Space Flight Mechanics

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Lecture - 41
Restricted 3 - Body Problem (Contd.,)

Welcome to lecture 41 we have been discussing about the restricted 3 body problem and in that context we have worked out the Lagrange points 4 and 5 and whose coordinate also the expression for the coordinates also be derived.

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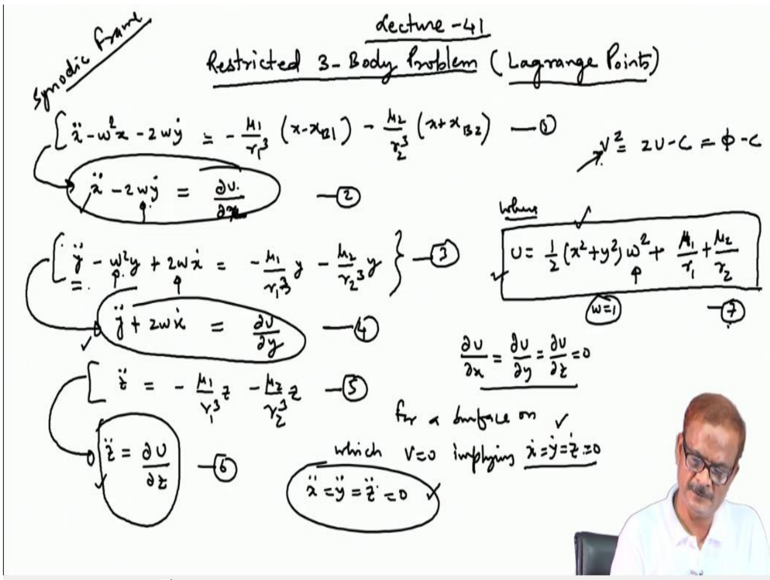

So, recalling what we have done earlier. So, these are the equations of motion, we have derived in the synodic frame. And later on as I told you that if we assume U to be this, the U given by the expression which is given in the equation number (7), so the same equation can be reduced to this form

$$\ddot{x} - 2\omega\dot{y} = \frac{\partial U}{\partial x}$$

Now, in the case ω = 1, so, you can see that this $\ddot{x}$ - 2$\dot{y}$ this gets simplified.

Similarly, this equation gets reduce to this format using this U and this gets reduce to this format and that is based on that and once we assume ω = 1, so, in that case ω basically we are eliminating from this becomes 1 here in this place and also it will become 1 and 1 here in this place and all these places wherever it is shown, so, you get a simplified form this and then we discuss the double points.

So, double points reiterated that these are the points where $\partial U/\partial x$, $\partial U/\partial y$, $\partial U/\partial z$ all these are 0 double points are the points where the curve self intercepts itself or it say the cross section surface self intercepts itself for this, what we see that for surface on which V = 0 implying

$$\dot{x} = \dot{y} = 0$$

. So, velocity is 0, so, all the components of velocity is 0 in this expression, so in that condition this will be satisfied if

$$\ddot{x} = \ddot{y} = \ddot{z}.$$

So both sides will be satisfied this equation, this equation and in this equation as we can see, so both sides will be satisfied this quantity is 0, so surface of 0 velocity and the acceleration also 0.

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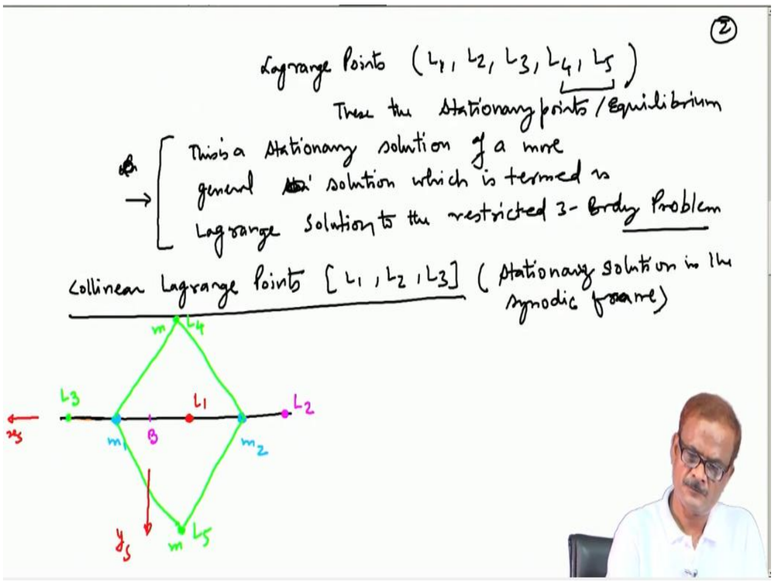

So, basically the, if we recall the Lagrange points we have derived. So Lagrange points L_1, L_2, L_3, L_4 and L_5 out of this 4 and 5 we have already worked out. So, these are the stationary points or the equilibrium points as we have written earlier and this is a special case of the Lagrange solution provided by Lagrange earlier. This is a stationary solution of a more general solution which is termed as Lagrange shall listen to the restricted 3 body problem.

If time permits I will take this otherwise I will provide just written material for this particular part. For now, let us go into the collinear Lagrange points, which are L_1, L_2 and L_3 and these are obviously the stationary solution in the synodic frame. So, if we have masses here m_1 and m_2, this is m1 and m_2 are the masses B is the barycenter as we have considered here to the L_1 point. As we know the other solution is the triangular one L_4 and L_5.

So, L_4 and L_5 we have already determined this is L_4 and L_5 mass m is here. This already we have worked out, so L_1 and L_2, L_3, so L_3 lies on this side. So, say at this point, let us say this is L_3 and somewhere on this side, we will have L_2 and L_1 is located here in between. So, these are the 3 other Lagrange points which are collinear that is they lie on the same axis or they are collinear with each other.

And x_s is how we have chosen, we have chosen x_s in this direction and in this direction we have taken y_s. Now, we look into this part the equation 3. So, what we can observe that for the stationary points $\ddot{y}$ is 0 and $\dot{x}$ is also 0.

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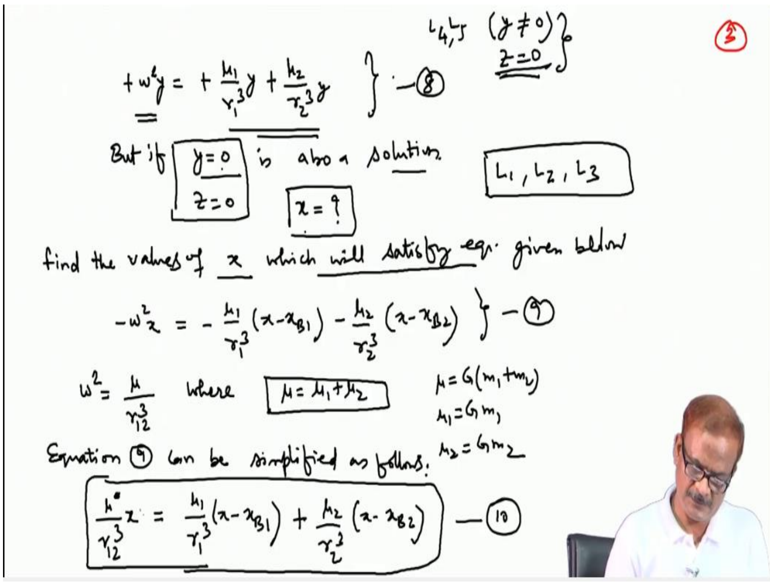

So, we are getting here minus

$$\omega^2 y = \frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3}\, y$$

and this minus sign can be eliminated we can make it plus. Then obviously, if y is not equal to 0 already the solution we have got and that was the context for L_4 and L_5. This was related to y not equal to 0. But if

$$y = 0$$

so, you can immediately see that both sides are satisfied this also becomes 0 and this also becomes 0.

So, therefore y is also a solution. So y = 0 is also a solution. This can be utilized for further and already we have got that for L_4, L_5, z was equal to 0. So this was the situation and this is also valid here. So, z is also equal to 0 in this case. Because in that case you have you are taking this equation and

$$\ddot{z} = 0$$

So, and it is bound to be then the z is bound to be 0, there is no other option because the r_1 and r_2 they are nonzero.

So, now we have these 2 conditions available

$$y = 0 \ \& \ z = 0$$

So, we have to find out what will be the value of the x? This is the question here. So, objective is find the value of x which will satisfy equation. Now, we take the first equation here and put here $\ddot{x}$ = 0 and $\dot{y}$ = 0 because that corresponds to the stationary point. So we will have minus ω^2 x = - μ_1/r_1^3 (x - x_{B1})- μ_2/r_2^3 (x – x_{B2})

So find the value of or values of x, which will satisfy the equation given below. And already we have observed that this quantity is nothing but μ/r_{12}^3. They are in the last lecture we have discussed there. So, again I am not discussing here. Here

$$\mu = \mu_1 + \mu_2$$

means μ = G (m_1 + m_2), μ_1 is Gm_1 and μ_2 is Gm_2. So, this equation then equation 9 can be simplified as follows,

$$\frac{\mu^*}{r_{12}^3}\, x = \frac{\mu_1}{r_1^3}\ (x - x_{B1}) + \frac{\mu_2}{r_2^3}\ (x - x_{B2})$$

And if we solve this equation for x, so we get the Lagrange points, L_1, L_2 and L_3 and here we need to simplify it, we have to write in terms of right now it is know in the non normalized form. So we will put it in normalized form so that the further processing or working with the equation will be much easier.

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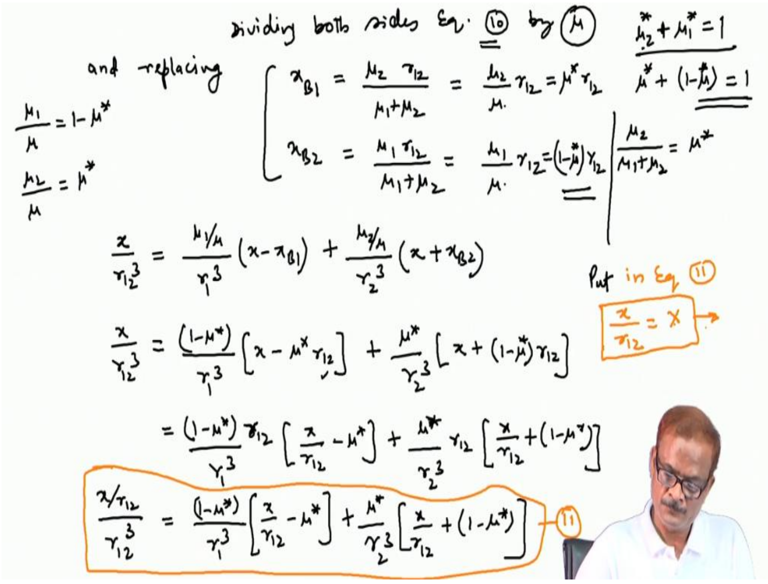

Dividing both sides of equation (10) by μ and replacing $x_{B1}/\mu_2\ r_{12}$ divided by (μ_1+ μ_2) and $x_{B2}/\mu_1\ r_{12}$ divided by (μ_1 + μ_2) this we have derived in the last lecture. So, again I will not repeat it. So, this is clear $\frac{\mu}{\mu_2} r_{12}$ and this is $\frac{\mu_1}{\mu} r_{12}$ and as we remember that μ_2 + μ_2^* + μ_1^* r on the normalized scale we take this as 1 and μ_2 is basically we denoted by μ^*.

So, this quantity then becomes 1 - μ^* so, that both sides are satisfied. So, we follow that notation, if we follow that notation so immediately you can see that so, what we have done? That μ_2/μ_1 + μ_2 we are writing as μ^*. If you follow this notation, so this becomes $\mu^*\ r_{12}$ and this becomes (1 - μ^*) r_{12}. So, we will use this information in the equation 10. So, in the equation 10 then we have x is on the left hand side.

So x will take here, μ is going on the right hand side we will divide it, so we will cancel from the left side. It is a very easy to see for what I demonstrating you must be understanding. So there is no need of further writing here. So simply I write x/r_{12} cube, this equal to and the right side μ I have already divided, so μ_1/μ that becomes 1 – μ. So 1 - μ^*.

Let me write the whole this so what I am trying to say that we can write this as x/r_{12}^3 this equal to $(\mu_1/\mu)/r_1^3$ (x - x_{B1}) and plus $(\mu_2/\mu)/r_2^3$ (x + x_{B2}) μ_1 by already we have written here, μ_1/μ is nothing but 1 - μ^* and μ_2/μ is nothing but μ^*. So we replace with that, so, we get here 1 - μ^* divided by r_1^3 and here this is x - x_{B1}.

So x_{B1} is μ^* r_{12} and plus μ_2 this becomes μ^* (1 - μ^*) r_{12} which we are taking from this place, then what we do that we take out r_{12} from this place. So if you do this μ^*/r_2^3. This is μ^*. Now divide both sides by r_{12} so this will become

$$\frac{\frac{x}{r_{12}}}{r_{12}^3} = \frac{1-\mu^*}{r_1^3}\left(\frac{x}{r_{12}} - \mu^*\right) + \frac{\mu^*}{r_2^3}\left(\frac{x}{r_{12}}(1-\mu^*)\right)$$

And then we will put at this equation we will name as (10), (11) this is equation number (11). So, put here put in equation (11) x/r_{12} = X.

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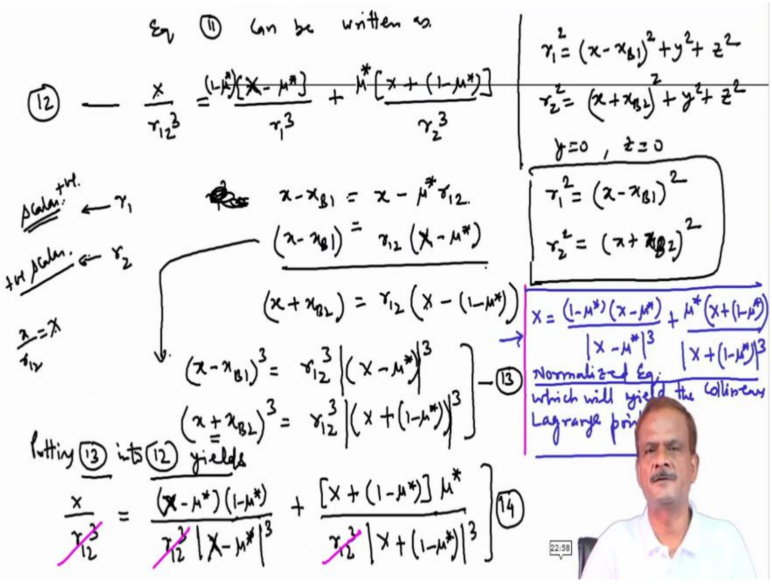

Equation (11) can be written as x divided by r_{12}^3 this equal to X - μ^*/r_1^3 + x + (1 - μ^*) divided by r_2^3, now r_1^3 and r_2^3 we need to replace, if you remember r_1 is what we have written as

$$r_1^2 = (x + x_{B1})^2 + y^2 + z^2$$

$$r_2^2 = (x + x_{B2})^2 + y^2 + z^2$$

So if y = 0 and z = 0, so r_1 is square this can be replaced by $(x - x_{B2})^2$.

And r_2^2 can be replaced by $(x + x_{B2})^2$ and here, then so we will utilize this expression here. So, this is our equation number r_{12}, $r_1$is a scalar quantity a scalar positive value. Similarly, r_2 this is a positive scalar quantity because this is just the distance so it cannot be negative so what we do now? That we take x B out of this and let us write this as r_1^3. So, r_1^3 will be x - x_{B1} here also we need to replace the same way.

As we have done earlier, we use this information $x_{B1}$and x_{B2} which we are utilized here, this is x_{B2} and this is x_{B1},

$$x_{B1} = \mu^* r_{12} = (x - \mu^*)\ r_{12}$$

let us write it this way. And then this we can right as so, if we take r_{12} of outside, so this becomes x - μ^* X because x/r_{12}. We are writing as X. So, therefore, x - x_{B1} can be written like this.

Similarly, x + x_{B2} this can be written as r_{12} times x – (1 - μ^*) therefore, from here immediately we can see that

$$(x + x_{B2})^3 = r_{12}^3\ |x + (1 - \mu^*)|^3$$

this also and say this is 13 for putting 13 into 12 yields x/r_{12}^3 and because this is distance involved. So, this should always be positive. So, this extra mod sign we will introduce.

So therefore you have x - μ^*. We have missed out here, the μ_1 also this part was there. So, this part we have missed out. So, here 1 - μ^* was missing, this part is there. So, we introduced that also. And in this place we have μ^* , for μ^* is also introduced. So, this way we have here (1 - μ^*) times r_{12}^3 |X - μ^*|^3

This is here plus because this is plus x_{B2} + (1 - μ^*)^3. And then this multiplied by μ^*. This is our equation number 14. And immediately we can see from this place that this terms because it is a nonzero for their dropout. And what we get to; finally we are left with the equation

$$X = \frac{(1 - \mu^*)(\mathrm{X} - \mu^*)}{|\mathrm{X} - \mu^*|^3} + \frac{\mu^*\big(X + (1 - \mu^*)\big)}{|X + (1 - \mu^*)|^3}$$

So, this is the equation we have got in normalized form. So this is normalized equation which will yield the collinear Lagrange points and we need to solve this equation. So, we end here this lecture and we will continue in the next lecture with the same stuff here. And we will solve this equation for L_1, L_2 and L_3. Let us see how do we work it out and there is an alternate method of doing the same problem, which is very straightforward. And I will also take up that issue of first let us conclude this part and thereafter we will take this.

So, I am doing a particular problem in different ways so, that you become aware of the fundamentals and because now, we have no scope for working out the problems because already we are lagging a lot. So, I will try to add some extra lectures related to problem solving for the fifth week and 6 weeks. Till 4 week, I have done certain problems, but for the fifth week I have not done so I will add as an addendum to that. So, thank you very much. We will start with the next lecture thereafter.