Space Flight Mechanics
Prof. Manoranjan Sinha
Department of Aerospace Engineering
Indian Institute of Technology – Kharagpur

Lecture – 36
Restricted 3-Body Problem (Contd.)

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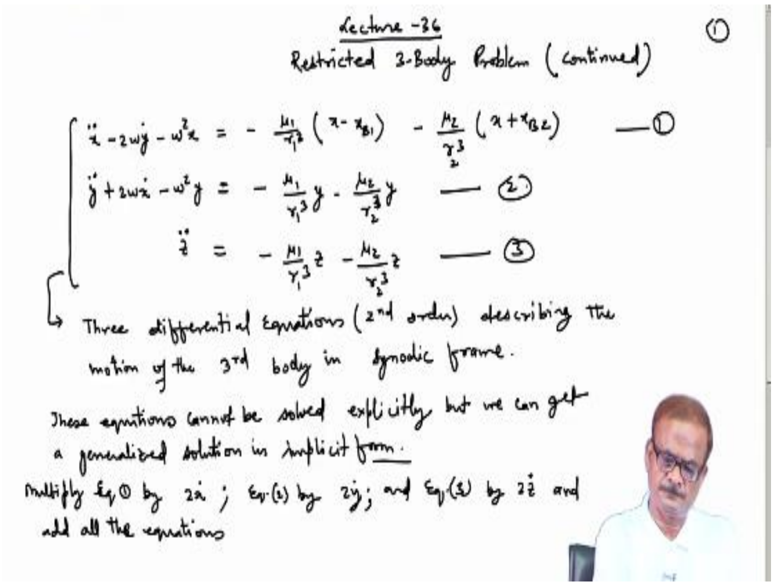

Welcome to lecture number 36. So, we have been discussing about the restricted 3 body problem and we derived the equation of motion in synodic frame. These are the 3 equations we have already derived now we need to solve it. So, these equations cannot be solved explicitly, but we can get a generalized solution in implicit form and for this what we need to do that we multiply equation (1) by $2\dot{x}$, equation (2) by $2\dot{y}$ and equation (3) by $2\dot{z}$ and add all the equations.

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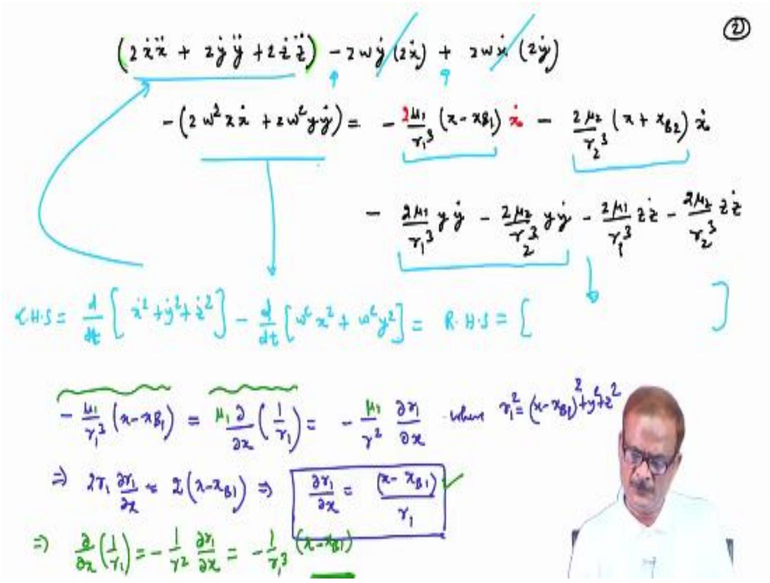

If we do that so we get here if we look into this so we get here $2\dot{x}\ \ddot{x}$ + $2\dot{y}\ \ddot{y}$ + $2\dot{z}\ \ddot{z}$. Here what we are doing currently we are taking this term multiplying it by corresponding with $2\dot{x}$, $2\dot{y}$ and $2\dot{z}$. Similarly, next we will take up this term and this term this is the second term and then thereafter we will take up this 2 terms because correspondingly in the z this and this second and third terms are not present on the left hand side.

So, this and then we have the other terms – $2\omega\dot{y}$ and $2\omega\dot{x}$ – $2\omega\dot{y}$ and then this multiplied by $\dot{x}$ and moreover we are multiplying it by $2\dot{x}$ so I will write it separately here $2\dot{x}$ multiplied by $2\dot{x}$. Similarly, the term from here will get $2\omega\dot{x}$ so this is + $2\omega\dot{x}$ and it is $2\dot{y}$ and thereafter the terms we take ω^2x so this becomes $2\omega^2 x\dot{x}$ and this is with minus sign.

And then in the y term ω^2y so this is 2 $\omega^2 y\dot{y}$. In the z these terms are equation pertaining to the z component, its zero; those parts are 0 therefore those are not taken into account here in this place. So, now on the right hand side we will have to multiply all these things. So we have – μ_1/r_1^3 (x – x_{B1}) this first term we have taken here and this multiplied by 2 times.

So we can put here and $\dot{x}$ we can put in this place. Similarly, the other term we will have 2 times μ_2 divided by r_2^3 (x + x_{B1}) and x_{B2} $\dot{x}$. This term we have picked up r_2^3. Thereafter, the rest of the terms from here next we pick up this term from this place so this is 2 μ_1/r_1^3 y $\dot{y}$ – 2 μ_2/r_2^3 y $\dot{y}$. Similarly, we will have 2 μ_1/r_1^3 z $\dot{z}$ – 2 $\frac{\mu_2}{r_2^3}$ z $\dot{z}$.

In this equation you can see that this term and this term they cancel out because here we have minus sign and this place we have the plus sign. These 2 terms we can write as d/dt ($\dot{x}^2$ + $\dot{y}^2$+ $\dot{z}^2$). You can check this if you differentiate you get this term. Similarly, this term can be written here as d/dt (ω^2 x^2 + ω^2 y^2).

And on the right hand side we have to copy all these things, but before that we need to work out little bit more to comprehend the right hand side. So, I will simply write here this is LHS equal to this, this equal to RHS quantity the whole thing is to be copied here I am just skipping, just copying from this place to that place. Now we need to work out first this term I am going to work out this term and let us look into the first term – μ_1/r_1^3 x_{B1}.

So this term is nothing but $\partial/\partial x_1$ ($1/r_1$) and on the right hand side this side if you expand this so you can see from this place this becomes – 1/ r^2 $\partial r_1/\partial$x where r_1^2 = x – x_{B1}^2 + y^2 + z^2. So this implies then 2r_1 times $\partial r_1/\partial$x will be equal to 2 (x – x_{B1}) and this implies

$$\frac{\partial r_1}{\partial x} = \frac{x - x_{B1}}{r}$$

Therefore, $\partial/\partial x_1$ (1/ r_1) = - 1 / r^2 $\partial r_1/\partial x_1$ this becomes $1/r_1^3$ (x – x_{B1})

So you can see that here we have to write the μ_1 so μ_1 is missing. So what we see that if we multiply this side by μ_1 already we have got $\partial r_1/\partial$x and if we insert here in this place this is what we are getting. Simply it is visible that this term is indeed equal to this term.

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$$\frac{2\mu_1}{r_1} + \frac{2\mu_2}{r_2} = f(x, y, z)$$

$$\mu_2 \frac{\partial}{\partial x}\left(\frac{1}{r_2}\right) = -\frac{\mu_2}{r_2^3}(x + x_{B2})$$

where $r_2^2 = (x + x_{B1})^2 + y^2 + z^2$ ②

$$\mu_1 \frac{\partial}{\partial y}\left(\frac{1}{r_1}\right) = -\frac{\mu_1}{r_1^3}y \qquad \mu_1 \frac{\partial}{\partial z}\left(\frac{1}{r_1}\right) = -\frac{\mu_1}{r_1^3}z$$

$$\mu_2 \frac{\partial}{\partial y}\left(\frac{1}{r_2}\right) = -\frac{\mu_2}{r_2^3}y \qquad \mu_2 \frac{\partial}{\partial z}\left(\frac{1}{r_2}\right) = -\frac{\mu_2}{r_2^3}z$$

Using above results in Eq ④ yields

$$\frac{d}{dt}(\dot{x}^2+\dot{y}^2+\dot{z}^2) - \frac{d}{dt}[\omega^2(x^2+y^2)] = \underbrace{2\mu_1 \frac{\partial}{\partial x}\left(\frac{1}{r_1}\right)\dot{x} + 2\mu_2 \frac{\partial}{\partial x}\left(\frac{1}{r_2}\right)\dot{x}} + \underbrace{2\mu_1 \frac{\partial}{\partial y}\left(\frac{1}{r_1}\right)\dot{y} + 2\mu_2 \frac{\partial}{\partial y}\left(\frac{1}{r_2}\right)\dot{y}} + 2\mu_1 \frac{\partial}{\partial z}\left(\frac{1}{r_1}\right)\dot{z} + 2\mu_2 \frac{\partial}{\partial z}\left(\frac{1}{r_2}\right)\dot{z}$$

$$\frac{d}{dt}\left[(\dot{x}^2+\dot{y}^2+\dot{z}^2) - \omega^2(x^2+y^2)\right] = \frac{\partial}{\partial x}\left[\frac{2\mu_1}{r_1} + \frac{2\mu_2}{r_2}\right]\frac{dx}{dt} + \frac{\partial}{\partial y}\left[\frac{2\mu_1}{r_1} + \frac{2\mu_2}{r_2}\right]\frac{dy}{dt} + \frac{\partial}{\partial z}\left[\frac{2\mu_1}{r_1} + \frac{2\mu_2}{r_2}\right]\frac{dz}{dt}$$

So, based on the same kind of derivations similarly we will have $\partial/\partial x_1$ (1/ r_2) and then we multiply it with μ_2. This quantity will be

$$\frac{\partial}{\partial x_1}\left(\frac{1}{r_2}\right) =-\frac{\mu_2}{r_2^3}\,(x\ + x_{B2})$$

where r_2^2 this equal to x + x_{B1}^2 and as we have written in the earlier lecture y^2 + z^2. Following the same notation you can see that whatever we have done on the previous page we can get this quantity here. In the same way we have

$$\mu_1 \frac{\partial}{\partial y}\left(\frac{1}{r_1}\right) = -\frac{\mu_2}{r_2^3}\,y$$

and along the same line we have

$$\mu_1 \frac{\partial}{\partial z}\left(\frac{1}{r_1}\right) = \frac{\mu_1}{r_2^3}\,z$$

After getting this now we insert all these results in the right hand side of the previous equation. So here let us mark this as equation let us mark this as equation (1), (2), (3) we have written earlier (1), (2), (3) we are using here so this we will mark as equation (4). Using above results in equation (4) yields d/dt ($\dot{x}^2$ + $\dot{y}^2 + \dot{z}^2$) – d/dt (ω^2 x^2 + y^2).

And then we use all these results so therefore all these terms we can replace. So this becomes this term will get reduced to 2 $\partial/\partial x_1$ (1/r_1) so here we write separately 2 μ_1 times and then $\dot{x}$ is there this term is there so therefore we write here $\dot{x}$. So this way we have 2 μ_1 $\partial/\partial x_1$ (1/r_1) times $\dot{x}$ minus sign will go and then we will have 2 μ_2 $\partial/\partial x_1$ (1/r_2) $\dot{x}$ + 2 μ_1 $\partial/$ ∂y.

And sorting the the above all these results in equation 4 this is what we are getting. Now we can write it in a little better way. From this place what we do that we rewrite all these terms, this term we rewrite as 2 times μ_1/r_1 because this is a constant ∂/ ∂x + 2 μ_2/r_2 , $\dot{x}$ which is nothing but dx/dt. Similarly, the next term this one can be written as ∂/∂y 2 μ_1/r_1 + 2 μ_2/r_2 and dy/dt and the third term we can write as ∂/∂z and let us write 2 μ_1/r_1 + 2 μ_2/r_2 this as f so this is a function of x, y, z.

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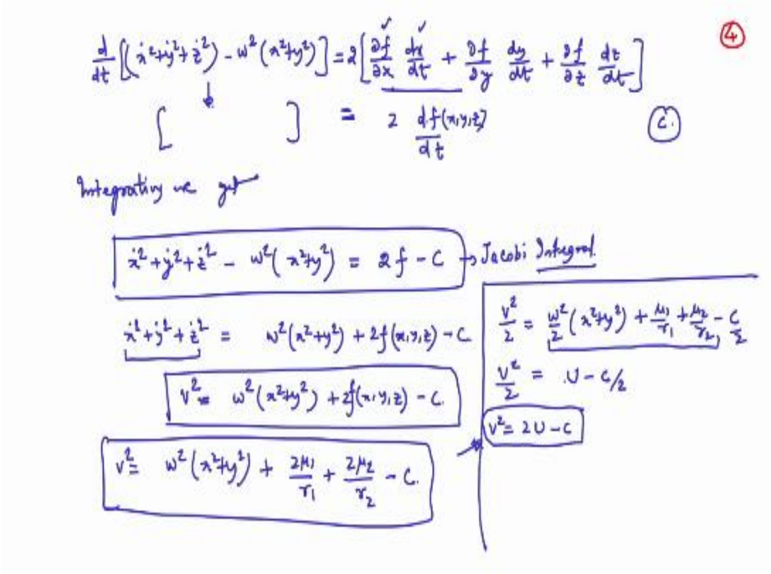

So if we do that. So, the above equation gets reduced to d/dt ($\dot{x}^2$ + $\dot{y}^2$+ $\dot{z}^2$) all these terms we are picking up from this place so this quantity we are writing as ‘f’ and therefore you can see that this is ∂f/∂x dx/dt. The next term then becomes ∂f/∂y dy/dt + ∂f/∂y dz/dt and of course there is the term 2 is there so let us remove this 2 from this place.

So then we can have a factor 2 here appearing like this and the right hand side is nothing, but 2 you can see this is the partial differential we have written of the function this part is partial differential. So the total differential here we get as df/dt because f is a function of x, y and z. Therefore, left hand side we copy here in this place so integrating we get

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 - \omega^2 (x^2 + y^2) = 2f - C$$

and this we call as the **Jacobi integral**.

Now left hand side we can recognize as this is the V^2 term and this equation is of great use in understanding what is the behavior of the 3 particle system especially we can write this as

$$V^2 = \omega^2(x^2 + y^2) + 2\frac{\mu_1}{r_1} + 2\frac{\mu_2}{r_2} - C$$

because f we have defined like that and let us put this as C’ instead of writing c here. If we divide both side by 2.

Okay this quantity we write as U and the same way this equation actually this is

$$V^2 = 2U - C'$$

because we are later on going to use this various values of the C we will remove this, we will use the term like the C_0, C_1, C_2, C_4 for C so we will remove it for the time being. So, it can be put here in this format. Let us forget about this what is this indicating and all other things related to this at this stage. Now we go further and first we normalize these things.

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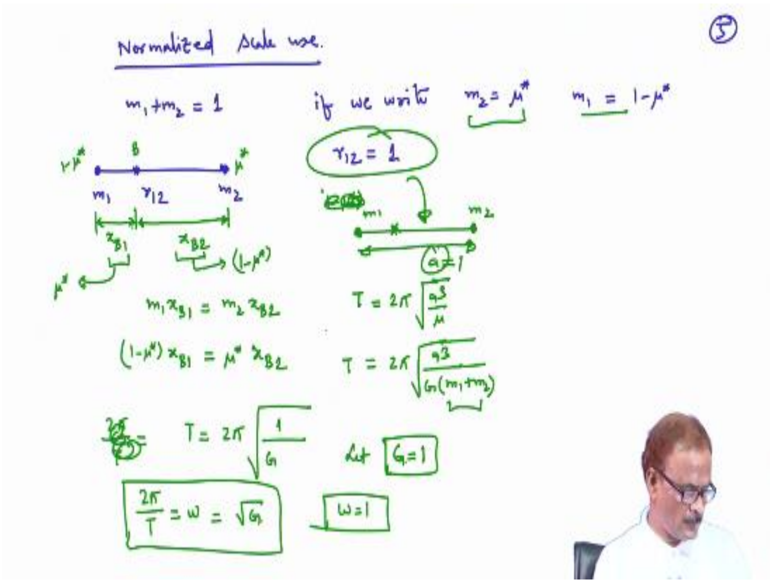

We want to look at the normalized scale what does it mean? Let us look into that normalized scale use. So, we are going to treat it in a way where we define

$$m_1 + m_2 = 1$$

. So, therefore you can see that if we write

$$m_2 = \mu^*$$

then m_1 will be equal to

$$m_1 = 1 - \mu^*$$

Similarly, the distance between mass m_1 and m_2 here we have mass m_1 this is the mass m_2. So this distance we can write as r_{12}.

So r_{12} we will take this as unity. So this also gets normalized this distance and m_3 of course it is somewhere and there is synodic frame located here in this place. Now this is your point B because this is the point B is the center of mass of mass m_1 and m_2 and therefore and this from here you are writing as μ^* and this as $1 - \mu^*$. So, immediately you can see that $1 - \mu^*$ or we know from our basic definition of the center of mass.

Because already we have considered to be $x_{B1}$and this quantity to be x_{B2}mass this magnitude wise these are magnitude wise. So, therefore from the center of mass definition if we are writing in magnitude why this will be equal to m_2 times x_{B2}. So m_1 from this place if we pick up this as $1 - \mu^*$ and m_2 is from this place is μ^* x_{B2}. So, immediately you can see that the x_{B2}then becomes this quantity becomes on the normalized scale $1 - \mu^*$.

And this quantity becomes μ^* this is magnitude wise. So, this is the normalization what we are following. Moreover, earlier we have learned that if there are 2 masses m_1 and m_2 and distance between them is given by a. So the time period of mass m_1 and m_2 either about the barycenter either about m_1 or either about m_2 it can be written as 2 πa^3 / mu under root where mu is nothing but G times $m_1 + m_2$.

So from here we see that 2 π/ T this quantity is okay let us write it this way only. Now in a^3 if we are writing a = 1 as I am indicating here this as on the normalized scale r_{12} I have chosen as 1 so this is now 1. So here we write 1 and G ($m_1 + m_2$) we have chosen as 1 so if we do that so this gets reduced to

$$2\frac{\pi}{T} = \omega = \sqrt{G}$$

And if we choose if we write

$$G = 1$$

so immediately we can see that

$$\omega = 1$$

So this is the normalized scale, it is a normalized scale discussion of derivation. So ω now we use this ω one here in this place. So you can see that this gets reduced to $(x^2 + y^2)$ 2 μ_1 r_1 + 2 μ_2 r_2 – c. So, once we are using the normalized scale everything will be normalized.

Now r_1 and r_2 also this quantity all of them will be in the normalized form. So, with this information our system gets reduced it to a simple format and we can treat it in a much simpler way which will be very useful later on and also I will do the same thing here I am using the normalized scale without normalization also we can work and I will do it little later on.

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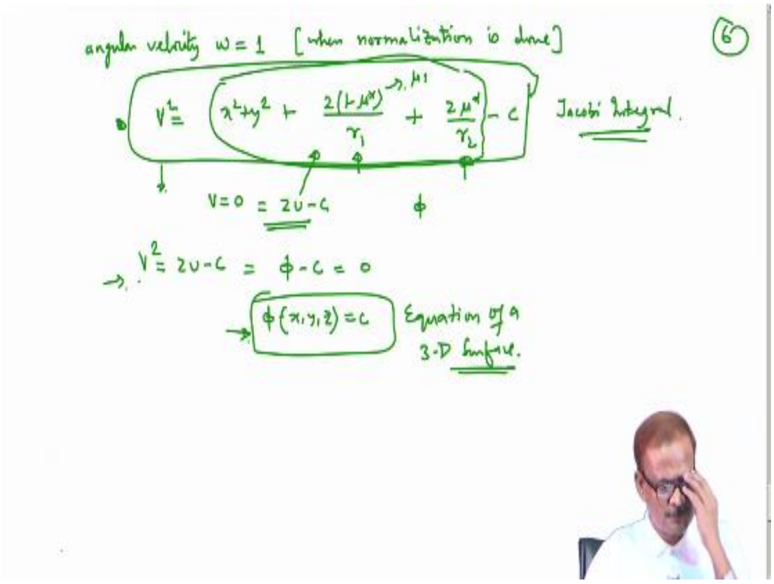

So, angular velocity ω this we are writing as 1 when normalization is done. So, finally our equation it appears as

$$V^2 = x^2 + y^2 + 2\frac{1 - \mu^*}{r_1} + 2\frac{\mu^*}{r_2} - C$$

So already we have written this is the Jacobi integral in the normalized form either this way or this way you write or whatever earlier we have write it is all the same.

We stop here finally what we are going to do if you see that if set V = 0 and this quantity already you have observed that we have written as 2u in the normalized form. So this is 2U – C and U is a function of then xy here also r_1 and r_2 they are dependent on x and y because this is the distance of the third body from the primary and the secondary body. So, what we observe that if we write this quantity as

$$V^2 = 2U - C$$

or either ϕ – C and set it to 0 assuming that V = 0.

So this ϕ which is a concern of x, y, z this equal to c this is the equation of a surface, equation of a 3-D surface. In the 3 dimensional space, this is the equation of the surface and using this we can get a lot of information which we will do little later before that we have to discuss about the Lagrange point. In next lecture, I am going to take up that issue and elaborate it. Thank you very much.