Space Flight Mechanics
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Lecture 34
Restricted 3-Body Problem

Welcome to lecture number 34. Today, we will start with restricted 3-body problem. Already, we have discussed about the general properties of the 3-body problems and we know that from our previous discussion 3-body problem cannot be solved. So in certain restricted cases, its solution can be worked out. Therefore, we are trying to discuss about this restricted 3-body problem, that is why this notion comes, restricted 3-body problem.

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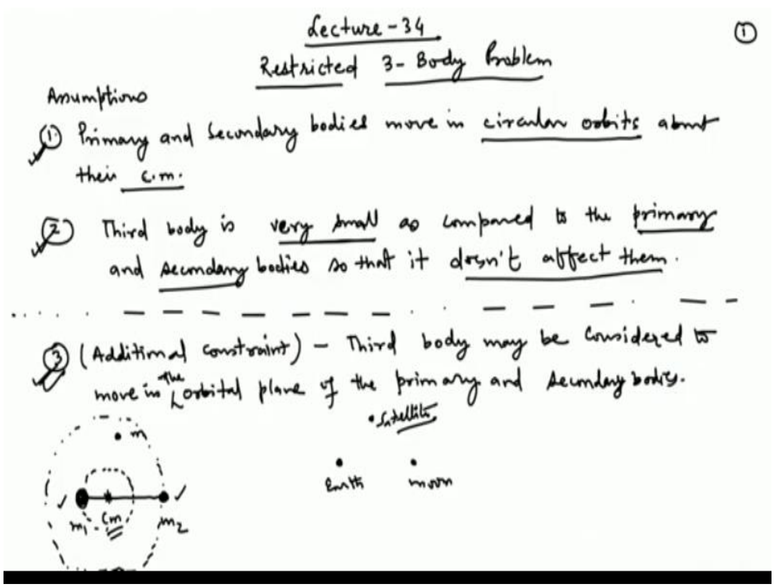

Now the assumptions we will make that the primary and the secondary bodies move in circular orbits about their center of mass. So we have one mass here and another mass here. So this we will call as m_1 and m_2 and the third body, is lying say here, its mass is written as m. The primary and the secondary body move in circular orbits. This is the assumption about their center of mass. So center of mass may be lying here in this place.

About the center of mass is the primary and the secondary bodies, they are moving and that too in the circular orbit. So this will make a circle like this, here about in the center of mass. Similarly, this will make a circle about the center of mass and they will be moving such that they are always opposite to each other. So this is the first assumption. The second assumption, the third body is very small as compared to the primary and the secondary bodies.

Say, this is the primary and this is the secondary body, so third body which is located here, this is very small as compared to the primary and the secondary, say in the case of the, I have here, earth and this point I have moon and I have a satellite here. So we know that satellite can be of few 1000 kgs. So while the earth and moon they are very massive as compared to the satellite. So in no way, though there is obviously gravitational attraction between all of them, but the satellite gravitational attraction on the earth will be feeble.

Therefore, it can be neglected all together. So therefore, the second assumption we make the third body is very small, so that it does not affect the primary and the secondary bodies. Additionally, we can have one more constraint, but it is not necessary. The third constraint, which I am showing here, this is written here, the third one. The third body may be considered to move in the orbital plane of the primary and the secondary bodies.

So in that case, you are assuming that the motion is planar, but it is not necessary. This is an additional constraint that you can implement, otherwise it is not required. So with these 2 assumptions, we start our work. Let us draw the figure first and thereafter we will.

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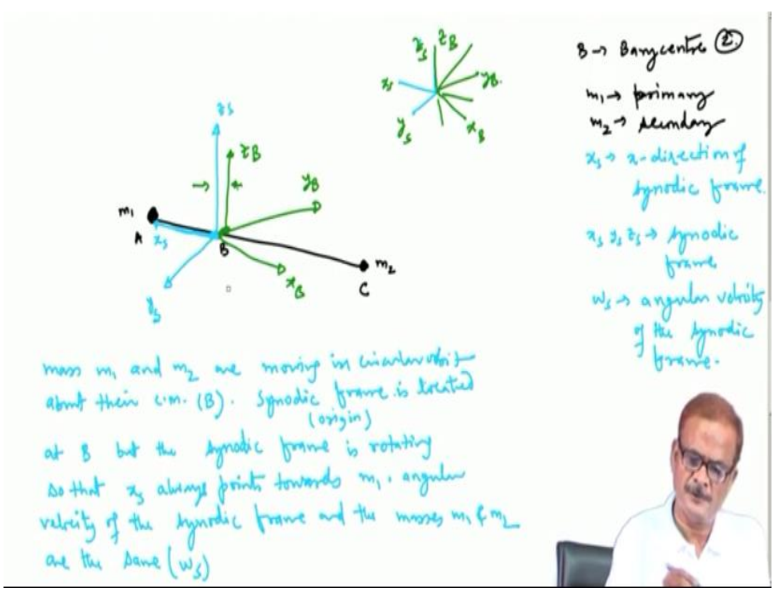

So we have here 2 bodies. This will show us the primary and this as the secondary, somewhere the center of mass is located here, which we will indicate by B and B is nothing but the barycenter, in this case we call the barycenter. This we will write as m_1, this is point A and this we write as known, then B we cannot write the other one. Let us make this as, B is okay for this place. We will make it C, A, B and C.

This mass we will indicate by m_2, this let us consider m_1 is the primary mass. It does not matter. You can assume m_2 also to be the primary and m_2 is the secondary mass. Now we have to fix a reference frame here in this place. So let us say, you put a reference frame. First, we define one synodic reference frame. So synodic reference frame, I am just putting this blue line here, which is indicating the synodic reference frame, x_s is in the x direction of synodic frame.

There is difference between the synodic frame and the barycentric frame, as I will explain you, but remember that reference frame and the coordinate frame, the difference already perhaps, I have explained. In the reference frame, we write the equation of motion. There Newton's law is applicable, but not in the coordinate frame. Coordinate frame is used to describe the position of the particle, or it may be whatever, but there you cannot write the equation of motion.

So reference frame especially refers to the frame in which we are defining the motion. So here in this case, this is the x_s direction and then in this direction, we show y_s and z_s will be vertically up. Therefore, x_s, y_s, z_s this constitutes the synodic frame. Already, I have stated that A and C, the mass m_1 and m_2 are moving in circular orbit about their center of mass B. Synodic frame is located at synodic frame origin is here in this case, this is the origin, is located at B.

But this synodic frame is rotating, such that or so that x_s always points towards m_1. So here in this case what is the effect? The effect is that the angular velocity of the synodic frame and the masses m_1 and m_2 are the same as a result of this and this is written as ω_s. So ω_s, this is the angular velocity of the synodic frame. Now barycenter also we need to define and actually we can do that. Let us say that barycenter; I show it along this direction.

This is x_v, then y_v can be here in this direction and similarly z_v we can show in this direction and it is located at B. I have separated it out here. Actually, the barycenter and the point for the origin of the synodic frame, which is here, they will coincide, but for the sake that it is visible, I have drawn it separately, little separation I have maintained. Otherwise, they are the same. This is your synodic frame. So your barycentric frame is shown like this.

It will overlap with this. This is your synodic frame x_v, y_v, and z_v and similarly here x_s, y_s, and z_s and x_v, y_v, and z_v and it is not necessary that both of them, this z_v direction is the same. It can be different or I can choose any direction. I can also take it like this using the right hand rule. With this description now, we are set to start the formulation of the equation, but for our formulation, I will remove this x_v, y_v, so that the figure looks neat and clean. So we do it on the next page.

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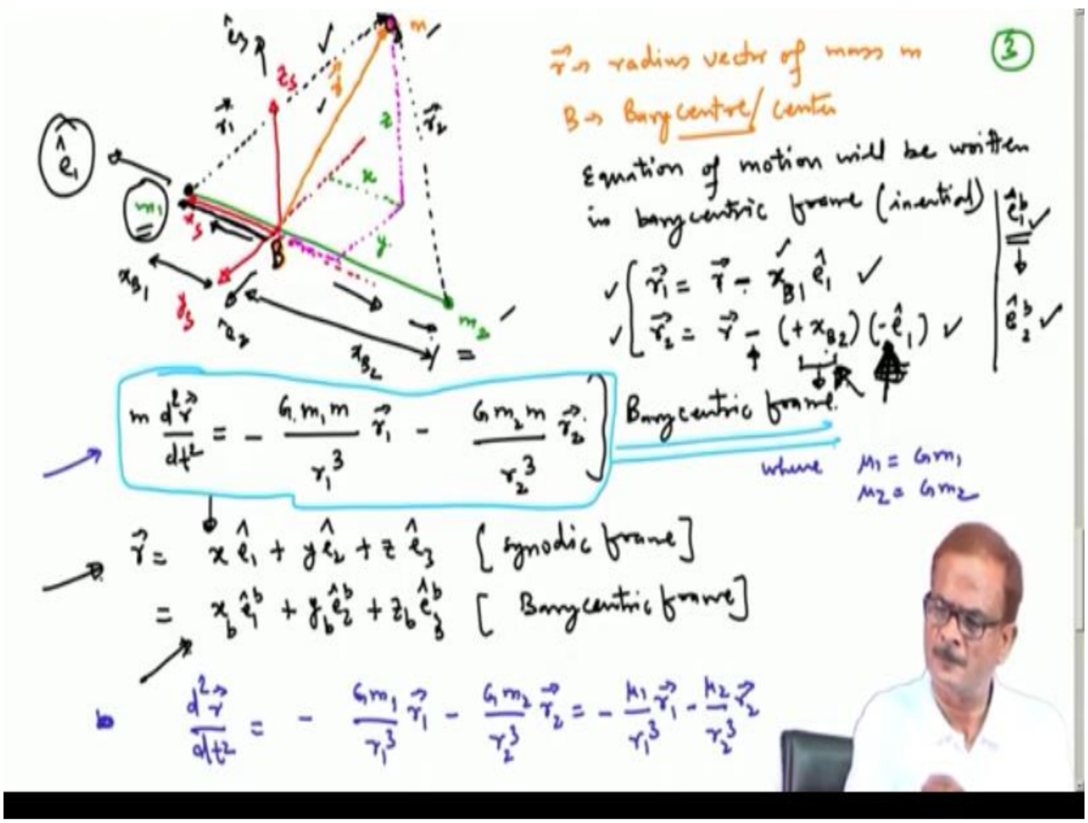

m_1, m_2 and then we have this is x_v, y_v, and z_v. Satellite is located somewhere here. So coordinate of this one, we can indicate like this, draw perpendicular in the plane of the orbit of the masses m_1 and m_2 and thereafter we draw perpendicular on this one like this. We can see that how the coordinates then will be defined and then we write this as r, r is the radius vector or the position vector of mass m, little more change I will do to make it more clear.

Because the axis I have to draw. This is mass m. Now, if we extend y, x axis here in this direction, obviously in this direction we will have the x_s. We have written x_s, y_s, this is the synodic frame we have written and x_v y_v already I have shown on the figure space. This is the barycentric frame and this is related to synodic frame and this point we write as O. So what we observe from this place, that the component because it is lying along the x axis, so we can write this as x; this becomes y and this component is z.

This is your vector, this place too, which is the radius vector. This is your r. So purposefully, we have removed the barycentric frame. You can write also the equation of motion in the barycentric frame, but we are going to represent in terms of the synodic reference frame components and result for that will be very soon visible to you. So x_s, y_s, z_s, already as stated that these are indicating the synodic frame.

This point we have written as B earlier, so we will write this as B. We have represented as the barycenter. So for writing the equation of motion, we need to find the forces acting on this mass. Mass is located here. We have the mass here in this point. So we need to find what are the forces acting on this, then we draw the radius vector from m_1 to the mass m and we write this as r_1. Similarly, we draw the radius vector, this is r_2.

So m times d square r by dt square, the equation of motion will be written in barycentric frame, which is inertial and now what are the forces acting on this. This we have to write and thereafter we will reduce it to the synodic frame. This is the process we are going to use. Now the forces due to the mass m_1 and m_2 we have to calculate. So therefore, due to the mass m_1, we will have the force acting on this particle will be given by this r_1^3.

$$m\,\frac{\mathrm{d}^2\mathrm{r}}{\mathrm{dt}^2} = -\frac{Gm_1m\overrightarrow{r_1}}{r_1^3} + \frac{Gm_2m\overrightarrow{r_2}}{r_2^3}$$

In which direction this force is acting, this is acting just opposite to the r_1 vector. So we write here with a minus sign. Similarly, you can see that the force acting due to the mass m_2 on the mass m, this g times m_2m divided by r_2 whole cube and r_2 in the barycentric frame. Remember that r_1 I can write this as, say the component in the synodic frame, we can write it like this, where r_1 is the vector or one more step is required here. This part we will change a little bit.

Actually what we can show here that r_1 is nothing but r – x_s or say r – the coordinate of mass m_1 and what is the coordinate of mass m_1? If I indicate it by, this is your r, so r minus this quantity and this part we write as coordinate with respect to the barycentric frame. So let us write this as x_B with respect to the B frame B to 1. As we have been indicating the notation we have been using, this is B located here and B to 1, in this point you have 1, mass m_1.

So it will go like this and the synodic frame direction. This is the way we will express it. Similarly, r_2 we can write as r minus, now here in this case you are taking positive direction of x along this direction, so this will be negative one. So therefore, this can be written as x_{B2} or if I put a plus sign and here, this is the coordinate, I will explain you what does it mean, if I write it like this, minus $\widehat{e_1}$.

So this is one is $\widehat{e_1}$, I am taking here in this direction and $\widehat{e_2}$ along this direction, $\widehat{e_3}$ along this direction in the synodic frame and this notation I am going to use. For the barycentric, you can put a symbol here in the upper one, like I can use a symbol e_{1B} indicating that this is a unit vector along the x direction of the barycentric frame. Similarly, $\hat{e}_{B2}$ it is a unit vector along the y direction of the barycentric frame, but this is not required.

The part I am going to explain you, that will make you understand what I am trying to do here. So if I write it this way, so this indicates this is the magnitude x_{B2}; x_{B2} is the distance from this point to this point, from B to m_2. This is your x_{B2} and similarly the distance from this point to this point, this is x_{B1}. So this is the magnitude and multiplied by the vector along this direction, so this vector is –$\widehat{e_1}$. Here in this direction, this is $\widehat{e_1}$ and therefore now this is consistent.

This is minus sign here; you can see that for getting this vector this r_1 I need to subtract from r the vector going from this place to this place up to the mass m_1, from here this place to this place, as I am showing in the figure. So this is done this way, the same way this is done and you have to remember that here whatever is appearing this is indicating the magnitude. If you do not want to write it in terms of magnitude, one more point here.

You can see that this sign here is minus, here also this is minus, but this minus sign here and minus sign ultimately it will make it plus. So this is the difference and if you put then the particular value of the x_{B1} and x_{B2} in the magnitude, so your r_1 and r_2 vector will be available to you. Otherwise, you do not also require putting it this way, you just keep it x_{B2} here in this place and here in this place as I am showing by this arrow, you can put a plus sign.

So if you do that, then you have to put here for x_{B2} while you are working the negative sign. These are the ways of doing and various authors can do the same problem in multiple ways. This is the approach I am taking, because it will be convenient to represent it. Now, this I have shown in the synodic frame. If I write the same thing, one more thing you should note that any vector r can be written in terms of the synodic reference frame, let us say its components are x.

So

$$\vec{r} = x\,\hat{e_1} + y\,\hat{e_2} + z\,\hat{e_3}$$

So this is in the synodic frame. The same vector, if you describe in the barycentric frame, so I will write this as

$$= x_B\hat{e_1}^B + y_B\,\hat{e_2}^B + z_B\,\hat{e_3}^B$$

So this is indicating in the barycentric frame. So components will change, but the vector r does not change. So the components here the structure is changing. Based on this, our whole analysis will depend.

So these 2, if you use this, you will get the result in terms of the barycentric frame and if you use this, then you get the result in terms of the synodic frame, but the treatment of the problem then will be little different for the barycentric frame. We have this equation written, the equation of motion first in the barycentric frame, this we have to do necessarily, there is no option to that, because we have to always write the Newton's law in an inertial reference frame.

Now we will reduce it to the synodic reference frame, which we have assumed, it is rotating with angular velocity ω_s. So this is the angular velocity of the synodic frame and it is nothing but also the angular velocity of the mass m_1 and m_2. So with this basic formulation, now we can proceed and work out the whole problem. In the next step, m we can eliminate from this equation. So this can be written as

$$\frac{(d^2r)}{dt^2} = -\frac{gm_1}{r_1^3}$$

And this we brief as $\mu_1/ r_1^3 - \mu_2/ r_2^3 \ r_2$, where μ_1 = gm_1, and μ_2 = gm_2.

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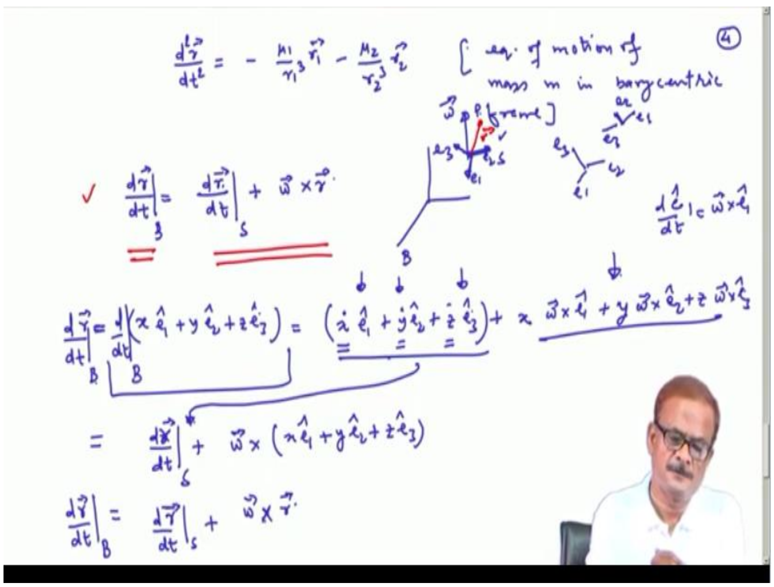

Thus, we have our equation in the initial frame as μ_1/r_1^3, equation of motion of mass m, barycentric frame and this can be reduced in terms of the synodic frame. For this, you need to look into this relationship, if this is in the barycentric frame, so dr/dt this with respect to the synodic frame ω × r. So with this notation, what this is indicating? That I have an initial frame here, which I am writing as a barycentric frame.

Another frame I have here, which I am writing as synodic frame, ω is the angular velocity of this frame and let us say, so what I am interested that I have one point here located here whose radius vector is r_1 or simply r. So rate of change of this r in the barycentric frame if it is written like this; it can be expressed in this way and this is pretty simple. Let us do this quickly here in this place r equal to, we express this in terms of the synodic frame components x_s.

Or simply we will write x not to complicate the whole thing $\widehat{e_1}$ y times $\widehat{e_2}$ + z times $\widehat{e_3}$. This is a vector given to us. If we differentiate this vector with respect to time in the barycentric frame and the other thing will be given as x times $\omega \times \widehat{e_1}$ + y times $\omega \times \widehat{e_2}$ + z times $\omega \times \widehat{e_3}$. So here the vector r, this is the point p here, it may be moving and because of this, these components are changing in the synodic frame.

You first just differentiate this, once you differentiate this part while breaking this up and applying the normal calculus rule, so we can break here in this part. Once we have broken it like this, this part then becomes dx or dr by dt with respect to the synodic frame. In the synodic frame, this is the synodic frame e_1, e_2 and e_3 these are the unit vectors located along. So unit vector in the synodic frame does not change. It is fixed and this frame itself is changing.

It is rotating. This frame right now it is like this after some time, it may be looking like this. So this is e_1, e_2, e_3, later on e_1, e_2, and e_3, it becomes like this, so it rotates because of presence of this ω. This indicates your motion with respect to the synodic frame and already I have discussed this part while discussing the central force motion. We are doing differentiation of de_1 by dt, which is nothing but $\omega \times \widehat{e_1}$ and this we have discussed while discussing about the central force motion.

So I am no longer expanding that again by $\widehat{e_2}$ times z $\widehat{e_3}$. Therefore, this gets reduced to dr/dt with respect to the synodic frame and $\omega \times$ r. So dr/dt with respect to the barycentric frame will be equal to dr/dt with respect to the synodic frame and plus $\omega \times$ r. So we stop here and we will continue in the next lecture, the same whatever we have left, we will complete that. Thank you.