac{T}{2} + 2$ means the 2 days more than the half of the period of the earth around the sun. So this problem we took. So this; once you know the basic mathematics, some of the basic mathematics you need to remember also because every time we cannot derive. Okay. It should not possible that every time we go and work out or either you can have a chart and work out the whole thing using a chart.

So; but if you remember few of the equations it becomes very easy to do the problems and the whole thing become spread and show. so we will conclude this lecture here in this place and continue in the next lecture. So next lecture maybe we can take one problem and also we are going to start with 3-body problem and restricted 3-body problem thereafter we will follow. Thank you very much.