Space Flight Mechanics
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Lecture – 37
Restricted 3-Body Problem (Contd.,)

Welcome to Lecture number 37 we have derived the Jacobian integral. Now we will be looking into the Lagrange point, what this is exactly.

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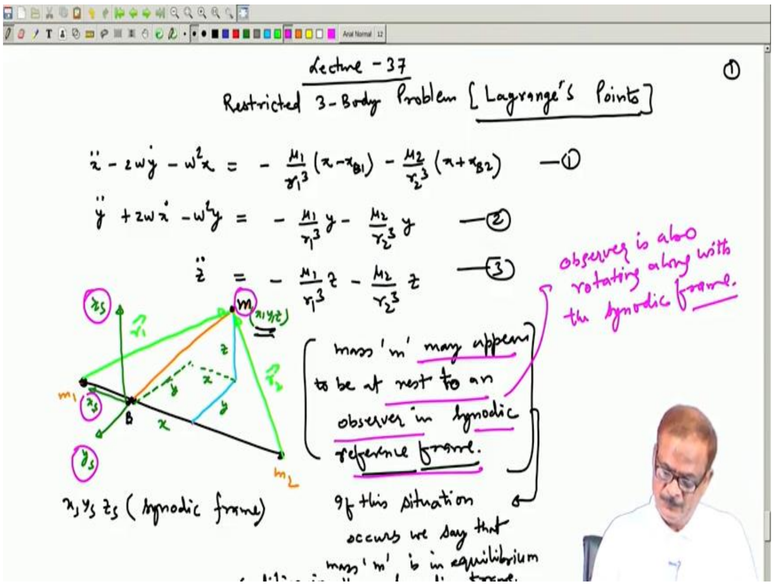

If you I have shown here the x_s, y_s and z_s so this is the synodic frame, synodic frame as I have defined earlier. Now, in this frame if the observer is sitting at say B this frame origin is located at B which is nothing but the centre of mass of the mass m_1 and m_2. So, if any observer is located at B and he is rotating along with the synodic frame the synodic frame is rotating at the rate ω and let us assume that xy plane is the plane in which the m_1 and m_2 is lying.

So it is possible that if any mass is located somewhere which is I have shown here by m whose coordinate is x, y, z. So this mass may appear stationary to the observer at point B in the synodic frame so mass m may appear to be at rest to an observer in synodic reference frame and if this situation occurs we say that the mass m is in equilibrium condition in the synodic frame.

So, under what condition this can happen or say, what is the location of the point m or the location of this elementary mass m in the synodic frame. Synodic frame is our x_s, y_s, z_s so in this frame where it should be located such that the mass m appears to be at least when observer sitting in the synodic frame so he is also rotating observer is also rotating along with synodic frame this is what it implies. So, this is the thing I am going to explore here now if I say equilibrium.

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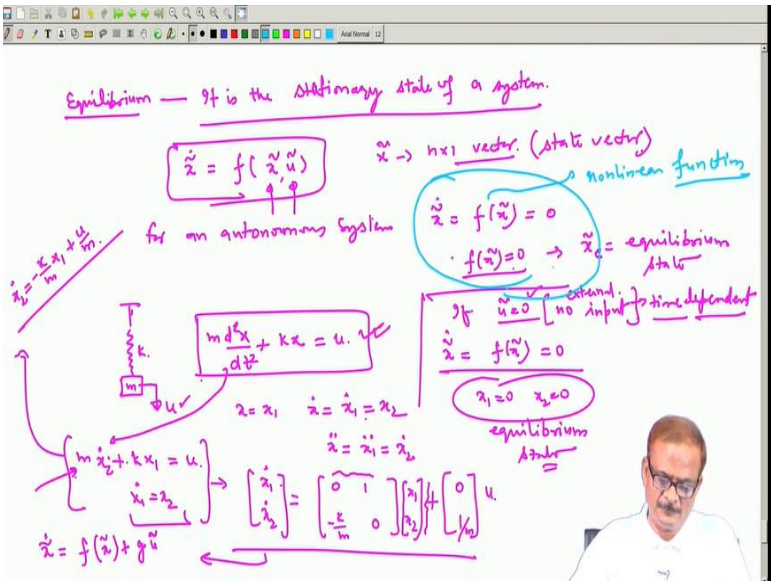

Okay the equilibrium has a very general definition we can write equilibrium as the it is the stationary state of a system this is a general definition and if your system is written like this f x, y, z or here I will remove this, this part and let us write this as

$$\dot{\tilde{x}} = f(\tilde{x}, \tilde{u})$$

So, written here in this format and there may be $\tilde{u}$ also this is called the input and the controls this is called the input and this is the state $\tilde{x}$ is the n into 1 vector this is called the state vector.

So, for an autonomous system which is defined in this format the equilibrium will be defined by $\dot{\tilde{x}} = 0$. So, that means we will have here

$$\dot{\tilde{x}} = f(\tilde{x}) = 0$$

and this we need to solve to get $\tilde{x}$ equilibrium so this is the equilibrium state. So, this is how we solve for the equilibrium point. For an example, if you have a spring mass system this is a linear system we assume that thickness to k this is mass m.

So we know that the equation of motion can be for a displacement it can be written as and u is the force supplied to this so you can write it like in this fashion and if we write x = x_1, $\dot{x_1}$ = x_2 and $\dot{x_2}$ then from this place what we observe that and $\ddot{x_1}$ this will be equal to here $\ddot{x}$ = $\ddot{x_1}$ this will be equal to $\dot{x_2}$.

So, from this place we can write this as m times $\dot{x_2}$ + k times x_1 = u and a long way this we have another equation $\dot{x_1}$ = x_2 while we have written in this format. So this is the state space form reduction and you can see that $\dot{x_1}$ and $\dot{x_2}$ then can be written in matrix format as from this place we can get the $\dot{x_2}$ this will be - k/m x_1 here we will have x_1, x_2 k - k/m x_1 and this quantity will be 0 here and in this place, this will be 0 and this will be 1 by if we divided the whole thing by m.

So this will be 1/m so this equation we are writing as $\dot{x_2}$ = - k/m x_1 + u/m so only - k/m x_1, x_2 part is 0 there and you get u/m in this one we have only x_1 = $\dot{x_1}$ = x_2 so we are getting from this place using this and this $\dot{x_1}$ = x_2. So, this is a linear second order differential equation and it can be reduced here in this format. And if you see I can write this as $\dot{x}$ = $f(\tilde{x})$ + g ($\tilde{u}$) here in this case u is a single input case. This is the only one-dimensional input maybe you can write here $\tilde{u}$ you can remove it is not a problem.

So, what we observed that if $\tilde{u}$ = 0 so where the equilibrium point will lie so that we have to solve using no external input. In that case you do not have any external input means the time dependent we do not give time dependent if you have done the controls course so you may be aware of all these things we are not putting time dependent.

So, if we take u as the state feedback okay so in that okay. We can skip all those things because many people may not be aware of all those things so it is better to escape this issue. So, the only thing that I wanted to point out that in the condition u dot $\tilde{u}$ = $\tilde{0}$= f(x) and this we need to solve and immediately we can see that x_1 = 0 and x_2 = 0 this is the equilibrium state.

So, this is for the this is a situation for the case where we have a linear system. The nonlinear system is there then in that case, you get your equation here in this format where this is a nonlinear function and therefore solving this it may be literally troublesome varying from case to case. So, this way we get the equilibrium point so with this; what we okay coming back to this place then we can see that using the notation what we have developed earlier.

So, using that we can reduce it in the format where it appears as $\tilde{x}$. So instead of doing this just note that $\ddot{x}$ = $\ddot{y}$ this $\ddot{z}$ = 0 and $\dot{x}$, $\dot{y}$, $\dot{z}$ = 0 this will define as equilibrium point. So in equilibrium state no acceleration in the synodic frame and no velocity also in the synodic frame means the point is not moving at all it is not accelerating it is not having any acceleration and no velocity also and therefore it will appear a stationary to an observer in the synodic frame, and then we can solve this equation. So, the first equation then gets reduced to.

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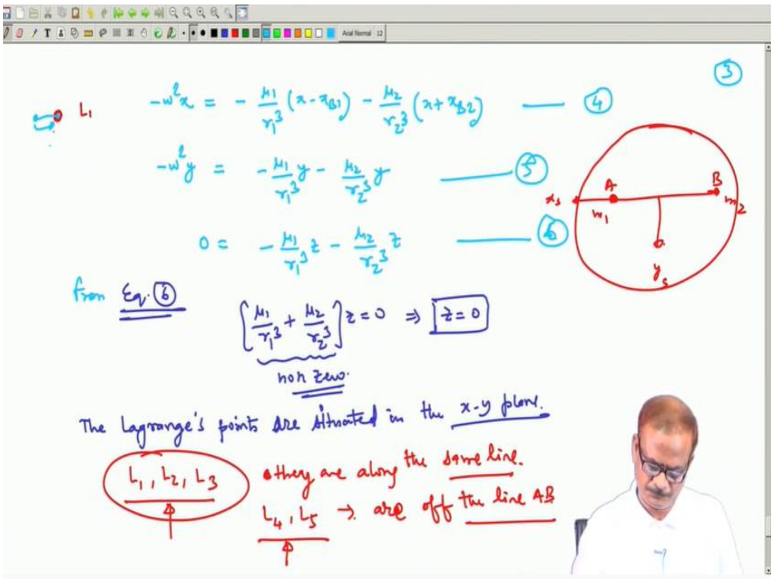

$$-\omega^2 x = -\frac{\mu_1}{r_1^3}(x - x_{B1}) - \frac{\mu_2}{r_2^3}(x + x_{B2})$$

the second equation gets reduced to ω^2 y and in the third term we do not have the ω^2 term so the left hand side, simply we write this as

$$-\omega^2 y = -\frac{\mu_1}{r_1^3} y - \frac{\mu_2}{r_2^3} y$$

$$0 = -\frac{\mu_1}{r_1^3} z - \frac{\mu_2}{r_2^3} z$$

this is equation number 6. From equation 6 we can see that it can be rewritten as and the quantity here in the bracket, this is a non 0 quantity because r_1 and r_2 they are non-zero and this implies therefore z = 0.

So, simply the first thing that we get from here is the Lagrange points are situated in the xy frame okay that means here what we have shown this z this quantity z will be equal to 0 the whole thing is going to lie I will show it by red colour whole thing will lie in this plane okay lying in the x_s, y_s plane.

This component this is going to be equal to 0 for Lagrange point. So, this is a first simplification we have got now the situation gets reduced into a form this is m_1 here and this is m_2 and all your points will be and there is x and y x we have taken in this direction this is x_s direction and this direction we have taken as y_s direction. So, all your points are going to line in this plane and our job is to find out all of those points.

So, all together there are 5 Lagrange point and they are named as L_1, L_2, L_3 these are colinear lie in the; they are in the same they are along the same line. They are colinear and L_4, L_5 are along the y direction means both x and y coordinates will be available for this lie we write this as the AB so we can write it in a better way lie along are off the line AB.

So our search if for all these 3 points, 5 points L_1, L_2, L_3, L_4 and L_5 these are five equilibrium points but our job does not end here, they used to relate it to this will be better all these points are stable or not because if you want to locate any say for a satellite over that point okay will it stay there, this will be the big question and this we also need to answer and that is called the stability of the Lagrange point and the stability how do we measure it?

So if this is the say the Lagrange point L_1 okay and if the satellite is lying over this i will make it little larger and then I will show the satellite with a blue dot okay and if the satellite is disturbed from this place to this place so can bit return back again to the red point which is a Lagrange point which I am going to work out later on. So, this issue we will be dealing in this topic.

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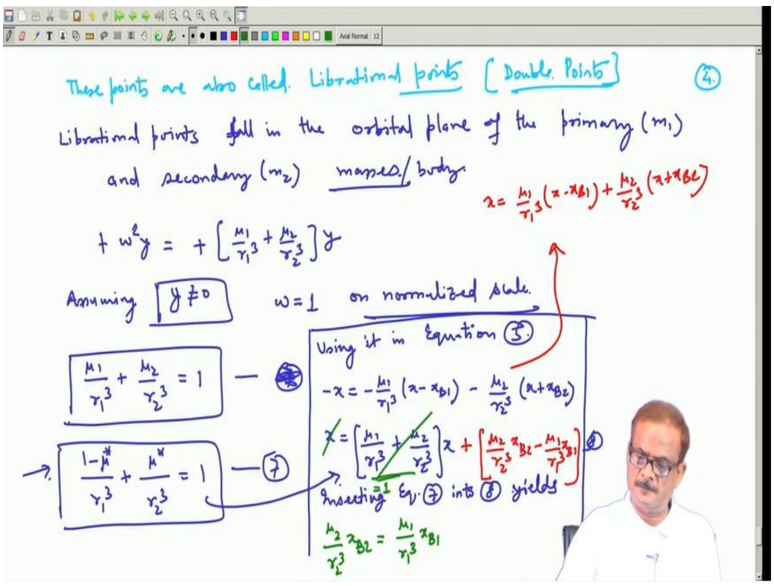

These points are also called Liberational points and also, if you look into the calculus or you say the differential geometry from there you will see that these are the double points. We will look into the for this definition double points what these are so our conclusion right now is Liberational points are the Lagrange points fall in the orbital plane of the primary and secondary masses primary mass m_1 and secondary mass m_2 or masses of body.

Now from equation (5) from this equation

$$\omega^2 \text{ y} = +(\mu_1/r_1^3 + \mu_2/r_2^3) \text{ y}$$

on both sides we have minus sign so that makes it plus we can remove it. Assuming why it is not equal to 0 assuming y is not equal to 0 and ω = 1 on normalized scale. So normalized scale solving it is a little easier as compared to the non-normalized scale and moreover μ_1 we have written as in the normalized scale as $1 - \mu^*$ this is equation number 7.

Now this we can utilize in equation number 5. So, equation number 5 we have ω^2 x so means equal to we will put 1 this is - x = these other things we have to copy from this place and remember on the normalized scale we have this representation μ_1, μ_2 already we have discussed. Hopefully we have replaced correspondingly with all those quantities.

So,

$$-x = \frac{\mu_1}{r_1^3}(x - x_{B1}) + \frac{\mu_2}{r_2^3}(x + x_{B2})$$

cancelling the sign and rearranging the terms what we observed that this is $\mu_1/r_1^3 + \mu_2/r_2^3$ x so we can make this whole thing in the next step, i will make it here let us say I write here in this place this is the x = $\mu_1/r_1^3\ x_{B1}$ okay and then rearranging the terms so you get terms like this and the next one we will have new $\mu_2/r_2^3\ x_{B2} - \mu_1/r_1^3\ x_{B1}$ and then inserting it here in this place.

This is your equation (8). I am inserting equation (7) into (8) yields this term will cancel out these two term in that case because this is equal to 1 this quantity equal to 1 therefore this term and this term they will drop out and we are left with

$$\frac{\mu_2}{r_2^3}\ x_{B2} = \frac{\mu_1}{r_1^3}\ x_{B1}$$

We rewrite this on the next page.

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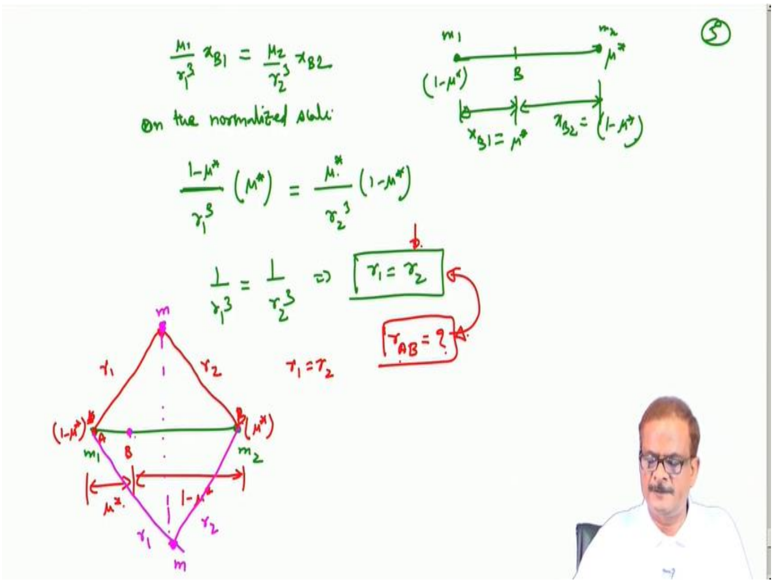

$\mu_1/r_1^3\ x_{B1} = \mu_2/r_2^3\ x_{B2}$ and μ_1 on the normalized scale μ_1 is $1-\mu^*$ and then we will have r_1^3 and then x_{B1} is how much? x_{B1} is see here this quantity we have written as μ^* and this is m_2 and m_1 was here and this is $1-\mu^*$ and B is here this distance which is your x_{B1} in magnitude wise. So, this quantity is μ^* and the quantity from here to here this is the x_{B2} this will be equal to $1-\mu^*$.

So x_{B1} is μ^* similarly on the right-hand side $\mu_2$is μ^* divided by r_2^3 and x_{B2} is 1 - μ^*. So, immediately what we observed that

$$\frac{1}{r_1^3} = \frac{1}{r_2^3}$$

and this implies

$$r_1 = r_2$$

So immediately what we observed that if you have points here, the primary and the secondary bodies in m_1 and m_2 so two of the Liberational points or the Lagrange points are going to lie.

And this will be r_1 and this will be r_2 whether $r_1 = r_2$ this is the immediate result what we get and somewhere here your barycentre is located and this we are writing as $1-\mu^*$ this on the normalized scale and this as μ^* this distance as $1-\mu^*$ and this distance as μ^*. But still our job is not over here we need to workout further because on the normalized scale what will be the value of r_1 and r_2 thar we have to get also we need to find out let us say this is point 1 and this is point 2.

So, either we write as A and B so if we write it like this r_{AB} what is the relation between all these. This also we need to derive. Moreover, what we observe that we will have this $r_1 = r_2$ will be satisfied for another point which is lying here just above on the other side of this then also this will be r_1 and r_2 here this is your mass m this is mass m. So, either mass m can be here and it can also be here and then it will appear to be stationary to an observer which is sitting on at the point B in the synodic frame which is rotating. So, in the next class we are going to work out the other relations so we will stop here thank you very much.