Space Flight Mechanics
Prof. Manoranjan Sinha
Department of Aerospace Engineering
Indian Institute of Technology - Kharagpur

Lecture - 40
Restricted 3 – Body Problem (Contd.,)

Welcome to lecture 40. We have been discussing about the restricted 3 body problem in that context we are working out the Lagrange point and already we have worked out the Lagrange point line of the line joining mass as m_1 and m_2. Now we are looking for the Lagrange point which may be lying. And of course, they lie as well derive there are 3 modal Lagrange point which lie along the x axis in the synodic frame.

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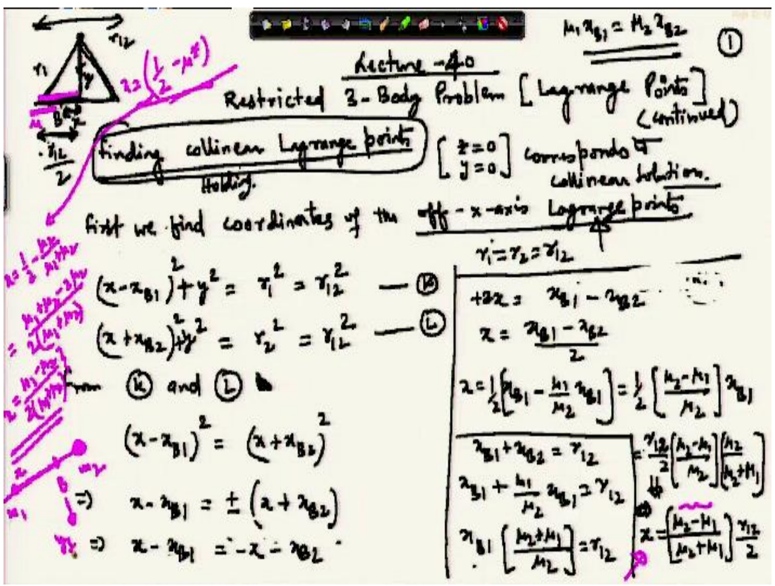

So to work out that; so, finding collinear Lagrange points. So we have the conditions already we have often z = 0 and y = 0 this corresponds to collinear solution. So under that condition, now we have $r_1^2 = r_{12}^2$ because, moreover before we do this, we do little more exercise and that will be better, we first look into the before we finding the collinear Lagrange points, let us also work out the coordinates of the Lagrange points. First we find coordinates of the off x axis Lagrange points which is still not done on the normalizer scale already we have done.

This is r_1 so, you know that this coordinate how we are calculating this one and this one this is y and this is x already we have about but in a formal way also we can get it and it will be good to look into this aspect also. So, we do it first. So, x - off x axis, so, this we are holding for the time being this we are holding not we are doing right now. So,

$$x - x_{B1} + y^2 = r_1^2 = r_{12}^2$$

From we are getting from the point that r_1 = r_2 = r_{12} for this condition. Similarly, x + x_{B2} + y^2 this is square, this is square here equal to

$$r_2^2 = r_{12}^2.$$

We write this is K and this write as L, from K and L we subtract. So, we get immediately we can check that x - x_{B1} square this will be equal to x + x_{B2}^2 and this implies and from here you will see that the plus sign is of no use. So using the minus sign we get here x_{B1} + x_{B2} this equal to r_{12}. So, we are here 2x = r_{12} and x = r_{12} divided by 2 which is very much visual. This is very much visual because in the equilateral triangle, this vertex will lie here, just at the half distance from this point.

So, total length is r_{12} here from this point to this point, this is r_{12}, so, half of this will be r_{12} divided by 2, then this is what we are getting these are very much visual, we do not have to do all this exercise to done, but this is a formal way of working. By doing this we do not do any sort of mistakes. Minus sorry for this we have already done a mistake here, we are looking for the x coordinate so the x coordinate we are measuring from here not from this place. So, we correct it here we are looking for the, this is okay this is r_{12}, r_{12} divided by 2.

But we are looking for x coordinate. So, this is once we take here one more step I will take to expand it one more step we make it x - x_{B1} minus or plus sign is of no use, so we write it this way x_{B2}. And then this result is not applicable, this is wrong. From here we get here 2x = x_{B1}, this is plus x_{B1} - x_{B2} or

$$x = \frac{x_{B1} - x_{B2}}{2}$$

we can use the relationship for the center of mass what we have used earlier. This is μ_1 μ_2 we have μ_1 times x_{B1} = μ_2 times x_{B2} and we are looking for x_{B2}.

So, this will be μ_1 x_{B1}, you can write it like this. This is 1/2 (μ_2 - μ_1)divided by μ_2 times x_{B1}, x_{B1} already we have derived earlier and also let us look here again into that problem the

$$x_{B1} + x_{B2} = r_{12}$$

and if we place your x_{B1} x_{B2} we replace in terms of from this place we can replace x_{B2} = μ_1/μ_2 x_{B1} = r_{12} and this gives us x_{B1} times (μ_2 + μ_1) divided by μ_2 = r_{12}. Using this result, the x can be written as 1/2 (μ_2 - μ_1) divided by μ_2 and x_{B1} from this place then becomes μ_2/μ_2 + $\mu_1 r_{12}$ = r_{12}.

So, this implies

$$x = \frac{\mu_2 - \mu_1}{\mu_2 + \mu_1} \frac{r_{12}}{2}$$

. So, this is the result for the x coordinate. And we have got it very systematically earlier what we did that the x coordinate will be simply because this distance is half here this distance is half, so, 1/2 and this distance is mu, so 1/2 - μ^* we wrote it and we got the x coordinate, but here in this case we see that how we are getting this result. And see this is not inconsistent, you should not consider that this is inconsistent

$$x = 1/2 - \mu^*.$$

So, what is μ^*? This is μ_2/μ_1 + μ_2 and if you do this, so, this gets (μ_1+) - 2μ divided by μ_1 + μ_2 times 2. So, this you get us, (μ_1 - μ_2)divided by 2 times (μ_1 + μ_2). So, this is your x. So, in that case what you are doing, you are not taking into a count the sign here this will appear all with the sign just the difference you can see here this is (μ_1 - μ_2) and here it is becoming (μ_2 - μ_1). So, it is a justice sign problem, if μ_2 is greater than μ_1 that means the masses m_1 and m_2 are there.

So if this mass is heavier than the mass m_1 so in that case x will be positive here barycenter is located in this place. And what we have done we have sown this x along this direction. And y is we have sown in the downward direction. Go to the next question. We will discuss this problem.

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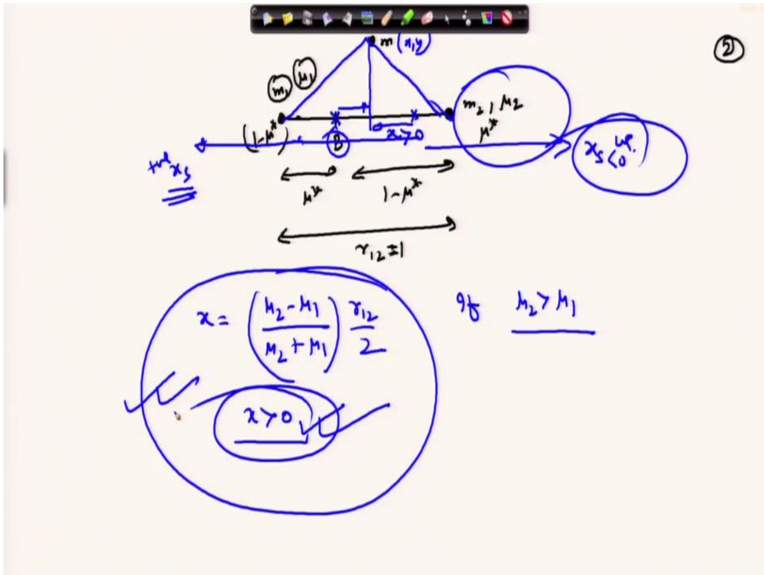

This is m_1 or equally we have written as μ_1 or equally we have written in terms of this is 1 - μ^*, this mass and this point we have taken as B and this point we have taken as m_2 or μ_2 or we can write in terms of μ^* and this distance we have indicated as μ^*, this distances 1 - μ^* and this whole distances r_{12} on the normalized scalars one. So, if μ_2 is heavier, so we are looking for the x coordinate of mass m, here this is the mass m the third mass. So, we are the coordinate will lie.

So, as you can see from the previous page if μ_2 is greater than μ_1 I am for marking it with this blue ring. Let us copy this on the next page

$$x = \frac{\mu_2 - \mu_1}{\mu_2 + \mu_1} \frac{r_{12}}{2}$$

So, if μ_2 is greater than μ_1, so, what happens? The x is greater than 0 that means, the coordinate of this mass which we are writing as x y, so, in that case if m_2 is heavier, so, your barycenter will not lie here barycenter will shift here in this place.

And because this is irrespective of this, this is going to lie on an equilateral triangle therefore you can see that this direction we have taken as positive x direction, this is positive x direction. So, that means, your x coordinate for this x is going to be greater than 0, which is you are getting here in this place. If you reverse the thing if μ_1 is greater, so in that case, you can see that this is barycenter will be here, and then this x will be negative because this is opposite to the x_s direction x_s, s is negative here, in this direction, this is less than 0, negative direction of x.

So, it is a consistent here, what we have got this result it is magnitude wise. While this result what we get this is signwise also consistent. So that is why I prefer when you have to solve certain problems very, it is a very critical design and other things. So you can use this expression rather than doing it by the Ad hoc method.

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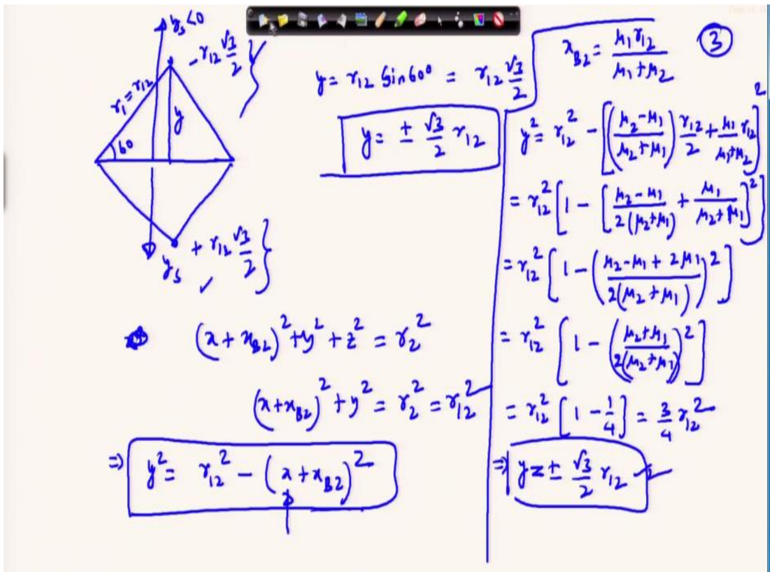

Now, y coordinate also, y coordinate is easy, because as we know that these quantities r 1 = r_{12} this angle is 60°. So here in this case y will be simply r_{12} sine 60° means, r_{12} √3/2 and in this direction from here to here y s is negative side and this is positive side for y s, for this coordinate here, y will be - r_{12} √3/2. And on this side this will be plus r_{12} √3/2. So, that means the y coordinates are given by plus minus √3/2 r_{12}.

And this is just by looking at this figure simply and working out. Otherwise, if you use this expression, what we have used earlier the $(x + x_{B2})^2$ + y^2 + z^2 square, this we have written as r_2^2 inversely for the Lagrange point z = 0, so, this gets reduced to $(x + x_{B2})^2$+ y^2 = r_2^2 = r_{12}^2. And therefore, from here we can write

$$y^2 = r_{12}^2 - (x + x_{B2})^2$$

x coordinate we know.

So, we can insert here and we can solve for this x coordinate we know in terms of r_{12}, okay. So, therefore, y^2 = r_{12}^2 -; x is (μ_2 - μ_1) divided by (μ_2 + μ_1) r_{12}/ 2 and then thereafter we have plus

x_{B2} , and plus x_{B2} we have to replace x_{B2} also we have worked out somewhere, x_{B1} we have written here, x_{B1} is coming in this place. You can see that from this place x_{B1} will be

$$x_{B1} = \frac{\mu_2 \, r_{12}}{\mu_1 + \mu_2}$$

So along the same line you can write, x_{B2} will be r_{12} - x B 1, so you get this as μ_1/r_{12} divided by $\mu_1 + \mu_2$.

So we can insert it. x_{B2} = $\mu_1 r_{12}$ divided by μ_1+ μ_2. So, we insert this in this equation r_{12} this is a square, so r_{12}^2 we can take it outside and then 1 minus here we get as μ_2 - μ_1 divided by 2 times μ_2 + μ_1 and plus μ_1/μ_2 + μ_1 and this is a square. This is not just a theoretical exercise; this is for proper understanding of how we go on solving the problem in a systematic way. This will help you solve the problems, this is 2μ_1 divided by μ_2 + μ_1 square and we just see that case in the denominator 2 is also there so this is 2 times.

So r_{12}^2 times 1 - 1/4 = 3/4 r_{12}^2. So this implies

$$y \; = \; \pm \frac{\sqrt{3}}{2} \, r_{12}$$

So, this we have done using so much of computation what directly it was visual from there because it is a collateral triangle. So, it will come here in this format, which is verified from this place.

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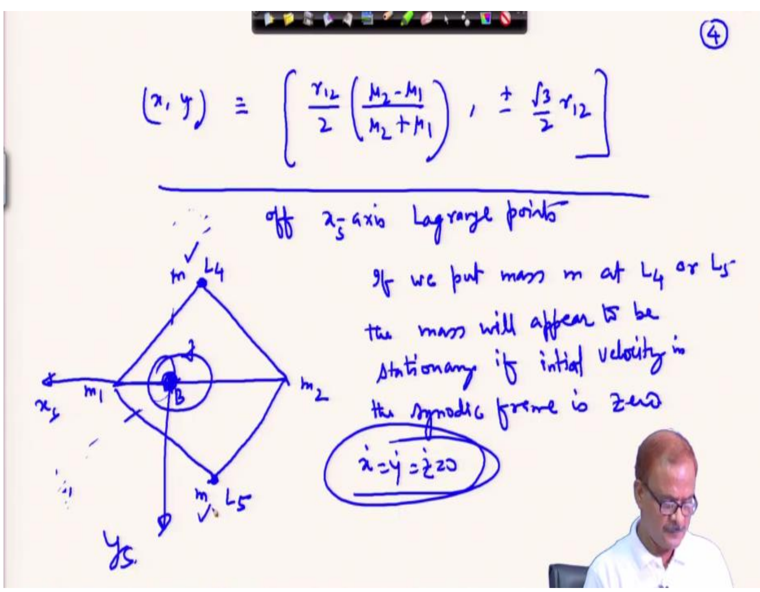

Thus the x y coordinates it can be written as

$$(x, y) = \left[\frac{r_{12}}{2}\,\frac{\mu_2 - \mu_1}{\mu_2 + \mu_1}, \pm\frac{\sqrt{3}}{2}\,r_{12}\right]$$

So, these are the off x_s axis Lagrange points, now I will name them on the equilateral triangle, this is L$_4$ and this is L$_5$. This mass we have written as m_1 this mass as m_2. And these are the 2 equilibrium points we are the mass m can, if you put so the synodic frame it will appear to be a stationary velocity synodic frame is 0, that is

$$\dot{x} = \dot{y} = \dot{z} = 0$$

So, if you keep it here, so, an observer which is sitting at the barycenter and your synodic frame is located here, so to this observer here in this case, we are discussing about the Lagrange points, so we need not take this part, otherwise this will confuse, so y s will now will indicate along in this direction. This is y s direction so to an observer who is sitting here in this place and rotating along with the synodic frame to him this mass m will appear to be here at least all the time. Now we go into the collinear solution.

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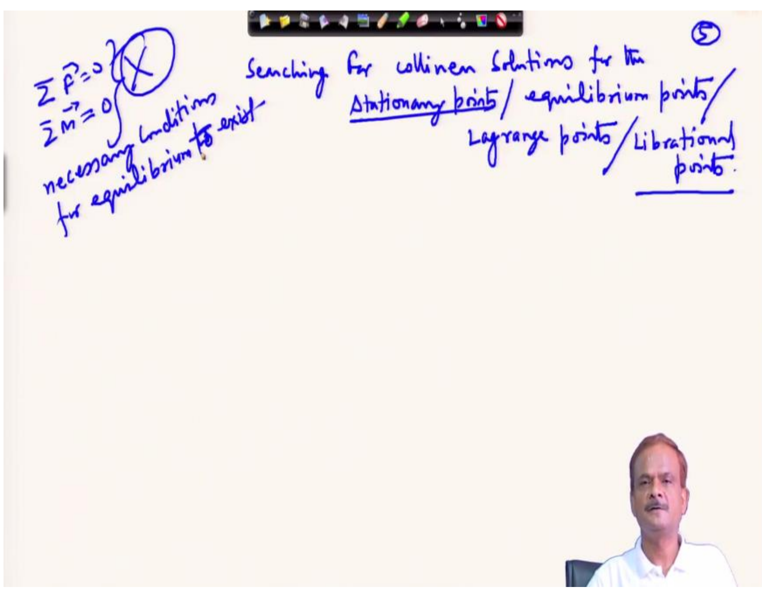

A stationary points why we are calling because by definition the equilibrium points are the stationary points of a system. This is very wide definition and what you will have learnt in your physics at the once the equilibrium points are defined as

$$\sum \vec{F} = 0$$

$$\sum \vec{m} = 0$$

this is not correct. These are just necessary condition for equilibrium to exist. So, what we will do that because a phrase up we will be doing this problem or phrase. So therefore, we will continue in the next lecture. And we will wind up this lecture here. Thank you very much.