Space Flight Mechanics
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Lecture – 43
Restricted 3 – Body Problem (Contd.,)

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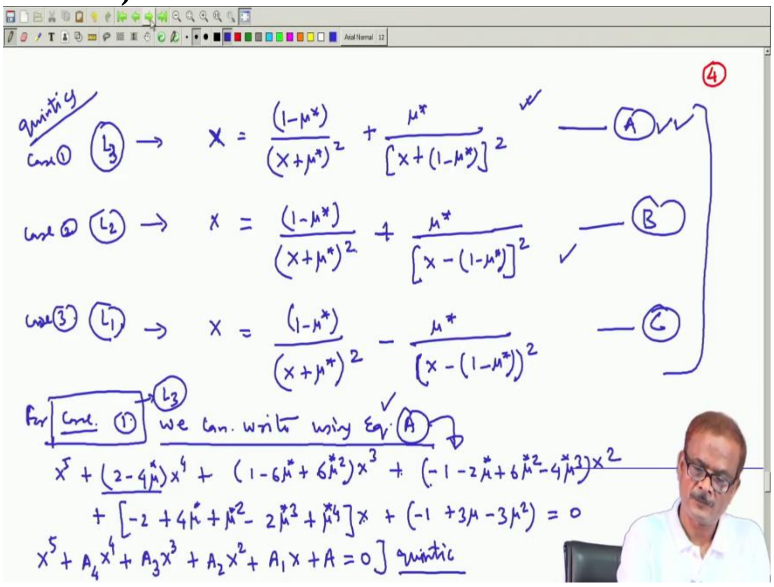

Welcome to lecture number 43. We have derived the equations for finding the Lagrange points L_1, L_2 and L_3 and we further work on that. So, going back if we look into the previous lecture L_1, L_2, L_3 these are given by these 3 equations and out of that the taking equation 1 we can write using equation A here. So, whatever has been derived below it says using equation A.

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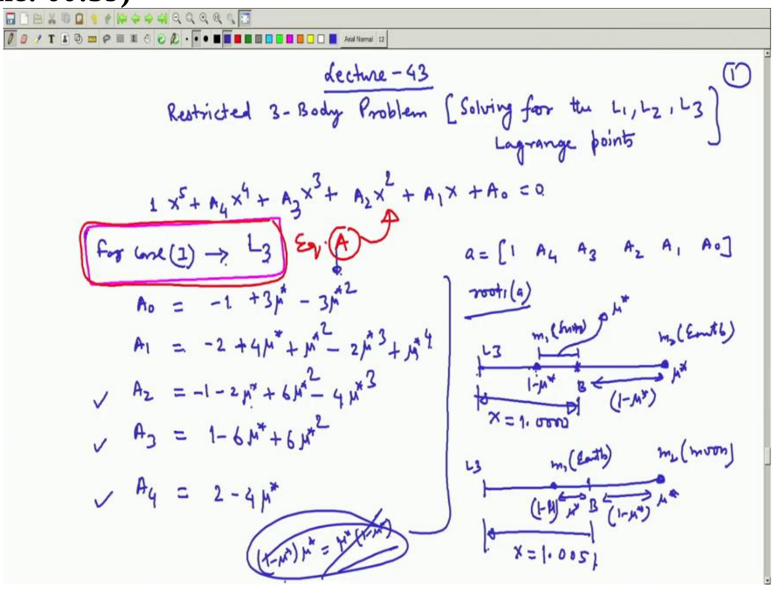

So, from this equation A what we are getting that our equations can be written here in this format

$$X^5 + A_4 X^4 + A_3 X^3 + A_2 X^2 + A_1 X + A = 0$$

And this is true for all the 3 equations (A), (B) and (C) all of them can be reduced into this format. So here in this case for the case 1, which is the case of L_3; the L_3 Lagrange point A_0, A_1, A_2, A_3 and A_4. So,

$$A_4 = 2 - 4\mu^*$$

$$A_3 = 1 - 6\mu^* + 6\mu^{*2}$$

And

$$A_2 = 1 - 2\mu^* + 6\mu^{*2} - 4\mu^{*3}$$

And A_1 is

$$A_1 = -2 + 4\mu^* + 2\mu^{*2} - 2\mu^{*3} + \mu^{*4}$$

a A_0 is here the quantity

$$A_0 = -1 + 3\mu^* - 3\mu^{*2}$$

So, these are the coefficients of the polynomial and coefficient of X to the power 5 this disappears as 1.

So, in MATLAB you can use this to solve this you have to define the coefficients let us say that you write A and then the coefficient A and then which is related to X^5,1 related to X^5 then A_4, A_3, A_2, A_1 and A_0. Defined like this and then thereafter use the command roots a to get the corresponding solution. So, it will depend on the value of the μ^*. How much is the value of μ^* depending on that you will get the solution. So, this is for the case for the case 1 which is related to L_3.

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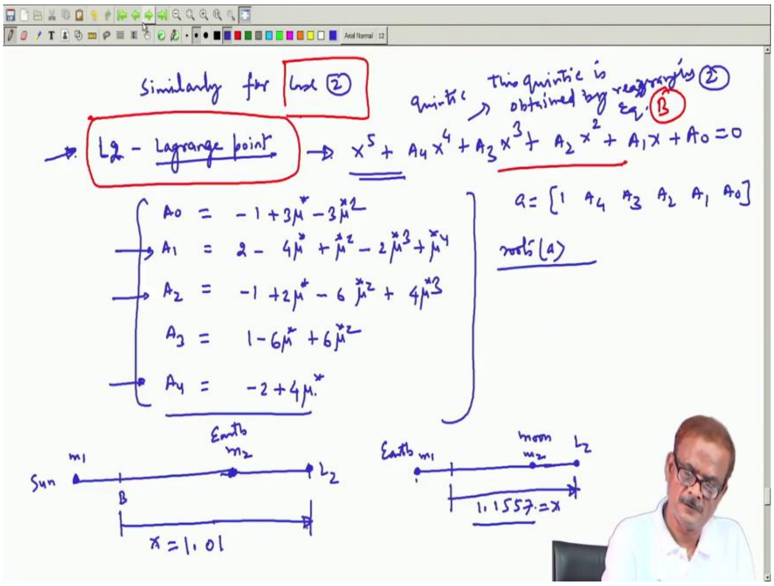

Similarly, for case 2 which we have taken as the L_2 point. The same kind of equation we have to write here, A_4 X to the power 4 A_3. This is again quintic and solve it where we have to define all these quantities. Now we can compare these coefficients if we look into the previous 1, so we find that A_0 coefficient is the same, which is for L_3 point but what is the difference between this and this. So, if A_4 is different then A_2 is different.

A_2 is minus 1 – 2 μ^*, we have minus 1 + 2μ^*, you can see that A_4, A_2 and A_1. A_1 is -2 -4μ^*. So here this is 2 – 4 μ^*. So, this way they are different. So, again, you have to devise the same trick that to define, a as a vector in MATLAB, where you this as 1, A_4, A_3, A_2, A_1 and A_0 and then find roots of A. So, this will list you all the roots of corresponding to this quintic and out of that the positive value we will choose and that will correspond to the Lagrange point L_2.

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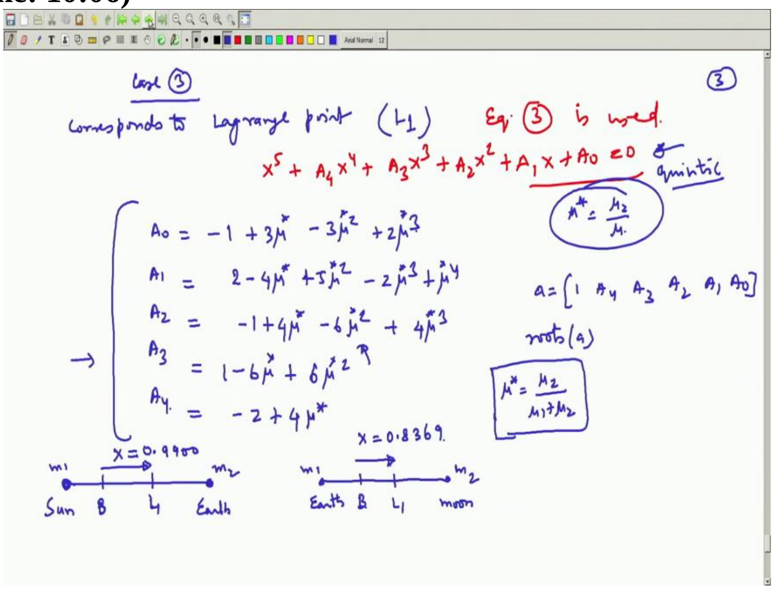

Case 3 in the case 3 this is corresponds to L_1. See in this quintic we will get from this quintic is obtained by rearranging equation B. This has been obtained by rearranging equation A as written here. This is by equation A, which is the case for L_3 case 1 L_3. In this case so for L_3 equation A this quintic is used for case 2 this is used quintic equation B from B we obtained and then for Lagrange point L_1, equation 3 is used and once rearranged.

So it will be represented in the same format. A_3 X cube, A_2 X square A_1 x + A_0 = 0. And here in this case then we list all these values A_0, A_1, A_2, A_3, A_4. So once we have worked it out, we can check that this is the value we get here. And then even we write as 2 – 4 μ + 5 μ^2 all these are into star indebted

$$\mu^* = \frac{\mu_2}{\mu}$$

For this is very important that we consider the mass into whose normalized mass is entering here all in this places.

And next we have A_3 and A_4 remaining. So A_3, A_4, this is 1 – 6μ+ 6 μ^2. So, these are the values and again you apply it right. Define a vector as 1, A_4, A_3, A_2, A_1, A_0 by supplying proper value of the μ^* and then solve it using the command roots of a, it will give you the roots of the quintic polynomial is the quintic fifth order polynomial fifth degree polynomial and after solving this we get this points.

Now, I will list some of the related values. So far this if we have suppose, Sun is here and earth is here. So L_1, let us say it is a line here, B is the barycenter for this distance x, this magnitude wise this is equal to 0.9900. While for the case of the Earth and Moon x is given by 0.8369. We have to use the proper value of the μ^*,

$$\mu^* = \frac{\mu_2}{\mu_1 + \mu_2}$$

and I will list the masses of the Sun and Moon.

This is your m_2, this is m_1, m_2. Similarly here in this case for the point L_2 is lying here. This is m_2 and m1 is lying here barycenter is here, this is your sun and this is Earth. This distance is x. So, magnitude wise magnitude wise L_2 is here. It is a line here in this place. So, this L_2 is then 1.01 and for the earth moon system, Earth and here, there is moon and L_2 is lying here this is m_1, m_2, m_1 m_2 we have written here also barycenter is here.

So, measured from the barycenter, this distance turns out to be 1.1557. This is on normalized scale this is magnitude wise. Similarly, in the case of L_3, this is m_1 sun and m_2 earth, this is a barycenter. Now we are looking for L_1. So L_3 is lying here and distance between this and this. This is your x, so magnitude wise x will be equal to 1.000 from your barycenter, this is the L_3 located on the left hand side of the sun.

Like this, while here in the case of the moon and our system measured from barycenter, this L_3 is located at a distance 1.0051. So, these are the values. Now, once we know the μ^* and these distances obviously once μ^*t is known, for these distances $1 - \mu^*$ immediately we can calculate, we can calculate this distance also distance this masses. This is μ^*. This distance is μ^*.

And these distances here, $1 - \mu^*$. While this mass we have represented as $1 - \mu^*$ and this mass we represented as μ^* according to our notation we have followed. This is $1 - \mu^*$. This mass these distances μ^* and these distances $1 - \mu^*$. So about the center of mass the first movement it balances $1 - \mu^*$ times μ^*, this will be equal to μ^* times 1 - μ^*. So, it balances.

So, this is the way the Lagrange points are solved for now before going any further, quickly, I will look into one alternative way of doing the same problem. And it is a much more easier in fact, than what we have done, we have done it systematically derived using our; the basic equations we have developed, but those basic equations are not necessary to work out this problem. It can be done in some other way also. So, this we have to explore.

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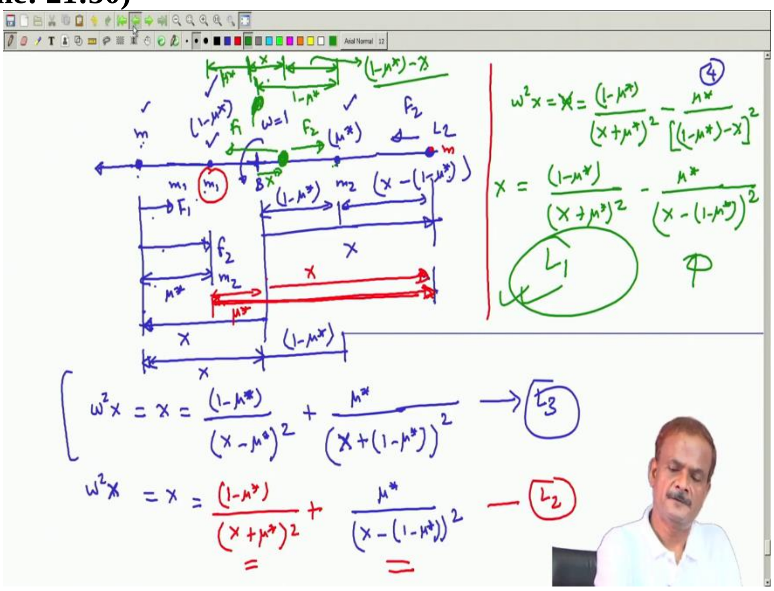

And this is based on the fact this one is worked out in the initial frame. Let us say this is mass m_1 and here the mass m_2 is located and let us say that mass m is located here in this case this is the barycenter B. Not the synoptic frame in the initial frame. So, the synoptic frame is moving rotating and along with the synoptic frame your mass m1 is also rotating. It is a stationary in that point, that means, this point suppose this is rotating about the barycenter.

Here like this the synoptic frame, we are this omega we have represented by 1 on the normalized scale. So, that means, if we write in terms of the forces acting on this mass, so say on this mass due to m_1 force will act which we can write as F_1 and due to this mass, the force F_2 will act. This we are looking in the initial frame because the barycenter it is the initial reference frame located at barycenter.

In that frame we can write the equation of motion. So, F_1 is the force acting on this due to m_1, this is due to m_1 and this is due to m_2 and this will provide the necessary acceleration for this mass to move in a circle about the barycenter. So, this distance you consider as your x, so, we can immediately write omega square x and this must be balanced by the corresponding force acting on the mass m.

So, how this force will be acting to under normalize the scale we will write this as x and then this mass is 1 – μ^* and this mass is μ^*. So force due to this on the elementary mass. So, obviously m will be here but m I will remove just indicating the acceleration which will be purely produced by this. So, this is 1 – μ^* this mass G = 1 on normalized scale. So, this G = 1 and therefore, this also I am removing I am not keeping it here.

So obvious and then the distance between these 2 points and a square of that, so, for distance between these 2 points is how much between the barycenter and this here. This is your distance μ^*. So, this becomes x – μ^* whole square. Here perhaps signs somewhere we have done the problem let me check here. For the first equation in this case in the case 1 we go to case 1, this is x minus, the case 1 we have written as x – μ^*.

And somewhere we have written with plus sign that we need to correct this is with for L_3 we have written here, this is with minus sign. Some for all others, some sign will get destroyed while copying from1 place to other place x – μ^*. This is point L_2 we are taking the point L_1 this is μ^* this is ok. So for L_2 this is ok. This is for L_3. So, this is minus sign here. While copying we have done the mistake.

So, immediately we can verify from this place then this is very easy to work. So, whenever you have to solve problem, you can use this method rather than what we have used earlier to develop the problem that we went systematically and worked out the problem. So, other one will be due to this μ^* the force acting on this mass. So, what is the distance from this point to this point? So, this distance from here to here, this is x and this is x and this is 1 – μ^*.

So, Then we need to add it. So, x + (1 – μ^*), this is whole square. So, this is for you are getting L_3. Similarly omega square x while your mass is line here in this place this is the L_2 point. Again take out the barycenter you will be measuring all the distances and working out. So, here no need of putting this sign of x either + or – just to work with the magnitude. So, the distance from this place to this place this is x.

And the force acting on this will be this is x – (1 – μ^*). This distance because this is distance 1 – μ^*. So, force acting on this will be this equal to x and due to μ^* we write here μ^* divided by x – (1 – μ^*) this whole square. This is acting towards the left on this side. So, this is due to this is F_2 and then due to F_1. So, F_1 all the way from this place to this place we have to write.

So, on that scale, for once we are doing because of the m_1 So, we have to write the force acting on this m because of this m1, so, what is the distance from this place to this place. So, barycenter this distance is x and this distance is μ^* and therefore, we can write immediately 1 – μ^* divided by (x + μ^*)^2. And this is for your L_2 and check it whether it is correct or not.

For the L_2 this

$$L_2 = X = \frac{1 - \mu^*}{(x + \mu^*)^2} + \frac{\mu^*}{\left(x - (1 - \mu^*)\right)^2}$$

. So, immediately we have got the solution. Similarly, for if the point is lying intermediate somewhere here in this place; let us say it is a line here in this place. So, due to m_1 the force will act here in this direction which is F_1 and due to m_2 force will act here in this direction, which is F_2.

So they are opposing hear each other. So, if we assume m_1 is heavier, so, we will write it like this ω^2x = X = 1 – μ^*, which is the mass of m_1, 1 – μ^* divided by the force acting on this. So, this distance from here to here, what will this distance will be? So, we know this distance from this place to this place here in this case this is x and this distance, so, this distance is x and this distance is this is μ^*.

So, this becomes x + μ^*. So, this is acting here in this direction and F_2 is acting in opposite directions, so, we put a minus sign here, μ^* is the corresponding weight involved here and then what is the distance between this and this. So, this distance now we have to sort out what this distance will be. So, this distance will be 1 – μ^* – x square, and which we can rewrite as

$$L_1 = X = \frac{1 - \mu^*}{(x + \mu^*)^2} - \frac{\mu^*}{\left(\mathrm{x} - (1 - \mu^*)\right)^2}$$

and this is for your L_1.

So, we have got L_1, L_2, L_3, this you can check here, which is the green one, I have shown. So the distance from barycenter to this point, this is 1 – μ^*. Barycenter to this and this distance if you are showing this as x from the barycenter for this distance will be (1 – μ^*) – x and based on this this formulation we have done so, and this is for L_1. So, this way, we have been able to work out for all the 3 Lagrange points which are collinear.

The expression and this expression can be developed, you can this is basically the equation of a quintic and whatever way we are developed here. So, this way you can write the same thing you will get exactly the same result, because the equations are we are developed and this equation all of them are same. And we get the solution for the Lagrange point magnitude wise at what distance it is aligned.

So, for the earth moon system I have already written. So, I will terminate the class today here and will continue in the next class. Thank you very much.