Space Flight Mechanics
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Lecture – 32
3-Body Problem (Contd.)

Welcome to lecture number 32 so we have been working with 3-Body problem we will continue with that. So, if you remember in the last lecture, we discussed about the Bary centric reference frame and motion with respect to that.

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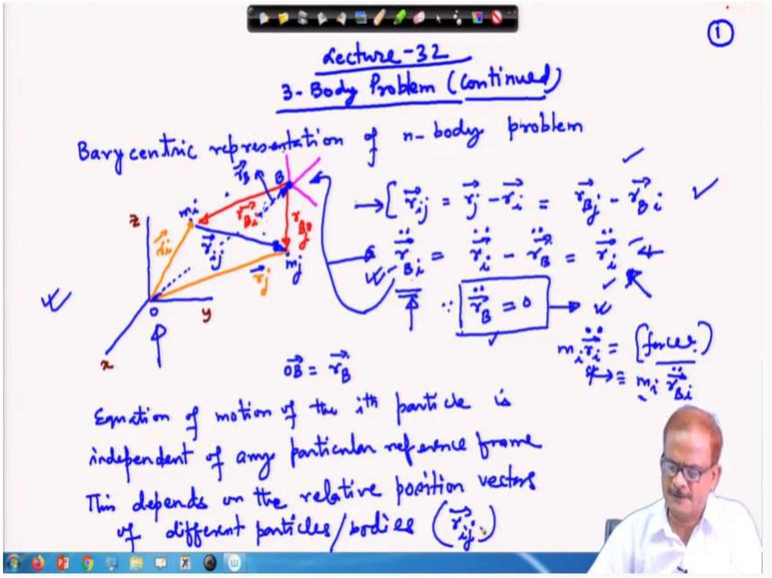

So in that context we worked out with this figure okay so here r_{ij} as shown here in this equation

$$\overrightarrow{r_{ij}} = \vec{r_j} - \vec{r_i}$$

as it is written here

$$= \overrightarrow{r_{Bj}} - \overrightarrow{r_{Bi}}$$

so this already we have written in the last lecture moreover r_{Bi} is this quantity that is here what I am showing by this symbol here let us say this one r_{Bi} this quantity it is a double dot means it is a second derivative with respect to the time.

So, this is given as $r_i - \ddot{r}_B$ and since

$$\overrightarrow{\ddot{r}_B} = 0$$

because with respect to the this centric reference the Bary centric frame centre of where it is located this point which is the origin of the Bary centric reference frame this is non accelerating.

So, the vector from this place if I draw from this place to this place to this point so this is your r_B this vector becomes from here to here and this is your r_B from this origin to this point.

And as we know that what a cluster of particles the centre of mass either most is constant velocity or either it is at rest and therefore that means the motion of any particular field are trying to write the equation of a motion for any particle with respect to any inertial reference frame the physics does not change it remains the same so r_{Bi} = $\ddot{r}_i$ this is the simple conclusion and already we have discussed about this particular request in the last lecture okay.

So, we can state the equation of the its particle what we are trying to write here m_i times $\ddot{r}_i$. On the right-hand side only, forces will come here in the vector format and because $\ddot{r}_i$ = r_{Bi} so this you are writing with this frame and r_{Bi} this you will write with respect to this frame the Bary centric reference frame. Because both are equal so you can just replace it that means this is also m_i times $\ddot{r}_{Bi}$. So, it is the same okay because you have this quantity turns out to be equal to this quantity therefore, I can write in terms of this or either we can write in terms of this.

So, equation of motion of the its particle is independent of any particular reference frame this depends on the relative positions only vectors of different particles slash bodies because the force will depend on that means it is the only it depends on r_{ij} position of one with respect to the other okay.

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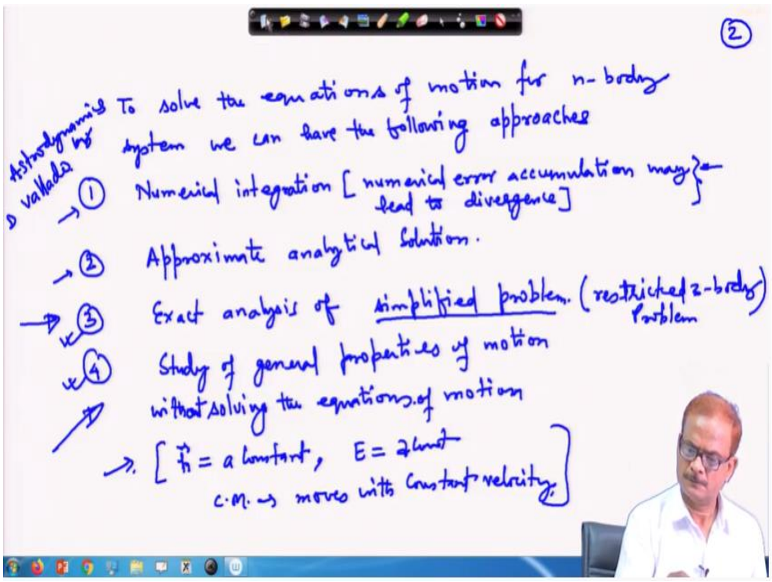

So now we will enter into some of the properties for the 3-body problem or maybe the n-body problem. To solve the equations of motion for n-body system we can have these approaches we can have the following approaches one is numerical integration of course we are aware of all these things well this is not as simple I just stated here because the numerical error accumulation may lead to divergence.

So, we have to be very careful in applying this and this part of course it is not part of our curriculum here so I will not go into this you are going to look into the Vallado astrodynamics by D. Vallado David Vallado approximate analytical solution okay this also if you have not doing the third one is exact analysis of restricted problem or the simplified problem exact analysis of simplified problem. Study of general properties of motion without solving the equations.

So simplified problem in this category we will take this as the restricted 3-body problem and in this part so this we are going to deal with and the third and the fourth. The study of general properties of motion without solving the questions of motion. So, in this what we are going to do already we have looked into we will prove that total angular momenta h is a constant total energy this is a constant and a centre of mass moves with constant velocity.

So, already we have done this for the two-particle system and here we will for n particle systems so I will do it very fast because there is no point in elaborating the same thing again and again because this is part of the curriculum so I wind it up fast. So, centre of mass most of with constant velocity so these are the three things in this we can prove so first we will do the fourth path and thereafter we will come to the third part.

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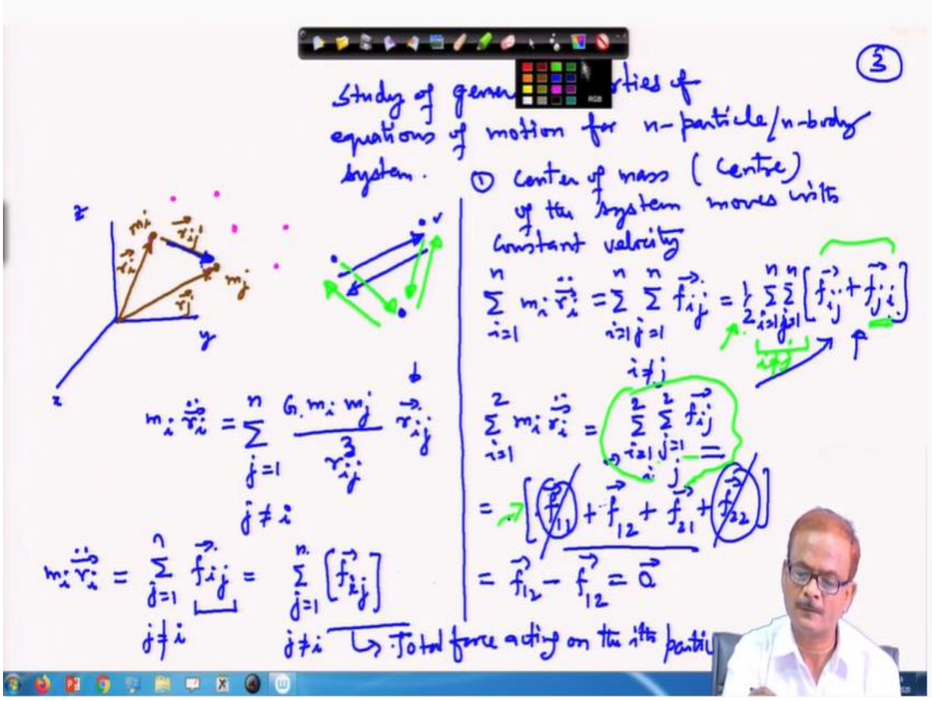

n particle or n-body system and for this we need to start with the scratch and the write the equations of motion. So we are starting here this is the difference frame x, y, z and we have multiple particles here and let us say that this is the its particle where mass is m_i and this is the jts particle m_j where mass is m_j here this is r_{ij} this vector is r_j and this vector is r_i.

So, if we write the equation for the its particle this can be written as

$$m_i \, \ddot{\vec{r}}_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{G m_i m_j}{r_{ij}^3} \, \overrightarrow{r_{ij}}$$

and this sign here will be positive because this vector is directed from this place to this place r_{ij}. So, r_{ij} is a vector in the direction of the force therefore here plus sign will appear.

So, this plus sign I will remove sum it over all the particles if we sum it over all the particles that will be j = 1 to n but because the its particle cannot apply the force on itself and therefore, we need to take care of that also. So j_0 = i m_i times $\ddot{r}_i$ therefore the quantity which is present here if you can write it in this way j = 1 to n f_j this term is nothing but the force action due to the its particle due to the jts particle and then we have to integrate over all the particle j = i.

This can also be written as

$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\overrightarrow{f_{ij}} + \overrightarrow{f_{ji}})$$

what does this mean? Say if we have two particles here okay the have to do one more step here before let me explain it completely first I will remove this right now this is not required in this part I will remove and therefore this also will remove here.

Now this is the force on the right-hand side we have the force acting on the its particle total force acting on the its particle. Now our objective is to prove that three properties I have mentioned there at the centre of mass moves with constant velocity total angular momenta it is a constant and another one was total energy of the system it is a constant.

So, the first one the centre of mass we also write this as centre of mass of the system moves with constant velocity. Here system means the system of particles this is the first issue here. So, what we do that we sum over all of the particles if you do this so on the right hand side we will have j = 1 to n f_{ij} now at this stage we write it this way 1/2 (f_{ij} + f_{ji}), j = 1 to n i = 1 to n. why we are doing this that will be obvious from the following thing.

Suppose we write here $m_i\ \ddot{r}_i$ and sum it from i = 1 to 2 so on the right-hand side we will have i = 1 to 2 and j = 1 to 2 f_{ij} okay. Now if I write according to this part so this will be 1/2 times summation these things I am not writing here just for shortcut. So, if we expand it let us expand it directly here in this place it will become if we write j = 1 to 2 so f_{12} and f. First, we can write i = 1 if we write i = 1 the way of expanding it is if I choose i - 2 and thereafter expand j =1 to 2 this will 1,1 j = 1 but already we have written that i = 0 = j.

So, this quantity will not be present okay thereafter we will have f = 1 and j = 2 so this quantity will be presented because i not equal to j the other part will be similarly i equal to then will choose 2. So, f_2 and j = 1 so this is okay and f_{22} then it will come this is not possible so this term drops out so once we expand it then here in that case you do not require this part so once we are writing here in this way and then we have got to expanding so 1/2 is not required here you will require once you write it in this fashion.

Therefore, f_{12} this is nothing but = f_{21} is nothing but = f_{12} okay and therefore this quantity is 0. $\vec{0}$. It is very simple so this way if you look into this part and to make it convenient I have written it in this way because there will be corresponding if you have one particle here so the force acting on this particle is in this direction so you will have also force acting on this particle in the opposite direction.

Similarly, one particle is here so you will have force acting on this particle in this direction and that also will be force acting on this particle in this direction. Similarly, force acting here in this direction force acting in here they will adjust in period. And therefore if I write using this notation and obviously here i is not equal to j this always we have to write we can expand it here in this way and 1/2 factor then we have to write it here you can check it is easy because extra term we have introduced here and for that reason we are putting here 1/2 but here it is not required in these places it is not required because we are directly expanding this term we are directly expanding it i is not equal to j.

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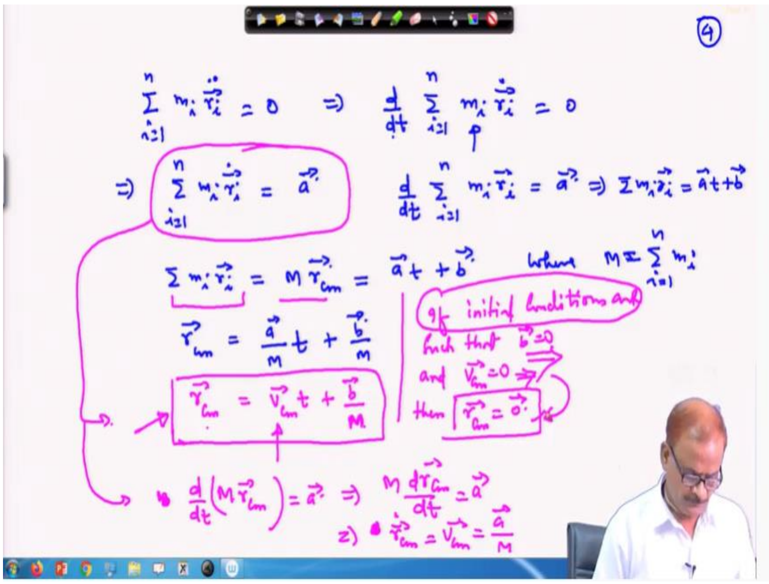

So, therefore what we see that m_i ri double dot this quantity summed over i = 1 to n this quantity is 0 and 0 is a vector here though always I will not put here this does not look good and therefore we will simply write here 0. Now if we take the differential sign outside it can be written like this okay?

Because the mass is constant and therefore this is applicable the differential operator and the summation operator they can be exchanged. So this implies

$$\sum_{i=1}^{n} m_i \dot{\vec{r}_i} = \vec{a}$$

at the next level then again we can take out the differential integrator outside write it this way m_i this implies $at + b$ and we know m_i r_i this is nothing but m times r centre of mass which we have written as the Bary centre.

So, this quantity is at + b where m = summation m_i i = 1 to n so this the definition of the centre of mass and from here what we can observe that this can be written as a/m t + b/m and from this equation we can observe that this quantity is nothing but the centre of mass you just ate it and this quantity can be written here at us c m times centre of mass you just rewrite it this quantity can be written here as m times r centre of mass d/dt because the quantity here we have written this quantity is M r_i.

$$\overrightarrow{r_{cm}} = \overrightarrow{v_{cm}}t + \frac{\vec{b}}{M}$$

So you can write her to get in this way this gets reduced into this format and this implies M times

$$\frac{d(Mr_{cm})}{dt} = \vec{a}$$

$$\dot{\overrightarrow{r_{cm}}} = \overrightarrow{v_{cm}} = \frac{\vec{a}}{M}$$

So, the position of the centre of mass it depends on the velocity of the centre of mass if initially initial conditions are such that if initial conditions are such that b = 0 and v_{cm} = 0 then r_{cm} also becomes 0 means it remains at rest.

But if v_{cm} is not equal to 0 in such case this keeps varying so only under this condition are such that v = 0 and v_{cm} = 0 then only you will get r_{cm} = 0 otherwise centre of mass of the n particles system it will move with constant velocity.

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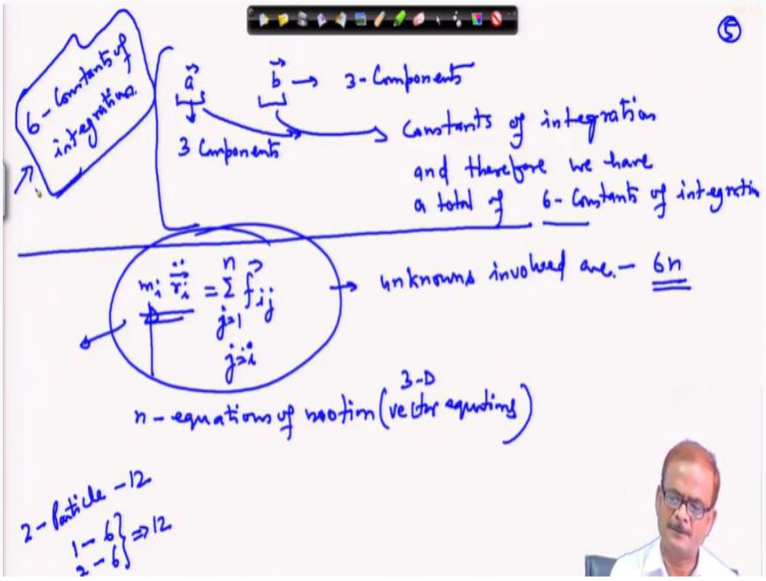

So what we have got here this constant a and b because it is a three dimensional vector we are dealing with r_i which is three dimensional so we can observe r_i this is three dimensional x_i it has three components r_i can be written as

$$\vec{r_i} = x\ \hat{e_1} + y\ \hat{e_2} + z\ \hat{e_3}$$

and we can put here I to indicate that this is corresponding to the its vector.

So this is a two dimensional vector so to be consistent on the right hand side all the vectors must also be three dimensional this has three components and this has also got three components and because these are constants of integrations all these are constants of integration and therefore we have a total of six constants of integration. So, this way we are able to identify in the first part the six constants of integration.

Next we have and how many constants are involved because of our system your equation m_i times $\ddot{r_i}$ this is our equation of motion on the left hand side and on the right hand side we have f_{ij} j = 1 to n and j_0 = i this is the thing given to us. So, this is the equation of motion and here because we have n number of particles so we will have n equations of motion and all these are vector equation 3-D vector equations we have three-dimensional x, y, z.

So, this gives us how many constants of integration are or unknowns are involved here. Unknowns involved are total of already we have discussed for the two-particle system what we have observed we have total 12 constants why because for one particle we have 6 constant and for the second particle also we have six constants so we have a total of 12. So, here in the three-particle case for each of them we will have six constants involved. So, unknown involved are 6 because there are n particles unknown involved are 6n.

So out of these 6n right now we had been able to identify just 6 constants of integration we are looking into the general properties of the motion so one of the general property is that the centre of mass moves with constant velocity and what else the other will be the total angular moment of the system or the particle system of the particles or the n body system is a constant and the lastly last one is the total energy which is nothing but the potential plus kinetic energy and that is constant. So, we quickly do that part also out of this only we are knowing right now the 6 constants.

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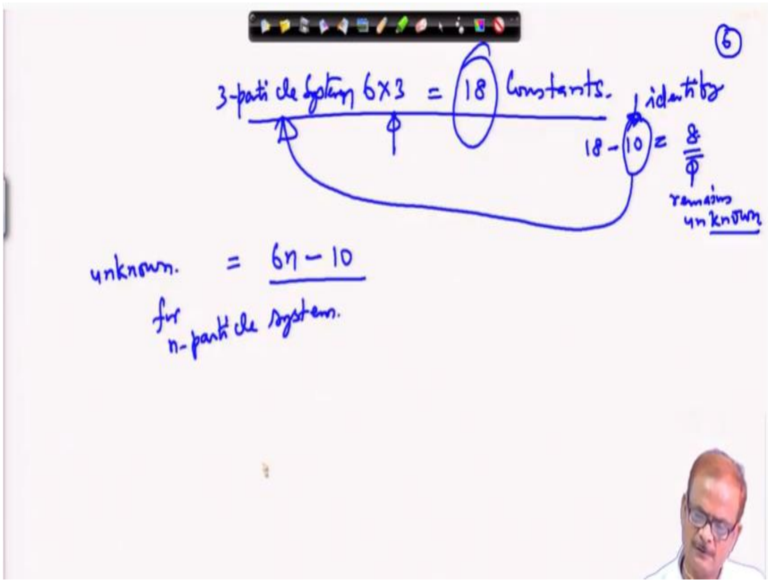

If we have three particles system so we will have 6 into 3 total 18 constants or 3 particle system or the 3-body system and all of that we will be able to identify 8 - 10 and we will be able to identify only 10 and 8 remains unknown. Again, I am stating that if the initial conditions are known so you can always numerically propagate the system of equations but the closed form solution you are not getting.

Once you get the closed form solution from the differential equation so for the three particle system 18 constants will manifest here but we are not able to identify those constants only 10 constants we are able to identify for the three particle system and for n particle system we will have 6n and out of that we will be able to identify only 10. So, this is the remaining unknown for n particle system we would not be able to identify all these constants.

So here in this part again I am stating that if you know the initial condition so as I told you that if you know the initial condition you can integrate the system equation and propagate the state. But also, you can check from this place that if the initial conditions are known you will be able to identify this constants a and b given the initial conditions, initial conditions are initial velocity and initial position vector.

And it is quite simple to see that last year 12th to B. Tech you might have all learned all these things so I am not going into all those details how to solve that equation and find out the solutions okay. So, we were here in this point now our next step is to prove that the angular momenta of the system remain constant.

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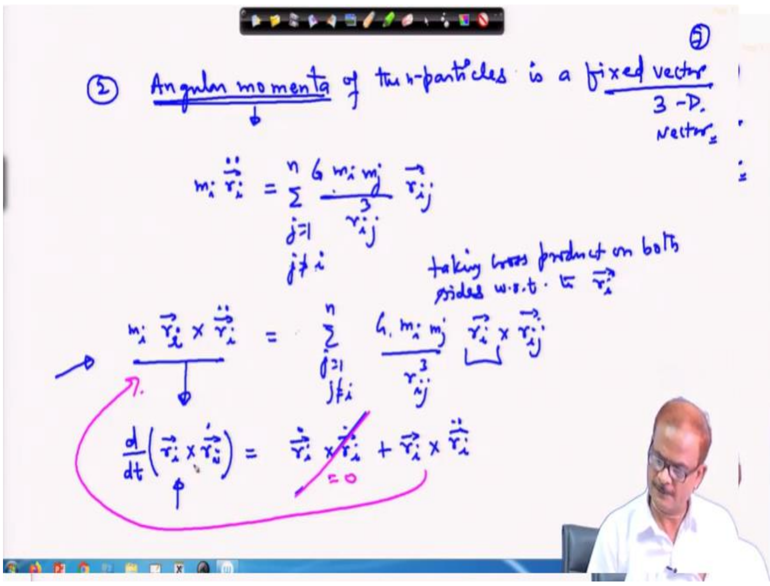

So the second general property of motion is angular momenta of the particles is a constant of n particles fixed vector or either a constant vector, constant means here because it is a angular momenta is nothing but summation of angular momentum once you sum it up along the x, y, z axis you get the component so this is also a vector of three dimension or three dimensional vector this is 3-D vector.

Now taking the equation of motion we have written this as

$$m_i \ddot{\vec{r}}_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{G m_i m_j}{r_{ij}^3} \vec{r_{ij}}$$

With this if you remember in the past for the two particle system we have used one technique that d/dt r_i cross $\dot{r}_i$ if we differentiate use this technique of taking the cross product on both sides with respect to r_i you can see that the left hand side can be generated from this differential if you differentiate so this becomes $\vec{r_i} \times \vec{r_i} + \vec{r_i} \times \ddot{\vec{r_i}}$ but the quantity which appears here is 0 this is 0 okay because it is a cross product of $\dot{r_i}$ and $\dot{r_i}$.

So, therefore this recovers here in this place so we used this technique and this has been a very useful technique you will find helpful this technique helpful in solving many of the (38:21) related problems especially to the planetary motion.

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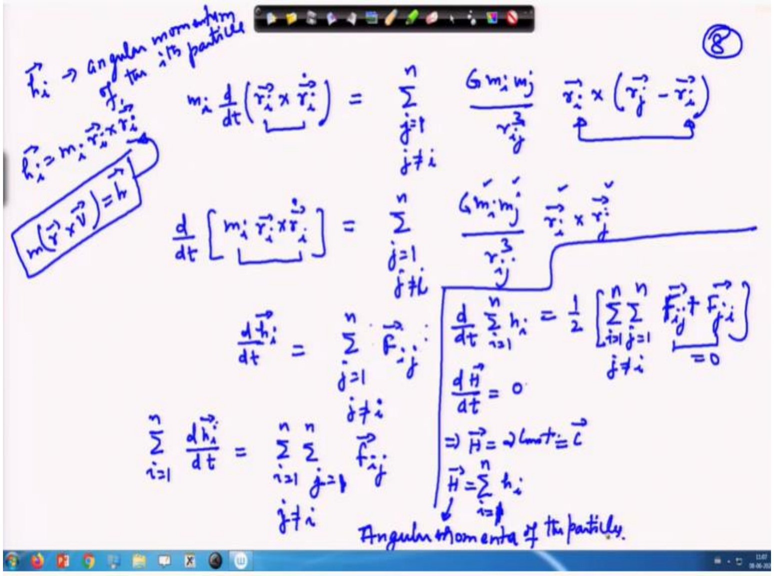

Okay on the left hand side we can write here

$$\frac{d}{dt}(m_i\, \vec{r_i} \times \dot{\vec{r_i}}) = \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{G m_i m_j}{r_{ij}^3}\, \vec{r_i} \times (\vec{r_j} - \vec{r_i})$$

So, therefore the right hand side gets reduced to j = 1 to n j_0 = i and you can see that this cross product this will vanish and therefore we are left only with $r_i \times r_j$ and on the left hand side we will have m_i and as you can see that we can take this d/dt outside.

So m_i times $r_i \times \dot{r_i}$ this is nothing but m_i times r_i cross you have to write it like this. This is the angular momentum of the its particle so this by so this is d/dt (h_i) you can write it this way angular momentum of the its particle. Now here look here in this place this is i and j here also r_{ij} so we can assume this to be some sort of function let us say I represent it using f_{ij} j = 1 to n j_0 = i and if we do the summation over all the particles j = 1 to n j_0 = i this is f_{ij}.

So, from here what we observe that this gets reduced to

$$\frac{d}{dt}(\overrightarrow{h_i}) = \sum_{\substack{j=1 \\ j \neq i}}^{n} \overrightarrow{F_{ij}}$$

F_{ij} in this case we are representing times F_{ji} and this quantity is obviously 0 so the right hand side is set to 0.

And therefore this gets reduced to

$$\frac{d\vec{H}}{dt} = 0$$

obviously this is a vector here which I am not putting and this implies H is a constant vector

$$\vec{H} = \vec{C}$$

which let us write this as C and so what H is H is summation over all the particles i = 1 to n h_i. So, h_i is your angular momentum of the its particle and h_i we have written as m_i times r_i cross $\dot{r_i}$, $\dot{r_i}$ is nothing but v. So, r times r cross v multiplied by m this is basically the equation of momentum of particle n mass all right dot is nothing but v so r times r across v multiplied by m and this is basically the equation of our angular momentum of a particle m.

So, following this notation we have h_i defined here and this is the total angular momenta therefore this h is total or simply you can say this angular momentum of the particles. So this is the second property we have proved here and h has three components, h is a vector which consists of H_1 $\widehat{e_1}$, H_2 $\widehat{e_2}$ and H_3 $\widehat{e_3}$ here these three constants are identified if you know the initial position and velocity of all the particles so we will be able to get H_1 , H_2 and H_3 . So how many constants we have identified.

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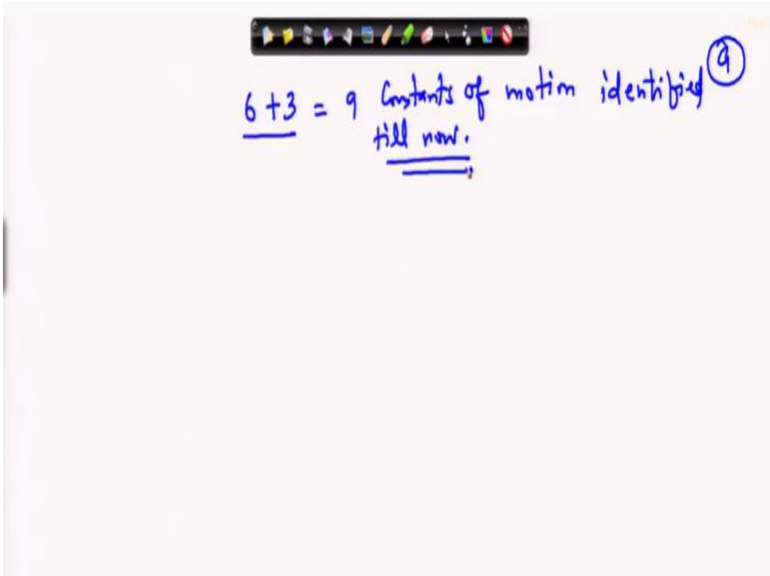

We have identified earlier the 6 and now we are identifying 3 so total of 9 constants identified till now 9 constants of motion identified till now rest one more we will be able to identify and thereafter we will discuss about this solubility of this problem of the n-body problem or the 3-body problem. So, we will continue in the next lecture we will stop here thank you very much.