Space Flight Mechanics

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Lecture – 42
Restricted 3 - Body Problem (Contd.,)

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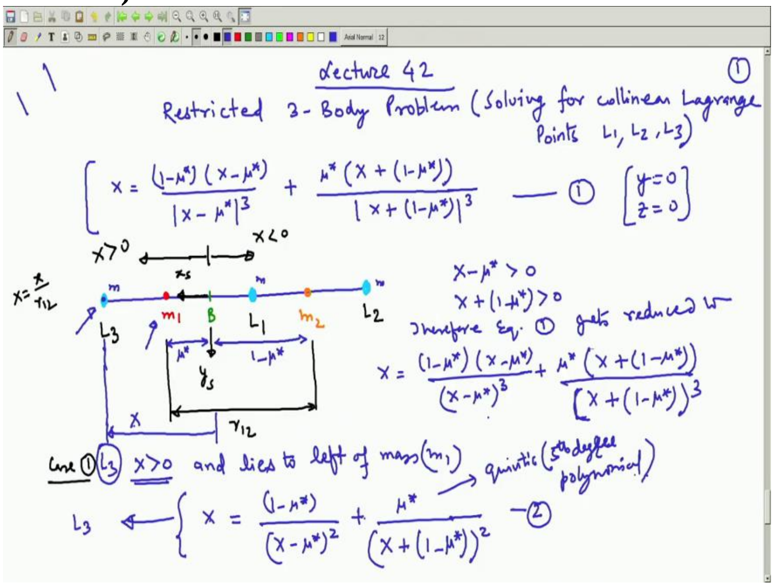

Welcome to lecture number 42. So, already we have solved we got this equation for the Lagrange points. Under the condition that y = 0, z = 0 when this we have got in the normalized form. Now we have to solve it. So, in the while solving for Lagrange points, what we realize that if we have mass m_1 here, mass m_2 here. So, I can have and barycenter obviously it is a line in this place.

So, in can lie either here in this place or either it can lie somewhere intermediate to this 2 masses, or either it can lie here in this place. And accordingly this positions are named as this is named as L_2, this is named as L_1 and this is named as L_3. So, these are the 3 Lagrange points L_1, L_2, L_3. And as you know that, at the barycenter we fix the difference frame. And in this direction we have taken x s to be positive.

And downward we have taken y s to be positive. So, here in this direction your x will be greater than 0 and from this place 1 what x will be less than 0, where x is the normalized value as we have written x divided by r_{12}, r_{12} is the distance between masses m_1 and m_2. So, here this distance is r_1. So, we consider the first case here, case 1, x is greater than 0 and lies to the left of mass m_1, which is shown here.

This is the mass m_1 here in this place and the Lagrange point the smaller mass m is either lying here, either lying here or either lying here. So, we are considering the first case, which is corresponding to L_3. So, x is greater than 0 and lies to the left of mass m_1. So, in that case immediately what we can see that x – μ^* and what is μ^*, as we know from our earlier discussion, this μ^* is nothing but this distance and this distance is 1 – μ^*.

So, therefore, you can see that if x is up to this point from here to here, if X is from distance from this place to this place this is X. So X – x this will be this X – μ^* this will be a positive quantity. So, this is a positive quantity and also x + (1 – μ^*) this is also going to be a positive quantity and therefore, equation 1 gets reduced to x = 1 – μ^*, x – μ^* divided by whole cube because it is a positive quantity therefore mod we can write like this, the mod function.

This is basically the sign for the mod modulus of any quantity μ^* (x + 1 – μ^*) divided by (x + (1 – μ^*))^3. And then this then get reduced to

$$X = \frac{1 - \mu^*}{(x - \mu^*)^2} + \frac{\mu^*}{\left(x + (1 - \mu^*)\right)^2}$$

this we name as equation 2. And this equation gives you the solution to this equation you will get L_3 and this gives you quintic. This is basically 50° polynomial. So we do not have much trouble in solving for L_3. Then we take L_2 which is also easier to work with case 2.

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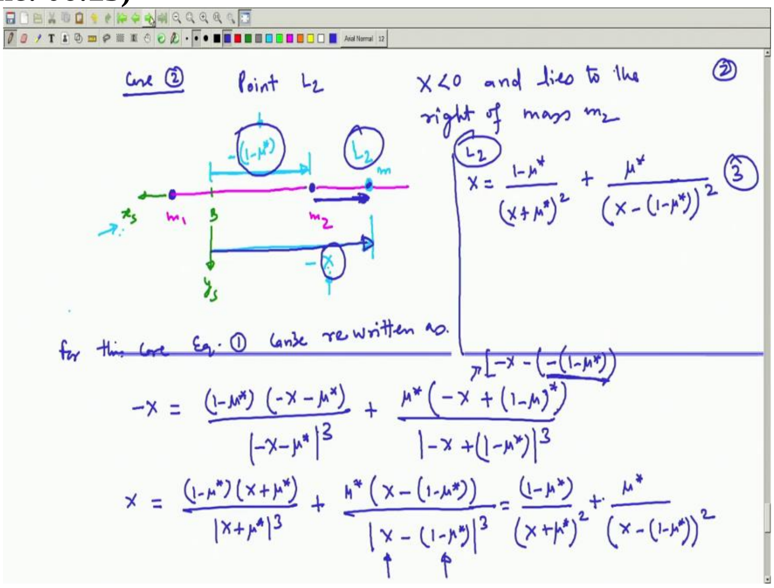

Point L_2 which is so here in this case, x is less than 0 and lies right of the here. It lies to the left of the mass m_1, here in this case lies to the right of mass m_2. So, again drawing the figure here, these are the masses m_1 and m_2, barycenter is located somewhere here. x s is positive in this direction and y s is positive in this direction as for our consumption. And the point L_2 now we are assuming that is a line somewhere here.

This is mass m and L_2. So in this direction, this is your X distance, the location of the third mass. Of course, the distance from this place to this place, this is 1 – μ^*. Now because we are solving for the values, so, what we will do instead of writing here because this is negative direction, so, I will insert a negative sign here in this place. So, this indicates that x we are taking as positive 1 – μ we are taking as positive and this minus sign implies that we are going opposite to the x s direction.

So, this way we will be able to get the once we get the equation and we can solve it. So, that will give us the proper value of the x. So for that case, then equation 1 can be rewritten as minus x. We replace this x y – x in this expression. – x = 1 – μ^* this remains as it is x gets replaced by minus (x – μ^*)^3 divided by minus (x – μ^*) + μ^*times minus x. Again this part this x we are replacing by minus x and then 1 – + (1 – μ^*).

So, this is plus 1 – μ^*. So, what we are looking for this is the basically the distance from this point to this point. This is total distance from here to here, this is minus x. And to this, this part you can also look like this. This is minus x – (1 – μ^*). So that we so from minus x, we are subtracting the quantity a negative quantity – 1 μ x. So that gives a positive value. So ultimately, this gives you vector along this direction.

And the same way here in the denominator we have minus (x + (1 – μ^*))^3. This minus sign we eliminate from both the sides, this gets reduced to 1 – μ^*. This gets x + $\mu^*$and here also this will get reduced to using the mods property. So here if we look here in this place, once the minus sign is removed this is indicating a positive quantity. Similarly, this is 1 – μ^*this is also a positive quantity.

Because the magnitude wise this x is greater than 1 – μ^*. So, we can break this bracket, the mod sign and we can write it like this. 1 – μ^*divided by x + μ^*whole square and + μ^*divided by (x – (1 – μ^*))^2. This is your equation number 3. So ultimately we have what we have concluded that for Lagrange point L_2

$$x = 1 - \mu^*/(x + \mu^*)^2$$

$$X = \frac{1 - \mu^*}{(x - \mu^*)^2} + \frac{\mu^*}{\left(x + (1 - \mu^*)\right)^2}$$

This is equation number 3. So, L_1 and L_2, L_3 and L_2 we are sort out. This is your L_2 point located here in this place. Now, we are left with the L_1 point which is intermediate to m_1 and m_2.

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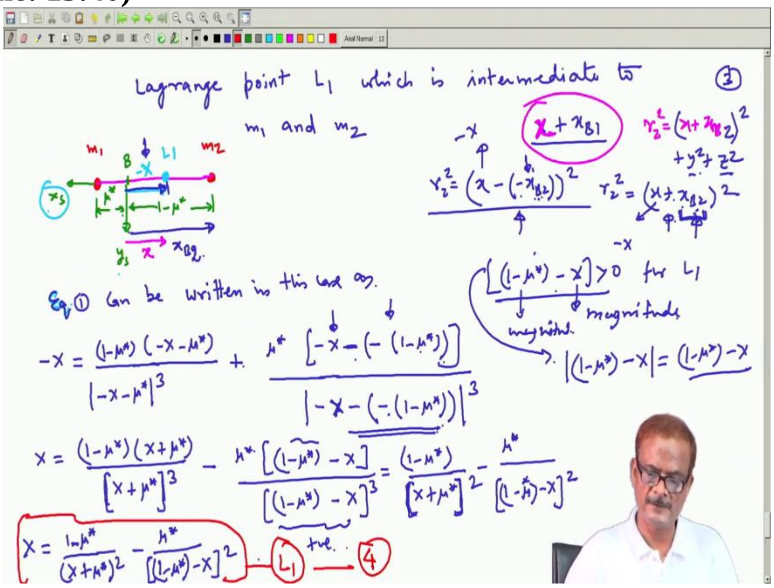

Now, we need to sort out this problem. So, as usual m_1 and m_2 these are the masses. Barycenter is located here in this place in this direction we have taken as x s. This is y s and barycenter is located here this distance we have written as μ^*. And this distance we have written as $1 - \mu^*$ from here to here and the point L_1 it may be lying somewhere let us say in this place.

So, this distance will indicate by x with minus sign. Why? Because it is lying toward the negative x s direction. This is our synoptic frame. It is lying along the negative x direction of the synoptic frame. This case is bit complicated and we have to work it carefully. Because it lies intermediate to the point masses m_1 and m_2. So, here in this case the primary problem see, we have what we are concerned with that, this and this should be taken care of.

We are not worried about this, because it is a scalar and it remains as it is. So, we have to worry just about these quantities. So, the equation 1 can be rewritten as in this case as x has to be replaced by minus x, what other things we have to be very careful while writing. So, here there is no change, but the other part we have to be careful. So, now we have this plus signs + sign I have taken this part we have to be careful in this particular one.

Minus x goes as it is minus x is this distance of whatever things were there X + x_{B1} was it was written like this. Now, here in this case, what we see that x_{B1} is the distance from here to here, while x is the distance from this place to this place, this is let us say the small x in terms of writing in terms of the small x. So, what this was indicating. This is part of the vector

$$r_2^2 = (x + x_{B2})^2 + y^2 + z^2$$

and when y and z they are 0, so r_2^2 and becomes $(x + x_{B2})^2$

And we have to consider it in this format. So this x we are replacing with minus x in the normalized form and this x_{B2} here in this case, so x_{B2}, it has to be replaced by we write it this way. See this minus from this point to this point, this is your x and that too with minus sign as indicated here. So, this is minus x. So, that we are choosing x as positive by putting with a minus sign here. So, this part has been replaced and thereafter we have the quantity x_{B2}.

So, x_{B2} we write it by minus see this quantity r_2^2, this is now here x – (– x_{B2}), because it was in negative direction, which said result of this, if you remember in the beginning once we started working for the restricted 3 body problems, so, we wrote out in this way. So, this x we are replacing with minus x, which is appearing here in this point and thereafter, this is minus and then minus x_{B2}.

Here once we are writing here in this way, so we are ensuring that this x_{B2} is appearing as a positive quantity once we have written that way, so we write it here in this way and then this divided by minus x – μ^{*3}. And then let us rearrange it. So if we take this minus sign, so this we can make your minus you can replace this with a minus sign μ^*. And here we have to be careful.

Now, in this case what we see that the quantity 1 – μ^*. This is the magnitude – x, this quantity is greater than 0 for L_1. Where x once we are writing, so we are considering the magnitude and also this is a magnitude it is a positive quantity here. So, we have to write in a way so that our problems get solved. Here, we are interested in breaking up this and this, we have to cancel out the numerator and denominator.

The upper 1 we can write as (1 – μ^*) – x, this is a positive quantity according to this and the lower 1 similarly we can write here, (1 – μ^*) – x whole cube, this is a positive quantity magnitude wise. So, therefore, we can remove the mod sign and write it in this way. Now, this can be rewritten as (1 – μ^*)x + μ^*, x + μ^* we can cancel out from the numerator and denominator and simply we get here (x + μ^*)^2.

Because this quantity is positive therefore, we can replace it by the square sign in the normal bracket and then this part we have to write. So, this is μ^* divided by (1 – μ^*)– x^2 because this quantity is positive and this quantity here also we can put it like this. So, we are itself. So, because this is positive and therefore, in the numerator denominator we can break it out for the quantity we are following this.

So, the this quantity mod of (1 – μ^*) – x, this quantity will be = (1 – μ^*) – x and this is what we have followed here in this place. Therefore to finally conclude x we get as 1 – μ^* divided by (x + μ^*)^{2}d– x^2 and this is our equation number this is 3. This Lagrange point L_2, Lagrange point L_3 we have got here. This is our Lagrange point L_1 and this we name as equation number 4. So, these are the 3 equations we are having for solving the Lagrange points.

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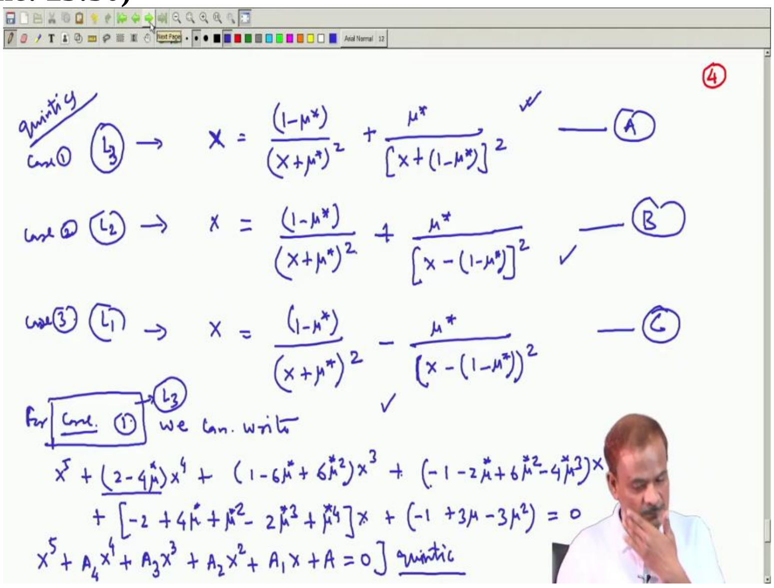

So finally I listed them in one place for easy reference. First we list for L_3.

$$L_3 = X = \frac{1 - \mu^*}{(x + \mu^*)^2} + \frac{\mu^*}{\left(x + (1 - \mu^*)\right)^2}$$

This is related to Lagrange point L_3, then I list for L_2.

$$L_2 = X = \frac{1 - \mu^*}{(x + \mu^*)^2} + \frac{\mu^*}{\left(\mathrm{x} - (1 - \mu^*)\right)^2}$$

Here the x is written as the numerator remains everywhere the same, only thing the denominator will change. This is (x – 1) whole the square and the third one this quantity. One thing you should note it here the quantity in this bracket.

This can also be written as x – (1 – $\mu^*)^2$, because this is always either you write this first or this first it does not matter because there is a square here. Squared means it is either whatever they said it and it will become positive. So it does not matter in that case and therefore we will write in that format. So, this becomes x – (1 – μ^*) whole the squared

$$L_1 = X = \frac{1 - \mu^*}{(x + \mu^*)^2} - \frac{\mu^*}{\left(\mathrm{x} - (1 - \mu^*)\right)^2}$$

this is equation C. So let us verify this 1 – μ^2 there is a minus sign here.

So, there is a minus sign here. Then we have this point x – μ^* that is a plus sign x – μ^* it is a plus sign. And finally, this remains as usual. So these are the 3 equations and if we solved them, and there they are quintics. So these are quintics. And if we solve them, we get the solution for x. And I did it on MATLAB. And so if you can try using your own calculator if it is a possible programmable calculator, you can do what on MATLAB you can find the roots of digit equations, once you have rewritten them, only then it is possible.

So say in the case of this is case 1, case 2, and this is here, this is L_1, the last one we have considered as L_1 and this is case 3. So for case 1, if we rearrange this equation, so finally for this you can do yourself I am not going to write it here. So this can be rewritten as

$$X^5 + A_4 X^4 + A_3 X^3 + A_2 X^2 + A_1 X + A = 0$$

These are all stars. I have not written here a star, but it is indicating a star times x and the last one will be plus.

Therefore, this equation can be written as x to the power 5 and this for this coefficient I will name as A to the power 4, A to the power 3. So, this is quintic and solve it to get the solution of x. So, this is for case 1, case 1 which is referring to L_3. The same way equation can be developed for others. So the rest of the things we will take in next lecture. We stop here. Thank you very much.