Space Flight Mechanics
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Lecture – 30
3 - Body Problem (Contd.)

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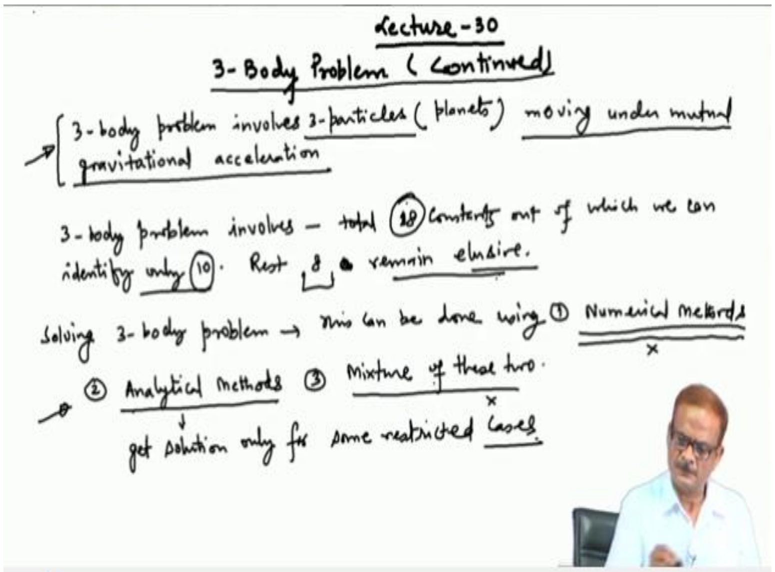

Now, welcome to lecture number 30, so we have discussed about 3-bodyproblem in the last lecture, so we will continue with that. So, 3-body problem this involves 3 particles basically, Newton's law it is applicable for particles and it so happens that once we take the aggregation of particles which we call a body, so in this, in that case also it becomes applicable, so you have to always be very careful while writing the equations.

Now, let us start with 3-bodyproblem involves total 18 constants, out of which we can identify only 10, rets 8 remain elusive. So, out of this 18 constant and how we are getting this 18 constants, we will discuss about that and we can identify only 10 constants and other 8 cannot be identified. Solving 3-bodyproblem this can be done using one the numerical method and the second, analytical method; numerical methods, analytical methods and there can be mixture of these 2.

Now, this numerical method, it is not a topic of our discussion, we are not discussing about this numerical method at all in our space flight mechanics, analytical method here you can get solution only for some restricted cases, again mixture of these 2, this also we are not going to discuss, this we are not discussing. So, as we have written this involves taking constant out of which we can identify only 10, rest 8 we can remain elusive.

That means, we cannot get a close form solution for 3-bodyproblem, now the issue is to get the analytical solution but before that we have to pass through a number of a stages, understanding various things and thereafter we will come to the analytical stage.

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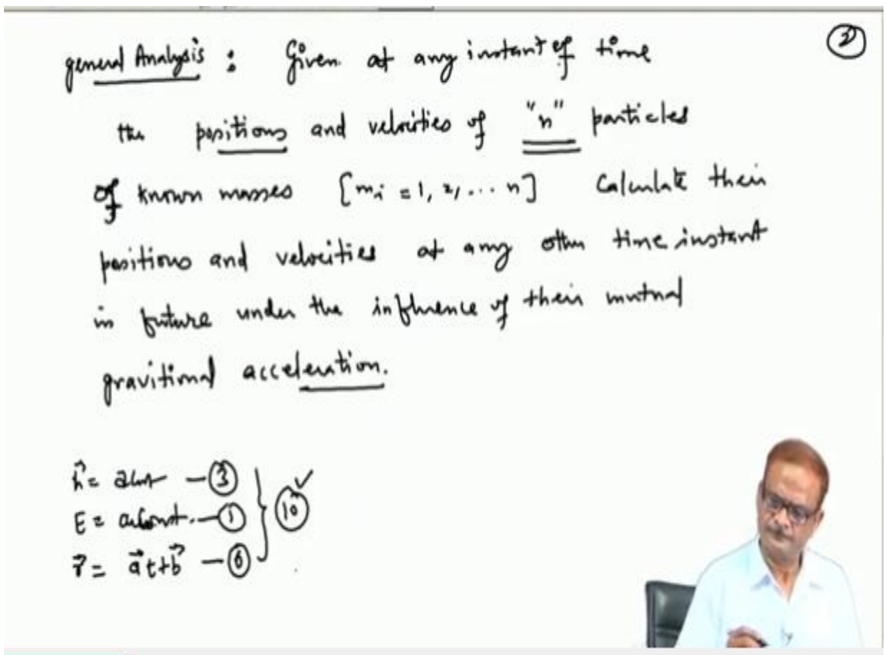

Given, so the general analysis is nothing but we want to get a close form solution but obviously, it is not possible, so I am just writing a definition here at this point; instant of time, given at any instant of time, the positions and; positions means it is a position vector, positions and velocities of n particles of known masses, so masses are known, let us say initial position and velocities are known, so obviously you can integrate the equation of motion, you can get the solution.

But that too has restriction that after some time it will start diverging, okay, we have various issues with numerical solution because your numerical method is not infinitely precise, it has finite precision, so error starts building up and especially if you look for the unstable systems, so if your system is unstable, so small error it will cause the system to diverge very fast while here in this case, this is not the case, you can propagate it for a longer time.

But ultimately, it is, it will start flowing anomalous behaviour, calculated their positions and velocities at any other time in future under the influence of; under the mutual gravitational acceleration, so here it is a general statement about n particles, so we will do some generalised presentation for this means, if you remember that for the 2 particle system we have got that the total angular momentum h is a constant, energy is a constant.

And also its centre of mass, it moves with constant velocity, so there we got 6 number of constants, so here this is 3 number of constant, this is 1 number of constant and r we have written as $\vec{a}$t + $\vec{b}$, so here total 6 number of constants are involved, so total 10 number of constant we were able to identify. In the same way here also we can identify 10 number of constants but before that it is better to write the equation of motion quickly and understand various points about this.

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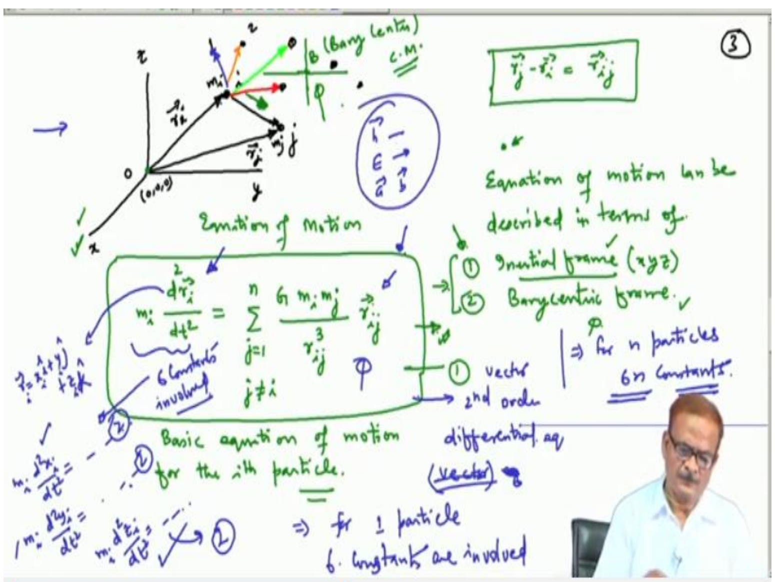

So, this is x, y and z reference frame, this is the origin, origin we can show this as may be as O and there are number of particles, so n number of particles, so let us say this is (1) and somewhere this may be (2), this is the i_{th} particle and this is the j_{th} particle, so here we so the radius, this position vector of this as r_i and position vector of j as r_j which along the same line what we have done for 2 particle system, its mass is m_i and this mass is m_j.

So, what we are interested in that we find the equation of motion of j with respect to i or maybe whatever way you want to say, here you want to find out the equation of motion for this with respect to this, how the motion will appear, also we can do the same thing in terms of the barycentric frame like say the centre of mass as we know; as we will derive that centre of mass in this case also does moves with constant velocity.

So, either it is a fixed or either it will moving with constant velocity depending upon that you can get from the initial; if the initial conditions for all the particles are available, you can calculate it, so this maybe, this is the barycentre. In the case of the solar system we call it the barycentre or we can call this as the centre of mass, okay. So, we will write the equation of motion for this and thereafter we will work out further.

So, dr_i; d^2r_i/dt^2, m_i is the mass, so on the i the force is acting here in this direction as shown by the this green arrow and the notation we are going to use, we will write

$$r_j - r_i = r_{ij}$$

this is the notation we are going to follow, okay. Some of the books may refer this as instead of writing r_{ij}, they may write it as r_{ji}, so irrespective of the notation, you use the physics will not change, okay.

So, here in this case what is the force acting on the particle i due to the z_{th} particle, so this will be m_i times m_j multiplied by the universal gravitational constant and then the distance between these 2 point particles, so r_{ij}^3 and it acts in the direction r_{ij}, it is directed along the arrow as it is shown.

$$m_i \frac{d^2\vec{r_i}}{dt^2} = \frac{G m_i m_j \vec{r_{ij}}}{r_{ij}^3}$$

So, this way if you see, so second particle this will also attracted, this particle will also attract, in the same way this particle will also attract.

So, as many particles are there, all those particles will attracted, this is the one, 1 here in this point, so this will also attract, so this implies that we need to sum it over all the particles, so we need to sum it from j equal to 1 to j equal to n, if n is the number of particles but particle on itself it cannot apply any force and therefore we need to write here i not equal to or j not equal to y; j not equal to y.

Because a particle cannot apply force on itself, so we have to eliminate that term, so this constitutes this is the basic equation of motion for the ith particle and this will utilise in our formulation, so our equation of motion can be described in terms of the inertial frame as shown here x, y, z in terms of Barycentric frame, so these are the 2 things you can; you need inertial frame basically to write the equation of motion.

So, we can do it in the inertial frame or either in the Barycentric frame, where inertial frame is x, y, z as shown here and your Barycentric frame is shown here in this place, however once we are looking for the relative motion, okay so motion can also be described with respect to the Barycentric frame for the individual particles and then also for; for the individual particle also you can write with respect to the inertial frame.

And we will see that there is no difference either if you choose either the Barycentric frame or either the inertial frame but once we are looking for how one particle appears to move with respect to the another particle, so that gets reduced to basically, the relative motion. So, anyway whatever we do, if we are trying to look for the relative motion as we have done earlier also, so in that case you cannot validate the Newton’s second law.

So, first we have to write the equation of motion in the inertial frame itself and here we have 2 inertial frames, one is the xyz, another one is the Barycentric frame, these 2 options are available to us, okay and thereafter you write the equation and then whatever the way you want to describe in terms of the motion can be purely in terms of Barycentre with respect to Barycentre how it is moving or it may with respect to some particle how the other particles are moving that can be the issue, okay so, this we write as our equation number 1, okay.

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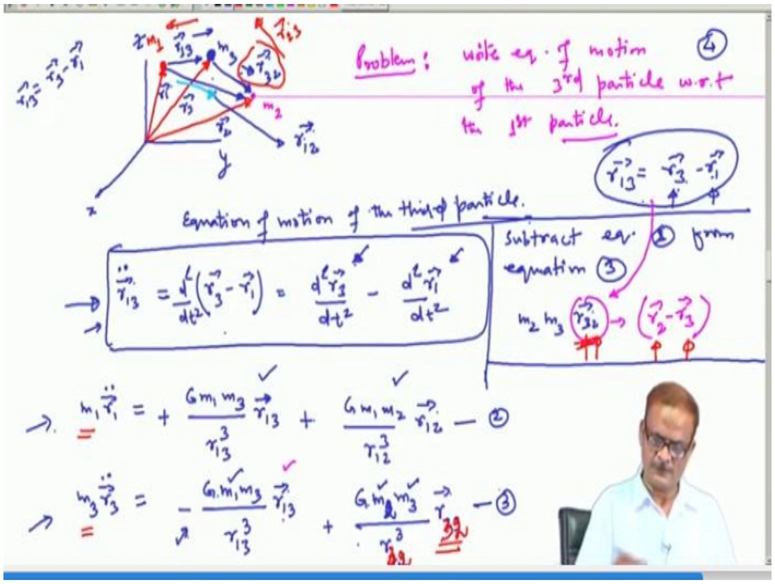

If we sum over all the particles; the previous equation we have written over and if we sum over all the particles, so we get in that case, we get something and shortly we are going to work it out, what does that mean. So, that will get reduced to of course the case where already I have stated, the centre of mass of the system moves with constant velocity, one more thing I would like to point out here, as we see here in this place, r consist of x$\hat{\imath}$ + y$\hat{\jmath}$ + z $\hat{k}$, where $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ they are the unit vectors.

Because you can see that if you break it up in terms of this is the vector equation of second order; second order differential equation, differential vector equation, let me write here, this is second order differential vector equation, differential equation this is vector or simply we can write vector second order differential equation instead of writing here. So, this implies that we can break into 3 parts, we can write it like this; m_i times d^2x_i/dt^2.

On the right hand side this we can break and we can write here, similarly m_i times d^2y_i/dt^2, we can break and write something on the right hand side whatever comes from this place and the same way, d^2z_i/dt^2, this equal to; so this is the second order scalar differential equation, this is also a second order scalar differential equation, this is also a second order scalar differential equation.

So, how many constants will be involved with each of them; say 2 with them, 2 with this, 2 with this and 2 with this, so total 6 constants, so for this you will get 6 constants involved. So, 1 particle

6 constants are involved, so this implies for 1 particle, 6 constants are involved, hence for 3 particle or n particles, how many will be there? So, this implies for n particles, 6n constants will be involved.

Because for each particle you can write 6, for each particle you can write this sort of differential equation and each of them it is giving you 6 constants and therefore, 6n total constants are involved in the system. So, before we write the equation in a generalised form like we have been writing here in this place, so we will return back to this equation again, we need this equation.

But before that let us look into; so, what I intend to tell here that we will look into the general properties of equation of motion, so general properties already we have, I have stated h is constant and E is a constant and then your centre of mass moves with constant velocity, so for that we got the constants a and b but right now we try to our intention is to describe the motion of a particular particle with respect to the another particular one.

So, first we will write for 3 different particles and then we will generalise, here one particle, this is another particle here and this may be another particle here in this place, so let us say this is m_1 and here this is m_2 and this is or m_3 we will write here, this is a m_3 and m_2 is here in this place. So, problem here is; write the equation of motion of the third particle with respect to the first particle, this is our objective.

And thereafter, we will generalised to n number of particles, so what we are looking for that the motion of the third particle about the first particle, this vector we will write as this is r_2, this is r_3, r_1, therefore this becomes r_{31}, this becomes r, this is 13, so here we need to correct, r_{13}; r_{13} means $r_3 - r_1$, so following this same sequence we get here r_{32} and then this becomes r_{12}.

Equation of motion of the third particle we need to work out; so r_{13} double dot this will be

$$\ddot{r}_{13} = \frac{\mathrm{d}^2(\overrightarrow{r_3} - \overrightarrow{r_1})}{\mathrm{dt}^2} = \frac{\mathrm{d}^2\overrightarrow{r_3}}{\mathrm{dt}^2} - \frac{\mathrm{d}^2\overrightarrow{r_1}}{\mathrm{dt}^2}$$

this implies that we need to get here the equation of motion for the third particle and the first particle separately and then if we subtract it, so we can get this quantity. See here in this form if I write it like this, this has nothing to do with the Newton's equation, okay this is just an expression written here; r_{13} equal to $r_3 - r_1$.

And if I differentiate twice, so we get this equation here in this place and nothing to do with Newton's equation now, we write the Newton's equation, so we have $m_1 \ddot{\vec{r_1}}$, this is the equation of motion for the first particle, so this is being acted by the other particles whole cube, r_{13}^3, r_{13} because we are writing in the vector notation, so we tend to write it in this way.

And remember here the sign is plus not minus because this force is acting along this direction not in the opposite direction, so here this is plus sign and what else, the other force is acting, so the other force is which is acting is I am showing it by the blue line here, it is acting along this direction by m_2,

$$m_1 \ddot{\vec{r_1}} = \frac{Gm_1m_3\overrightarrow{r_{13}}}{r_{13}^3} + \frac{Gm_1m_2\overrightarrow{r_{12}}}{r_{12}^3}$$

this is our equation (2). In the same way we have $m_3 \ddot{\vec{r_3}}$, this can be written as G, so force is acting on this will be due to 1.

And which direction this is acting; this is acting opposite to the r_{13} direction, so now here in this case, it is acting here as I am showing by this red line, so therefore here this will appear with a minus sign and then on this particle, the second particle, the third particle the force acting due to the second one, so this also we have to write, so G m_3 or first we write m_2 m_3 divided by r_{23}^3 and vector from 2 to 3 and we have to see whether the plus sign will exist before this or minus sign.

So, we are writing in terms of r_{23}, also we can write in terms of r_{32}, it is not a problem both are okay, so on the third particle the forces are acting because of the first; so here because of this the negative sign appear and on the third particle again the force is acting along the by this force acting due to the second particle it is along the r_{32} direction, 23 whatever you want to show, you can show.

$$m_3 \ddot{\overrightarrow{r_3}} = \frac{Gm_1m_3\overrightarrow{r_{13}}}{r_{13}^3} + \frac{Gm_2m_3\overrightarrow{r_{23}}}{r_{23}^3}$$

If you write r_{23}, so this will come with a positive sign, if you write here r_{32} this will be coming with a negative sign, so both are okay, so it is just a matter of later on you can convert, now this is our third equation okay. Now, if we subtract what we are looking for; for this particular form, it should come in the form of r_{13} and for that we need to subtract from r_3, the r_1. So, here what we are going to do?

We are going to subtract equation (2) from equation (3) and if we do that so we can write here and before that we need to do something more, we need to divide both side by m_1 here in this case and m_3 here in this case and if we do that, so this will cancel out because it is a nonzero quantity, so we can eliminate it, we can go to next page and write in one more step rather than compressing here.

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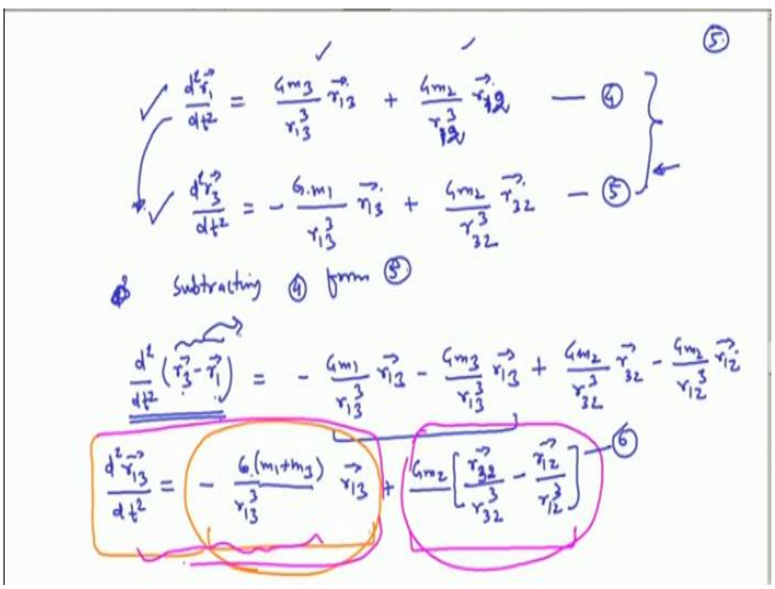

So, we have

$$\frac{\mathrm{d}^2\overrightarrow{r_1}}{\mathrm{dt}^2} = \frac{Gm_3\overrightarrow{r_{13}}}{r_{13}^3} + \frac{Gm_2\overrightarrow{r_{12}}}{r_{12}^3}$$

this is between 1 and 2, so we will have r_{12}, this is between 1 and 3 and similarly, we will have

$$\frac{d^2\overrightarrow{r_3}}{dt^2} = \frac{Gm_1\overrightarrow{r_{13}}}{r_{13}^3} + \frac{Gm_2\overrightarrow{r_{23}}}{r_{23}^3}$$

these are actually very small things and we can write it quickly just for explanation, so it is I am taking time, so subtracting (4) from (5), so this is the equation written in inertial frame, this is also the equation written in inertial frame and we have just remove the mass by dividing both sides.

And no way this is defined Newton's law and therefore it is a 100% correct, only thing we need to do that we have to subtract 4 from 5 and once we subtract, we get this equation, therefore the left hand side this becomes r okay, here if you look here in this place, $r_3 - r_1$ we have written as r_{13}, so therefore this is r_{13} by dt^2. So, this quantity is nothing but the position vector of 3 with respect to 1.

So, how the position vector of mass 3 with respect to 1, it is changing, this is what it say implying and if we take the first derivative, this is the velocity with respect to the 1 and if we take the second derivative, so it shows the acceleration, okay in no way, it is an incorrect and now, these 2 terms can be combined together, G we can take it outside, r_{13}^3 $(m_1 + m_3)$ and if you can recognise, this term is what it appears like, suppose the second body was not there.

So, in that case this particular part let us say you are referring to the 2 body problem, how the mass 3 is moving with respect to mass 1 but if the other mass is also present which is here in this case is the mass 3, then the situation changes and then we can write here, Gm_2 can be taken outside and r_{23} divided by $r_{23}^3 - r_{12}$ divided by r_{12}^3 and with this equation , the extra term which is appearing here, this extra term, this is the perturbative term.

$$\frac{d^2\overrightarrow{r_{13}}}{dt^2} = -\frac{G(m_1 + m_3)\overrightarrow{r_{13}}}{r_{13}^3} + Gm_2\left(\frac{\overrightarrow{r_{32}}}{r_{32}^3} - \frac{\overrightarrow{r_{12}}}{r_{12}^3}\right)$$

So, you have 2-particle system here and the third particle let us say acting like it is distorting the motion of the these 2 particles with respect to each other, okay so we are going to wind this part; wind up this part now, we will name this equation as equation number 6.

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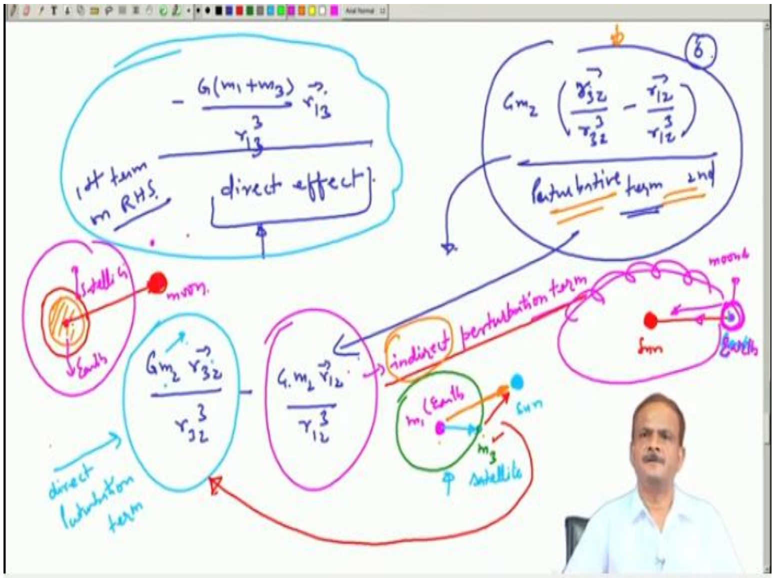

This is also called a direct effect, this is 3 direct effect the first term, this is the first term on RHS; on the right hand side and the second term which we have written as Gm_2 r, let us check this, r_{23}, okay here just I am verifying this particular part; r_{23} we have written here, r_{23} is a vector from this side to this side, this is r_{23} and r_{32} is shown here, it is from 3 to 2, so the force acting on the third particle due to the second one and this force will be acting along the along this direction from 3 to 2, so this is r_{32} m_3, m_2 and this equation we are using here.

So, this notation we are picking up, about the notation, we have to be little careful that we do not do any mistake, therefore this quantity is according to this notation, $r_2 - r_3$, so we need to do the correction here, so as we were following this notation, this r_{32} will be the vector directed from 3 to 2 that means, we have to subtract from 2 this 3 and therefore we should write here as r_{32}, rather than writing r_{23}, this is r_{32}.

So, if we follow this notation, so about the notation be careful because we can do this error, r_{32} and here also in the denominator also, we will have r_{32}, now it is okay. So, here also we need to change in these places, r_{32}, therefore the second term which we have here Gm_2 ($r_{32}/r_{32}^3 - r_{12}/r_{12}^3$), this is the second term and this is perturbative term.

And what does this mean; this is direct effective on this term, this is just acting like a 2-body system which is very much visible, this term here which I have written as a perturbative term there are 2 terms involved in this, so we break it up, Gm_2 r_{32}, if you look here in this particular term, this is the effect of the second body on the third body, so this is the direct perturbation term.

And here this term, this is indirect perturbation term, suppose I cited by taking one example, let us say this mass m is earth, m_1 is earth and there is a satellite moving around this which I will show by some other colour, around this another satellite is moving, this is m_3 and sun is located somewhere here, this is your sun and this is satellite, so motion of m_3 this satellite about 1, so this term is taken care here in this place, this particular thing, this is your just 2 body problem.

But the extra bodies present here which is sun, so because of this sun, you get one force here in this direction and that term this appears here as this is the direct perturbation, so it is perturbing the motion of the particle m_3 and this term which I have written as the indirect perturbation term, this acts by acting the effect of sun is manifested on the earth, okay this will affect; the sun will affect earth and in turn, the earth will affect m_3 which is the satellite.

And therefore, this is not a direct term, this is indirect term, as I have written here but overall these 2 terms together except all the perturbative term and you can see here that because of the presence of sun, the motion of the satellite around the earth, if this is the earth and about this the satellite is moving and about the sun, an earth mass system which we call here in this case the Barycentre or the centre of mass of the sun earth system, so they are moving about their mutual centre of mass and which lies inside the sun itself.

Because sun is very heavy around 30 lakh kilometre sun’s diameter is and it is a very massive as compared to the earth, therefore what we can observe that earth is moving around this but because of the presence of the sun, the motion of the satellite will get affected and you can also consider this from the point of the view of; this is the sun here and earth moving around this, this is the earth and then the moon is there about this.

So, how the motion of moon will appear? Now, both the systems, here the moon is also accelerating toward the sun and earth is also accelerating toward the sun but primarily, the moon is; if we look from the earth, so moon is circulating, it is going around in an orbit about the earth, this is your earth and here this is moon, moon’s orbit and therefore, moon orbit as the it goes around in the ellipse, in the elliptic frame.

So, moon’s orbit will appear like this, okay, it is going around the Sun also and simultaneously, it is moving around the earth and because of that it appears like this, so moon’s orbit it also gets perturb because of the sun. Similarly, if you have a satellite here and if suppose this is moon and this is earth, so moon will also affect the satellite motion and then the other planets also they will affect including sun.

So, all these things it is affecting the motion of this satellite, therefore it is a very important to if you are looking for a long term propagation of the satellite that say after 3 months, where my satellite will be in the space, what its orbit will be; we need to account for all these perturbation and one of the objective of our discussion will be also this parameter, variation method which is part of general perturbations, so we will get the differential equation of motion for the say if any satellite which is perturbed by other planets.

And it is a matter of derivation and that derivation is pretty difficult also and pretty lengthy also and whatever we are developing here it is going to help us in understanding the parameter variation method, so thank you very much and we will continue in the next lecture.