Space Flight Mechanics
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Lecture - 39
Restricted 3 – Body Problem (Contd.,)

Welcome to lecture 39. We have been discussing about the Lagrange points on the normalized scale, so we start with that.

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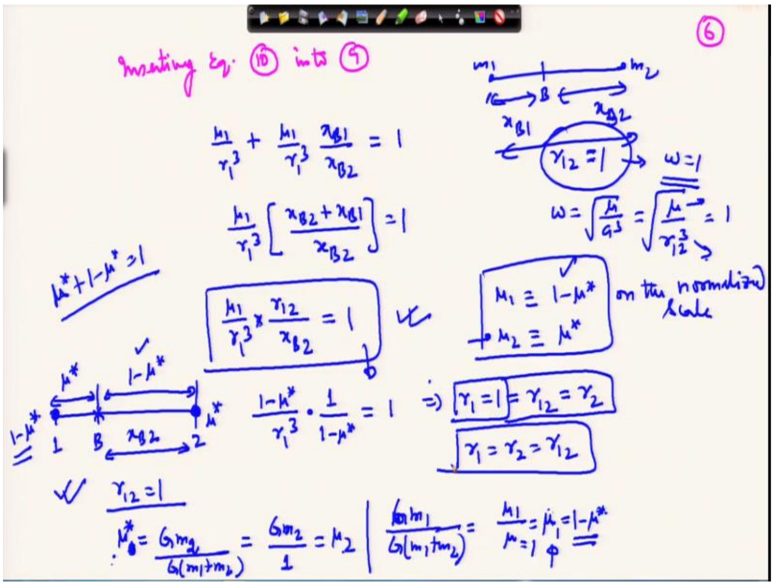

So if you look in the previous lecture, we have derived this. So μ_1/r_1^3.

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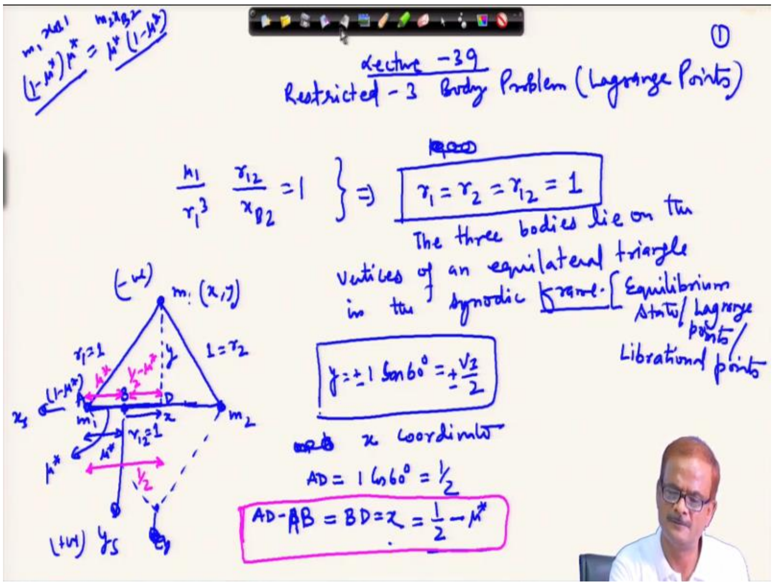

And then

$$\frac{\mu_1}{r_1^3} \times \frac{r_{12}}{x_{B2}} = 1$$

. So if you look into this μ_1 on the normalized scale.

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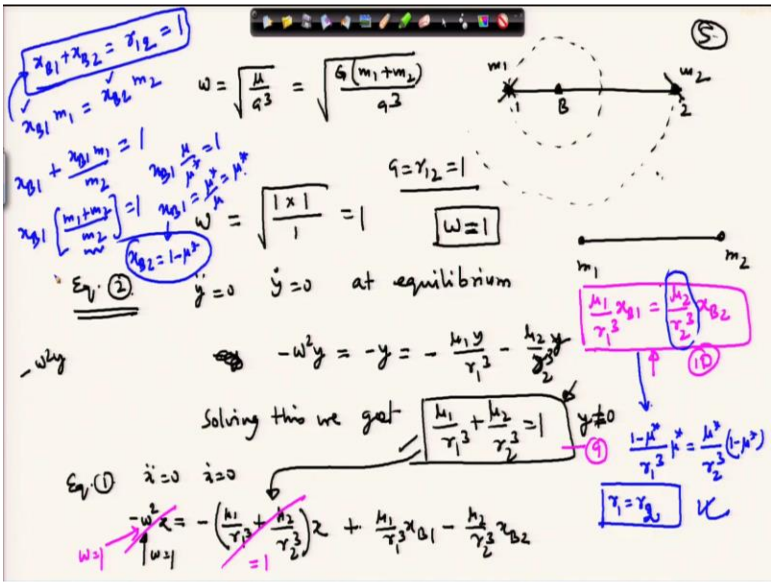

Once we are choosing this on the normalized scale already we have mentioned somewhere this μ_1. So if you use the normalized scale for this. So μ_1 needs to be replaced by

$$\mu_1 = \mu_1 - \mu^*$$

So μ_1 is on the normalized scale, 1 - μ^*, and μ_2 is μ^* on the normalized scale. Already the ω, we have eliminated using the normalized scale what these things were remaining. Now, also the distances already we have mentioned that this is B.

So this distance is μ^* and this distance will be equal to 1 - μ^*, because if you add 1, μ^* times + 1 - μ^* so this gives you 1. So this is unit distance, this is 1 and this is 2. So

$$r_{12} = 1$$

this mass is μ^* here and this mass is (1 - μ^*) and depending on the heaviness of the mass with respect to each other, due to barycenter, it will shift either to the left or to the right. So, r_{12} = 1 and similarly this μ_1 then is yours because G = 1, remember your μ_1 = Gm_1 and then you are dividing it by G (m_1 + m_2) to get μ^*.

Remove this, so this is and here let us write m_2, μ^* you are getting it this way. So, already quantity in the denominator is 1. And Gm_2 is nothing but your μ_2. So

$$\mu_2 = \mu^*$$

So this fact we are going to use and I have written here, in the same way the G times m_1 divided by G (m_1+ m_2). We can check this the upper one is μ_1 and lower one is denominator is μ.

And this we have written as, this is nothing but your μ_1 star using normalized form which is nothing but 1 - μ^*. Here, once we have written here μ_2 = μ^*. So this is μ_2 and divided by 1,

$$\mu = 1$$

so the same way we write here or write it like this μ_1 equal to because on the normalized scale μ = 1 and therefore just gets reduced to μ_1 = 1 - μ^* which I have written here in this place.

So if I insert here so this becomes (1 - μ^*)divided by r_1^3 and r_{12} is already 1, and x_{B2} from this place it we look x_{B2}, so x_{B2} is that this distance. This is x_{B2} and which is nothing but (1 - μ^*) already I have written. So we get this as 1 from here. And this implies

$$r_1 = 1$$

and this equal to r_{12}. Also, if we go back and look here into this equation, so you will find that μ_1 equal to on the normalized scale

$$\mu_1 = \frac{1 - \mu^*}{r_1^3}$$

And x_{B1} is nothing but μ^* on the right hand side you have μ_2 which is nothing but μ^* and r_2^3 and x_{B2} is nothing but (1 - μ^*) as per the figure we have made here. So from here, immediately we can conclude that

$$r_1 = r_2$$

So one result we are getting here, another result we are getting here in this place r_1 = $r_2$and therefore this must be equal to r_2. So therefore, r_1 = r_2 . This is the result on the normalized scale.

Something more I have to write it here, so x_{B1} + x_{B2} this we have written as r_{12}.

$$x_{B1} + x_{B2} = r_{12} = 1$$

And on the normalized scale, this quantity is 1 and also x_{B1} times m_1, this is a center of mass property, x_{B2} times m_2, these are magnitude wise not with sign. So from here x_{B2} we can replace in the above equation for we get x_{B1} + x_{B2} from here become x_{B1} m_1 divided by m_2 this equal to 1.

And you can check x_{B1} can be taken outside. So this becomes m_1 + m_2 divided by m_2 = 1. And on the normalized scale, if I choose m_1 + m_2 = 1, so x_{B1} and m_2 we write as μ^* as we are writing and this as μ. So this becomes μ divided by μ^* = 1. And immediately we can see that x_{B1} = μ^* divided by μ and μ = 1 on normalized scale. Therefore this gets reduced to μ^*. And using this, then we can also write this implies

$$x_{B2} = 1 - \mu^*$$

, because on the normalized scale we have x_{B1} + x_{B2} = 1.

So immediately we can write x_{B2} = 1 - μ^*. So this is your result on the normalized scale. So what we are getting here that using this implies as from the previous discussion we are getting here

$$r_1 = r_2 = r_{12} = 1$$

So the 3 masses the 3 bodies lie on the vertices of an equilateral triangle in the synodic frame. So and this is a configuration for Equilibrium state or Lagrange points. We are writing this as Lagrange points or we are writing this as the Liberational points.

This is your mass m_1 this is mass m_2 here and mass m_3 is present here. This is 1, this is 1 and this is 1. So, you know well that on this side also r_1 = $r_2$will be satisfied and this is r_{12}. And immediately, if the barycenter is here, and it depends on which mass is heavier if m_2 is heavier it will lie on this side if it is m_1 is heavier it will lie on this side. But in all the figures, I have shown it on the left hand side.

So I will continue with this figure only and it is coordinate can be then immediately determined. This is y coordinate and from here to here, this is the x coordinate. So this is your mass m whose coordinates are abscission and coordinates are x and y, therefore y = 1 cos 1 sin 60° this equal to root 3/2, and we give here plus, minus to indicate that here in this direction this is x_s and downward we have the along this direction we have y s.

So on this side this is positive, on this side we are showing the negative distance so accordingly we can choose from this point. So, this is your ordinate on the normalized scale. And the x coordinate similarly can be obtained from this point, this distance is known to us. This is μ^* and this distance is also known to us, let us write this point as D and this as A. So, AD is also known to us which is 1 cos 60° that is 1/2.

So, AD - AB that gives you BD which is your x here in this case. So, 1/2 - μ^* this distance from here to here, this distance is your μ^*. This mass is 1 - μ^*. You can check 1 - μ^*, the left hand side mass m_1 times x_{B1} which is μ^* you can see on the right hand side m_2 x_{B2}. So m_2 is μ^* and x_{B2} is 1 - μ^*. So they satisfy they are equal to each other. So there is no problem in that.

So this way we have got the solution here in this place. So x coordinate is given by 1/2 which is the distance from this point to this point, this is half of the r_{12}. This is 1/2 and minus this distance which is μ^*. So this turns out to be 1/2 - μ^*. So this used x coordinate and what we have got on the normalized scale. Now instead of doing this way alternatively, we can work using the formal method.

Which I have told you that the formal method in that will involve where ω we do not make it to 1, we carry it and then solve it. But in that case, depending on how you are trying to proceed, it can be done in a little shorter also but, I will take a little longer root and finish this part so that you understand it better what is that? What are the differences between all of them?

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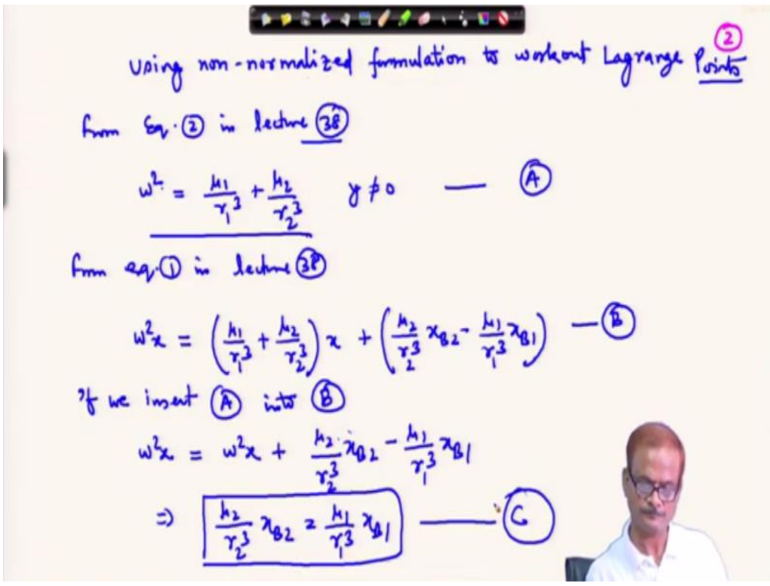

From equation 2 here in the lecture 38 this equation. Already we have written that in the case ω = 1, we made it 1 and solve the problem on the normalized scale. Now, I am not going to normalize it. From equation 1 lecture 38 similarly ω^2 x will be equal to. So from this place, we insert A in B into B we get ω^2 x = ω^2 x + same result as we have derived earlier till this, it is the same but thereafter the things has start getting a little lengthy and this implies.

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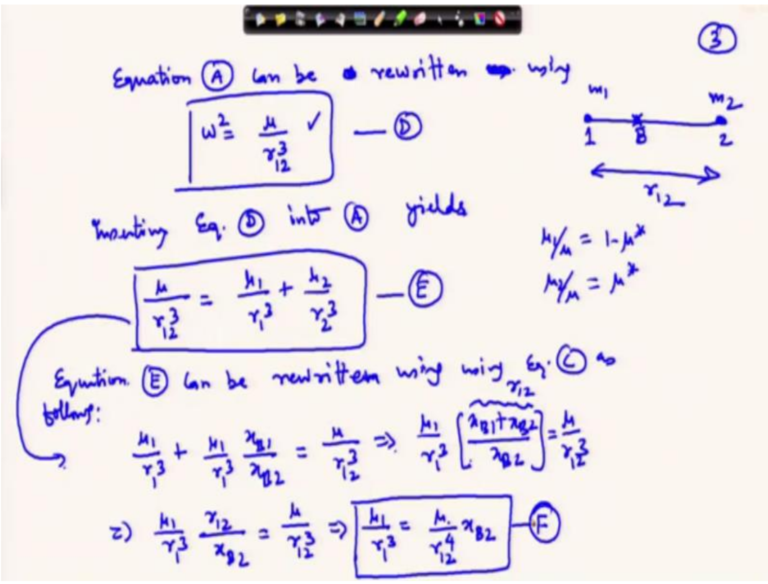

Now equation this equation. Equation A can be rewritten as rewritten using

$$\omega^2 = \mu/r_{12}^3$$

. This already I have explained that the period of these 2 particles or$_2$bodies m_1 and m_2 about each other or either about the barycenter will be given by $\omega^2 = \mu/r_{12}^3$ here this is 1 and this is 2 this distance we have written as r_{12}. This we name as inserting equation D into A.

$$\frac{\mu}{r_{12}^3} = \frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3}$$

And immediately you can see that if you tend to normalize by dividing both sides by μ and writing this as 1 - μ^* and $\mu_2/\mu = \mu^*$, so this gets reduced to the original form what we have started with. Equation E can be rewritten using the earlier formulation we have here, using C, using equation C as follows. This equation we are rewriting here $\mu_1/r_1^3 + \mu_2/r_2^3$ this we have to replace.

So, we are going to replace from this place μ_2/r_2^3 μ_1/r_1^3 times x_{B1}/x_{B2} and this is equal to μ/r_{12}^3 and if we rearrange now this implies $\frac{\mu_1}{r_1^3}\frac{r_{12}}{x_{B2}} = \frac{\mu}{r_{12}^3}$. Remember that this quantity here, this is nothing but r_{12}. So therefore, from this we get

$$\frac{\mu_1}{r_1^3} = \frac{\mu}{r_{12}^4} x_{B2}$$

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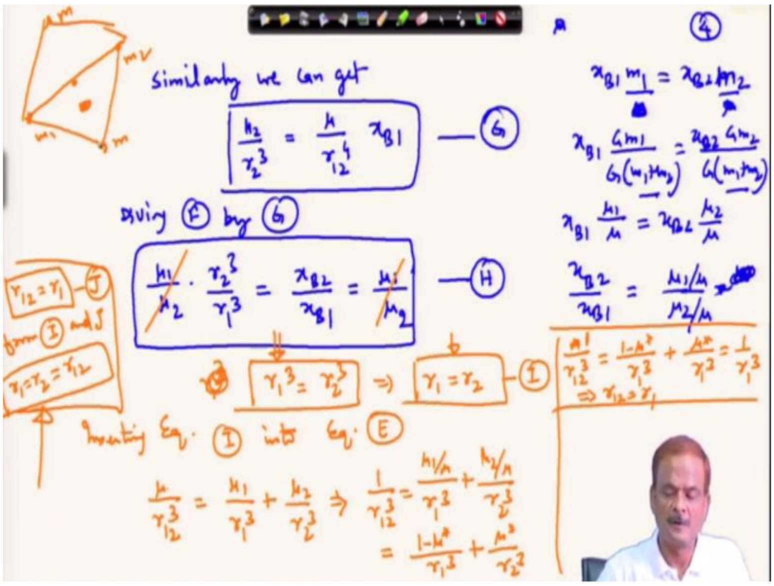

Similarly, we can get this step I am leaving to you get

$$\frac{\mu_2}{r_2^3} = \frac{\mu}{r_{12}^4}\, x_{B1}$$

. Now, if you divide F/G dividing F/G we get μ_1/μ_2 times, r_2^3 divided by r_1^3 on the right hand side μ r this quantity is common, so this we get as x_{B2}/x_{B1}. And as we know that x_{B1} times m_1 this equal to x_{B2} times m_2. And dividing it by μ which is μ is nothing but we can what we can do we have finished the stage is already I have done, but I keep repeating the things for your convenience.

G times B_2 G m_2 divided by G times. So it does not make any difference, m_1 and m_2 is nonzero on both sides. We have divided by it and G is present throughout. So this leads you to G_1 and this quantities μ_1/μ and x_{B2} and this is μ_2 divided by μ. So x_{B2} divided by x_{B1} is nothing but μ_1/μ_2 or we can write it like this. So this is nothing but yourr_1 - μ^* divided by μ^*.

So this quantity is μ_1/μ and now this is G. This part cancels out leaving us with r_1 whole cube this implies r_1^3 = r_2^3. So on the non normalized scale, this is how we get the result and this implies your

$$r_1 = r_2$$

. And then once we put back this result into this equation E for here let us write this which this is I inserting equation I into equation E and this equation for the left hand side is μ/r_{12}^3 hand side is μ_1/r_1^3 .

And μ_2/r_2^3 replace this. So we write it like this $1/r_{12}^3$ this equal to $\frac{\mu_1}{\mu} r_1^3 + \frac{\mu_2}{\mu} r_2^3$ and this implies μ_1 by this is (1 - μ^*) r_1^3 + μ^*/r_2^3. Now already r_1^3 and r_2^3 we have seen that it is equal, therefore, μ/r_{12}^3 this becomes equals to 1 - μ^* divided by r_1^3 + μ^*/r_1^3 and you see that this gets us to μ already we have divided this is 1.

So r_1^3 and this implies $r_{12} = r_1$. So what we have got here?

$$r_{12} = r_1$$

. This is I then this equation we write as J. So, from I and J we see that

$$r_1 = r_2 = r_{12}$$

So we have got to the same result, what we got using the equilateral triangle the normalized scale and the unnormalized scale also we get the same result. There is not nothing much different. But we can observe that doing by this method.

It has taken us a lot of time and we do not also get anything extra in terms of understanding of the system. Therefore, normalized scale working it is always preferable. But still, I will complete this part on the non normalized scale. And also we have to go and find out these are the non collinear solution means m_1 and m_2, they are lying here and m is lying here. Here your m is lying either m is lying here; this is the solution on the equilateral triangle.

Now we look for the collinear solution. Why we look for the collinear solution because the equilibrium can also exist in the collinear form, equilibrium states that will be visible from if you go back look here into the equation.

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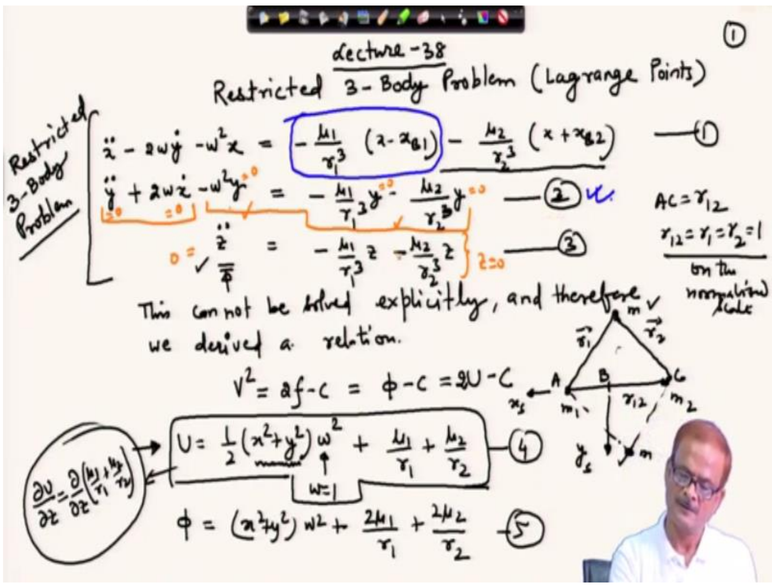

See in the equilibrium state $\dot{x}$ and $\ddot{y}$ this will be 0 and what we are left with this term. So there we assume that y is nonzero. But if y is 0 so the right hand side also equal to 0 and left hand side also equal to 0. That means y = 0 is also a solution for the equilibrium state and z = 0 already we have got. From here we have got that z = 0 by assigning this quantity to 0. Similarly here y = 0, if you put here y = 0, y = 0, y = 0 and this already for the equilibrium these things are 0.

This equal to 0, this equal to 0. So therefore these things are eliminated. And the equation 2 is then satisfied with y = 0 also. So, therefore looking for the collinear solution is also important.

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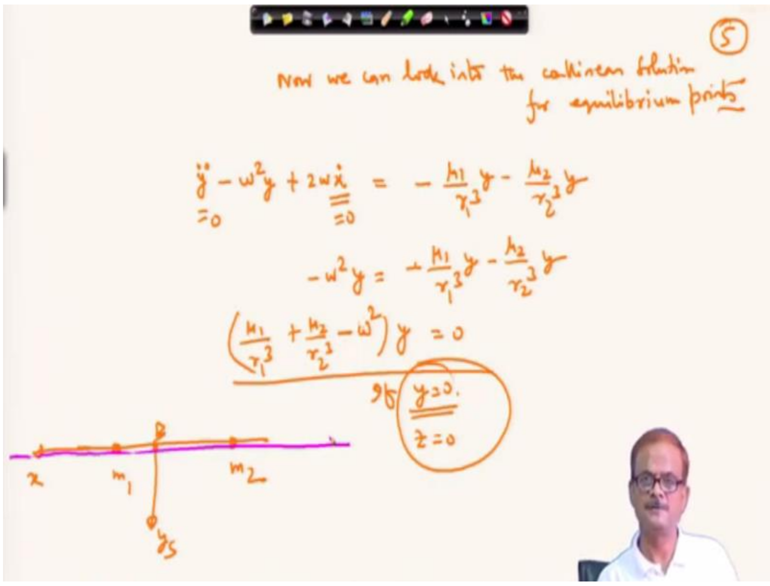

Now we can look into the collinear solution for equilibrium points or the Lagrange points. And this we are doing on the basis of writing $\ddot{y}$ - ω^2y + 2ω$\dot{x}$. Here this quantity we are setting to 0 this quantity we are setting to 0. So minus ω^2 y and as we can see this can be written as

$$\left(\frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3} - \omega^2\right) y = 0$$

So this can be satisfied if y = 0, this will immediately get satisfied. This is minus here.

So if I = 0, we are getting the solution that means now we have to look for the solution where m_1 and m_2 are lying like this y = 0 and also z = 0. That means we have only the x_s and here this is B so y s we have taken it down side. So here in this situation our equilibrium points will lie only along this axis. And we have to search for where those equilibrium points are lying. So we will do this in the next lecture. Thank you very much.