Space Flight Mechanics

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Lecture – 35
Restricted 3-Body Problem (Contd.,)

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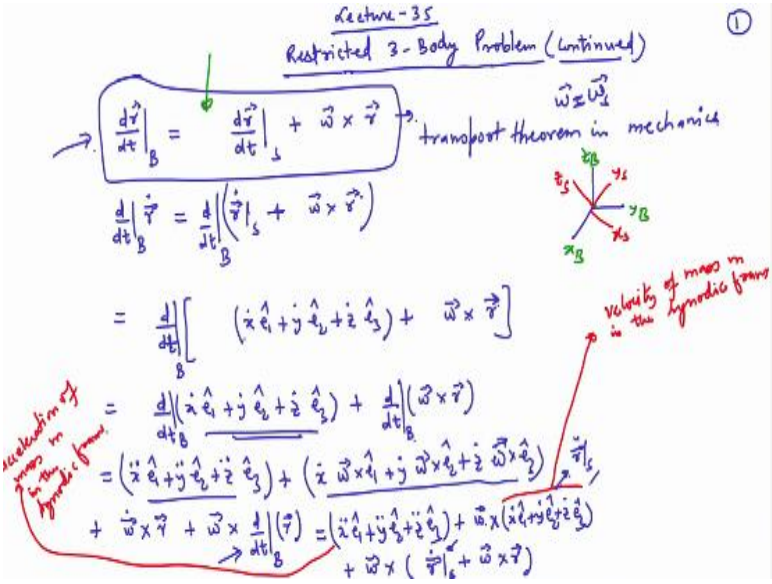

Welcome to the lecture number 35. We have been discussing about the restricted 3-body problem. So, we will continue with that. So, if you remember we worked out this expression. So we have written dr / dt rate of change of the vector r in the barycentric frame this will be equal to rate of change of the vector in the synodic frame plus $\vec{\omega} \times \vec{r}$ where ω is nothing, but ω_S.

$$\left.\frac{d\vec{r}}{dt}\right|_B = \left.\frac{d\vec{r}}{dt}\right|_S + \vec{\omega} \times \vec{r}$$

'S' I have removed just to facilitate just for the ease okay I do not have to carry the subscript. If I differentiate this once more and write this as $\dot{r}$ this is with respect to the synodic frame. So if I represent this as let us mark this as with respect to the synodic frame. This is from your basic mechanics I should have skipped it, but for your purpose I am working out those who have not done it.

So this part is your d/dt so following this notation now whatever the notation this equation we have developed here this is called transport theorem in mechanics and we just need to expand it further to solve the problem. One more step we will do so what we can observe that this becomes d/dt this we are doing with respect to the B frame. In the same way as we have done earlier we can expand this part and write here $\dot{x}\ \omega \times \hat{e}_1 + \dot{y}\ \omega \times \hat{e}_2 + \dot{z}\ \omega \times \widehat{e_3}$

So this part we have expanded and thereafter the other part we have to do and this is with respect to the B frame, we have to remember this part. This is according to this part is according to the normal calculus rule. Once we have done this so one more thing I would like to point it out here that this is our barycentric frame and here itself will show the synodic frame. This is x_s, y_s and z_s and the barycentric frame x_B, y_B and z_B.

So, therefore this distance which was shown here this distance is 0 in that case and that part I have not indicated in this place. So only the rate of change of this vector I am trying to estimate how much this will be while this point and this point both of them they coincide otherwise we will have some extra term appearing there which is not necessary here in this case.

So $\dot{\omega}$ cross r and then ω cross d/dt with respect to the B frame r and then we can summarize this as $\ddot{x}\ \widehat{e_1}$ + $\ddot{y}\ \widehat{e_2}$, $\widehat{e_3}$ and plus now you can look here in this term. This is nothing, but ω cross $\dot{x}\ \widehat{e_1}$, $\dot{y}\ \widehat{e_2}$ + $\dot{z}\ \widehat{e_3}$. This particular term is the velocity of mass m in the synodic frame and this is the acceleration. Okay one more term I have to write so here I will not write.

This particular term we can write here this is the acceleration of mass m in the synodic frame other term is remaining as this one the third one. The third term can be written as ω cross and from here we can insert because this is with respect to dry by dt. So we can insert from this place $\dot{r}$ with respect to the synodic frame and plus $\vec{\omega} \times \vec{r}$. So this quantity is nothing, but here in this place this is your $\dot{r}$ with respect to the synodic frame. So you have the same term appearing here also and in this place also.

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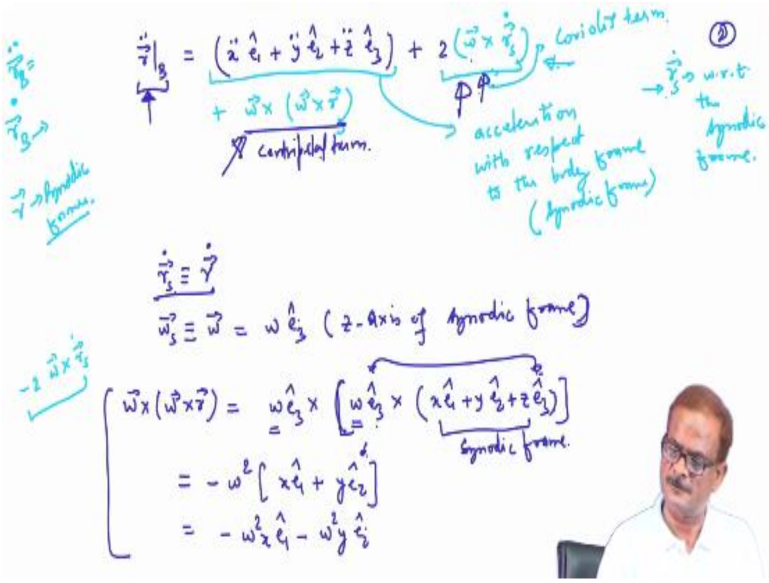

In the next page we go and therefore the $\ddot{r}$ with respect to the barycentric frame this gets reduced to

$$\ddot{r}|_B = \ddot{x}\hat{e_1} + \ddot{y}\hat{e_2} + \ddot{z}\,\hat{e_3} + 2(\vec{\omega}\times\dot{\vec{r}_s}) + \vec{\omega}\times(\vec{\omega}\times\vec{r})$$

$\vec{\omega}$ we can take it outside so this makes it 2 times ω cross $\dot{r}$ where $\dot{r}$ and particularly because we are denoting it with respect to the synodic frame.

So, we can put a symbol here to indicate that this is with respect to the synodic frame and I would like to get rid of this symbol rather than using this if we use the symbol $\ddot{r}_B$ as acceleration or $\dot{r}_B$ as a velocity with respect to the synodic frame then simply r will be indicating with respect to the synodic frame. I want to get rid of this symbol so that we are at ease while writing the equation.

The other term is ω cross $\vec{\omega}\times\vec{r}$ plus ω cross so that is what we are getting here in this place. Now this part is if you are aware of your basic mechanics so this part this is referring to the acceleration with respect to the body frame in which in this case we are calling as the synodic frame and if you remember this term is Coriolis term with a minus sign if you take it on the left hand side.

So, the body acceleration can be represented in terms of this particular term which appears as a Coriolis term. So – 2(ω × $\dot{r}_S$) this is your Coriolis term or the Coriolis acceleration we call this and this is your centripetal acceleration on the centripetal term or we can say so pick this at our disposal. Now we can write the equation of motion what we have developed by proper formulation.

So $\dot{r}_s$ this notation I will drop and I will write here simply as $\dot{r}$ and wherever the B is appearing it will appear it will indicate it is with respect to the barycenter. Here, in this case if I am dropping this so it is indicating it is with respect to the synodic frame and this is for our convenience. Similarly, ω_s we have dropped and written simply as ω and this we can write as ω times $\widehat{e_3}$ which is along the z direction of the synodic frame.

Z direction of Z axis of synodic frame. We have fixed our frame such that the angular velocity of the synodic frame it lies along its Z direction. To further process, we need to expand this one and this one and once we expand it and write it then we can write the equation of motion what we have used written in terms of the barycentric reference frame. So let us expand it $\vec{\omega} \times \vec{r}$ and these are the components in the synodic frame.

Once we expand it this part I am leaving it to you once you expand you get here minus ω_s^2 this ω and this ω that makes it square and rest of the things that gives you x times $\widehat{e_1}$ + y times $\widehat{e_2}$ that means it is - ω^2 x $\widehat{e_1}$ $- \omega^2$ y $\widehat{e_2}$, $\widehat{e_3}$ as you can see from this place this drops out and e_3 times e_1 that gives you e_2 which appears here and then okay you can check it I will not take time I will not spend time over this. This is the result that you are going to get. Another part remaining is ω times $\dot{r}_s$ this two.

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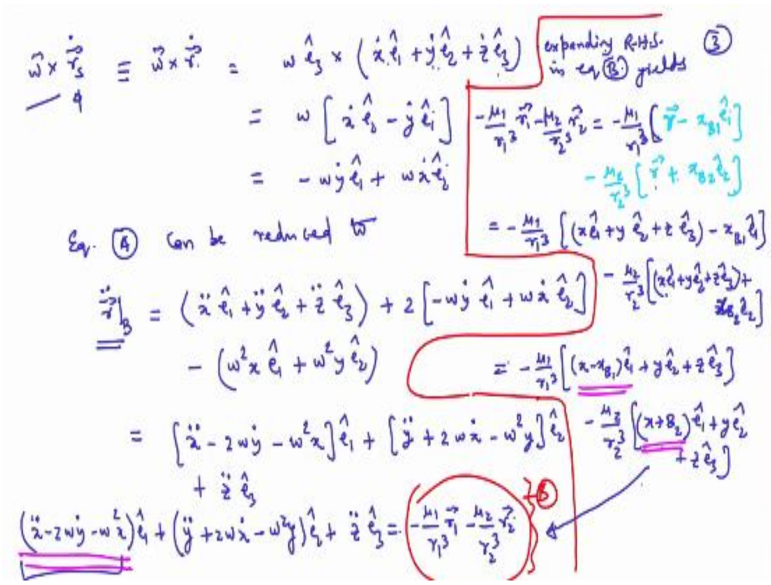

$$\vec{\omega} \times \dot{\vec{r}}_s = \vec{\omega} \times \dot{\vec{r}} = \omega\, \dot{y}\, \hat{e_1} + \omega \dot{x}\, \hat{e_2}$$

Now we assemble all of them in this equation let us name this equation as (A) $\ddot{r}$ with respect to the barycentric frame.

This gets reduced to all the terms we have to write $\ddot{x}\, \hat{e_1}$ + $\ddot{y}\, \hat{e_2}$ then we have this term here this is + 2 times ω_s times r_s. Here we will insert the values and the third term is ω cross ω this part ω here. So, if we insert we have to insert this value there. So this is minus with minus sign $\hat{e_1}$ + ω^2 y $\hat{e_2}$. This is what we have written here in this place.

Now we combine the corresponding vector together this gets reduced to ($\ddot{x}$ – 2 ω $\dot{y}$ – ω^2 x) $\hat{e_1}$ + ($\ddot{y}$ + 2 ω $\dot{x}$– ω^2 y $\hat{e_2}$ + $\ddot{z}\, \hat{e_3}$) So, this is your $\ddot{r}$ with respect to the barycentric frame. So, therefore as we have written earlier this gets reduced to $(\ddot{x} - 2\,\omega\, \dot{y} - \omega^2\, x)\, \hat{e_1}$ + $(\ddot{y} + 2\,\omega\, \dot{x} - \omega^2\, y\, \hat{e_2} + \ddot{z}\, \hat{e_3})$ and on the right hand side we have to insert the term here which we have written in this place. So this is your with respect to the barycentric frame $\ddot{r}$ with respect to the barycentric frame. So, on the right hand side we have this term so we will insert that term there – μ_1 / r_1^3 r_1– μ_2 / r_2^3 $r_2$and then $r_1$and r_2 values we have to insert there and separate out the terms.

$$(\ddot{x} - 2\,\omega\, \dot{y} - \omega^2\, x)\, \hat{e_1} + (\ddot{y} + 2\,\omega\, \dot{x} - \omega^2\, y\, \hat{e_2} + \ddot{z}\, \hat{e_3}) = -\frac{\mu_1}{r_1^3}\vec{r_1} - \frac{\mu_2}{r_2^3}\vec{r_2}$$

So, we will get (3), second order differential equation which we need to solve. So, let us first develop this part if we develop this part it will be easy to work out on the next page. So here we do it in quickly and let us name this equation as this whole thing as equation (B). Rewriting or expanding RHS right hand side in equation B yields. So, we have – μ_1/r_1^3 r_1 – μ_2/r_2^3 r_2 this can be expanded as μ_1/r_1^3.

Now $r_1$we have already written if you remember this term was x minus this term. This particular part we can expand and write it here. Let us go and write there itself in this place it is there. Now write systematically so this is r – x_1 with respect to the barycentric frame or with respect to the origin B we have written x_{1B} or x_{B1} anything we can write here it is okay so r – let us write this as x_{B1} and – μ_2/r_2^3 (r + x_{B2} $\hat{e_2}$).

And then we expand it this is from this place $x_{B1}\ \widehat{e_1}$ and (r + x_{B2}). Therefore, this right hand side then we can further expand and write this as μ_1/r_1^3 ((x $\widehat{e_1}$ + y $\widehat{e_2}$ + z $\widehat{e_3}$) – $x_{B1}\ \widehat{e_1}$) and – μ_2/r_2^3(z $\widehat{e_3}$ + $x_{B2}\ \widehat{e_2}$). So the right hand side for that part it becomes - μ_1/r_1^3 (x – $x_{B1}\widehat{e_1}$ + y $\widehat{e_2}$ + z $\widehat{e_3}$) – μ_2 / r_2^3.

And this will be (x + $B_2\ \widehat{e_1}$) this term we are arranging or we are putting look here in this part this is x + here this is x_{B2}. So x_{B2} has appeared rest other terms are simply (y $\widehat{e_2}$ + z $\widehat{e_3}$) and then we can insert here in this place and thereafter we can shift the other terms. So, at this stage instead of copying the whole thing on the next page again I will take liberty and this term will be equated with I will show it by different color. This term will be equated with the term here which contains $\widehat{e_1}$ and here also the $\widehat{e_1}$. So this two terms can be combined together and if we combine them together.

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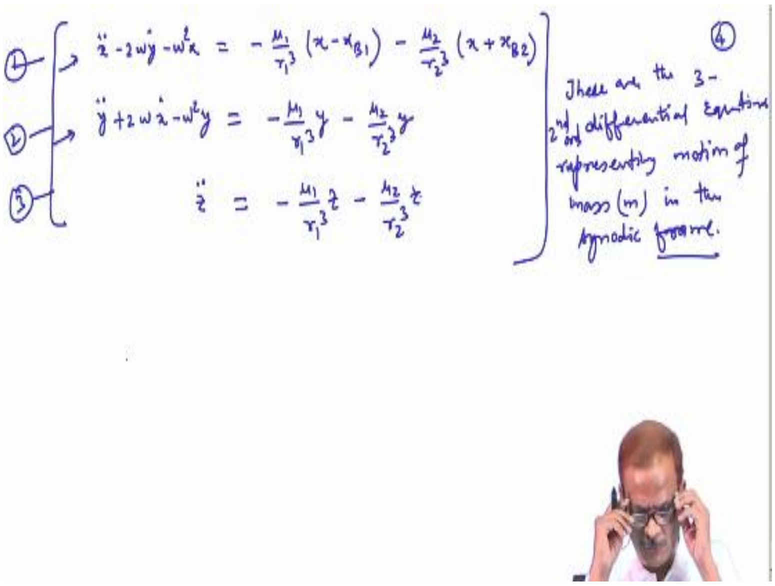

So in the next page we write it

$$\ddot{x} - 2\omega\,\dot{y} - \omega^2\,x \;= -\frac{\mu_1}{r_1^3}(x - x_{B1}) - \frac{\mu_2}{r_2^3}(x + x_{B2})$$

So this is what we have picked up this term and this term we have picked up here and along with this common multiplication terms are also there so they will enter in that place. Thereafter, we take the next term we will take up this term.

So

$$y + 2\,\omega\dot{x} - \omega^2\,y \;= -\frac{\mu_1}{r_1^3}\,y - \frac{\mu_2}{r_2^3}\,y$$

and similarly we have $\ddot{z}$ this is the last term here existing. So $\ddot{z}$ with this there is nothing no other terms are there so we write here

$$\ddot{z} = -\frac{\mu_1}{r_1^3}\, z - \frac{\mu_2}{r_2^3}\, z$$

which we have picked up from this place and this place.

So these are the 3 second order differential equations representing motion of mass 3 or mass m motion of mass m in the synodic frame why because $\ddot{x}$ $\ddot{y}$ it is now represented in terms of the synodic frame which we are telling as the body frame not as the inertial frame and we need to solve all these 3 equations. So, I will mark them as equation number 1, equation number 2 and equation number 3.

These 3 equations need to be solved, worked out in order to get the solution. So, the solution for this it is solved in a different manner instead of getting explicit solution we get for this an implicit solution which represents the behavior of the system and the solution to this then we are going to look into the next lecture. So we stop here. Thank you very much.