**Space Flight Mechanics**

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**Lecture - 45**
**Restricted 3-Body Problem (Contd.,)**

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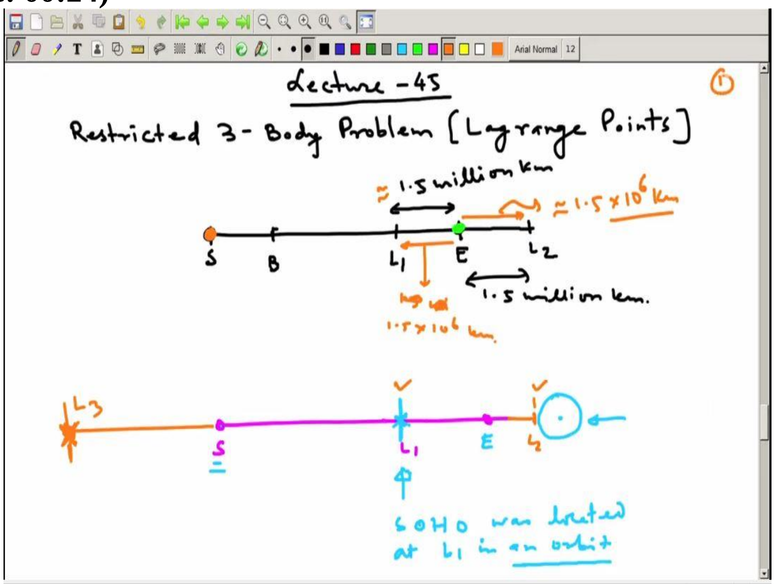


Welcome to lecture number 45. We have been discussing about the Lagrange point. So in that context, we will look into the solar system very briefly, because already we have discussed about the earth moon system. Accordingly, the solar system is there. So, here in this case, if you look into the sun is located here, and the earth is located here in this point. So, here is your earth and the sun is located in this point. So, $L_1$ point it lies around 1.5 million km from the earth on the left hand side.

So, this is your 1.5 km 1.5 million km or $1.5 \times 10^6$ km. Similarly, on the right hand side the $L_2$ is located at around $1.5 \times 10^6$ km and this you can compute all the distances already I have listed here. So, using these distances and the mass of the sun and mass of the earth, as it is mentioned here, so, this details are given here. So, using these details, you can work out these distances.

So, there have been mission though the $L_1$ and $L_2$ are on a stable point but mission has been sent to see the sun is here and the earth is located here this is your point $L_1$. So, mission has been sent to this point where perpendicular to this line joining in the orbit it will look like. So, I have drawn the orbit like this, but if you I will rub it out and if we look from the side, so, if this is your relevant point, so, the orbit this is a side view of the orbit.

And if you look from this point so, orbit will appear like this. So, at $L_1$ satellite has been sent and satellite orbits around $L_1$ point $L_1$ is still on the stable, but by orbit keeping you can keep the satellite here in this place. This is your earth this is sun and this is earth so on mission called So Ho was located at $L_1$ in an orbit of course this has to be maintained it because it is on a stable one.

So, it will not stay there but mission has been sent here and this is the sun observation satellite search for the study of the sun this has been used similarly, the satellite can be sent here to the $L_2$ point also so if you are $L_2$ is located say here in this point $L_2$ is located then the same kind of orbit can also be placed on this site. But if you have on this site, the $L_3$ is located this is of no use because this will get eclipsed by the sun. And any satellite, if somebody tries to place here at $L_3$ this will be of no use. So $L_1$ and $L_2$ have been used for locating the satellite and observing the behavior of the sun or studying the sun.

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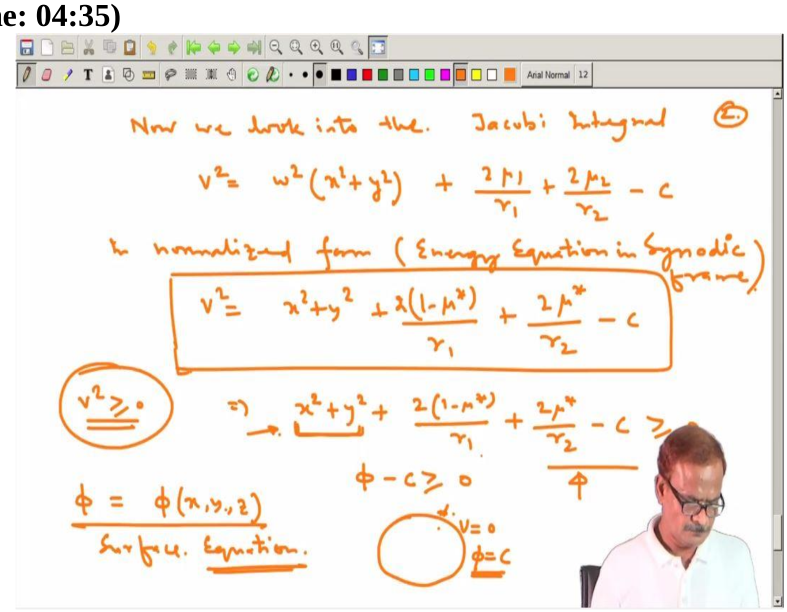


Now, we will look into the Jacubi integral we have derived earlier in a normalized form V once we put omega = 1, so we got the normalized form. Otherwise in the non normalized form we have written it like this. So, how the Lagrange points they emerge from this equation can be studied this is basically an energy equation. In Synodic $v^2$, this is a positive quantity which is always greater than or equal to 0. So this implies that

$$V^2 = x^2 + y^2 + \frac{2(1-\mu^*)}{r_1} + \frac{2\mu^*}{r_2} - C$$

On the normalize scale if we write, this will be always greater than or equal to 0 it cannot be negative. And this quantity also we have written as ϕ - c, this is greater than 0. We got ϕ is the function of, of course, here in this case jet is not appearing because its gets reduced to a planner case as we have discussed earlier in this point the jet is not appearing this what I mean to say now this is the equation of the surface so its $v^2$ is always greater than or equal to 0 because it is a positive quantity at which it can be 0.

So that means your particle is not allowed to cross this surface. So, if I have a surface and if on this surface v = 0 means ϕ = c this is indicated by. So, then this surface cannot be penetrated you cannot go from this side to this side on unless it is allow. So, this we are going to a study and from this how the Lagrange points they emerge, but that we will look graphically.

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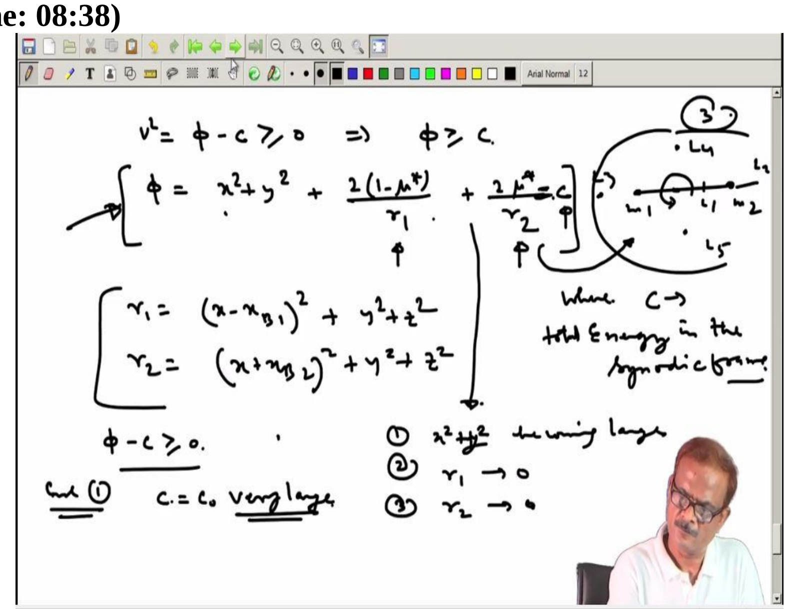

So, one thing we have I will remind you again and again that if we have 2 masses 1 primary mass 1 and the secondary mass 2, so, the Lagrange points are located in the plane of the orbit. So, if this is rotating like this about this point is the orbit or vital plane of the $m_1$ and $m_2$. So, the $L_4$ and $L_5$ are located like this $L_3$ is located on this side. $L_2$ is located on this side and $L_1$ is located here.

So, all these are co planner while here in general this is $r_1$ $r_2$ as you remember that $r_1$ we have written as

$$r_1 = (x - x_{B1})^2 + y^2 + z^2$$

$r_2$ we have written as

$$r_2 = (x + x_{B2})^2 + y^2 + z^2$$

So, ϕ is not a planar surface it is a 3 dimensional surface. But using this we can get to know that there are certain points to which letters write here on this site ϕ is equal to C to which this will get reduced as the value of C is change.

Where C indicates energy of the system in the Synodic frame, this is stands for total energy in the Synodic frame. So, ϕ - C this is always greater than or equal to 0. And let us say case 1 we start considering case 1 and C is called to $C_0$ is very large and how it is possible, if we look here in this equation so, this is possible by $x^2 + y^2$ becoming large or $r_1$ tends to 0 or $r_2$ tends to 0 in either of the 3 ways that is possible.

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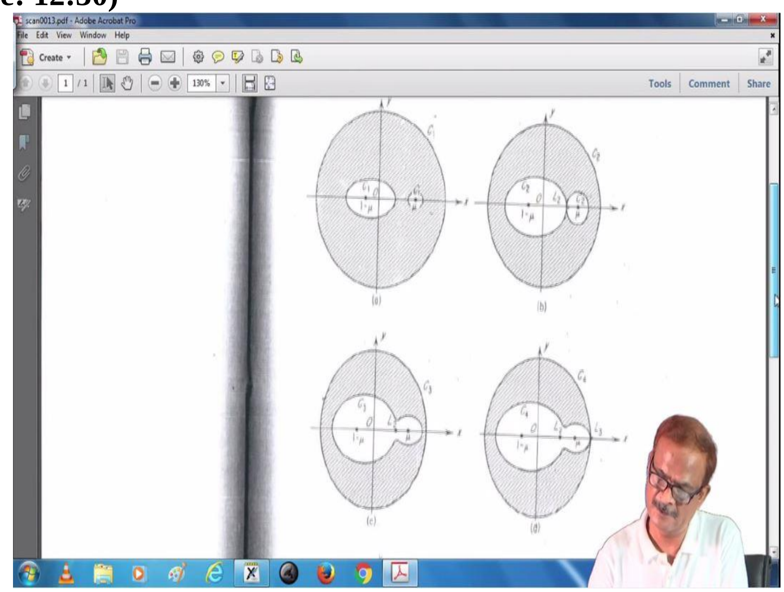

So, if we looked from that point of view. See here in this case this figure has been taken from the addictions Archie Roy Astrodynamics by Archie Roy already I have given names of all the books. So, here this is showing your $C_1$ in this point you can view the cursor this is your $C_1$. So, this is one of the surface being shown here, inside this you have the points 1 - $\mu^*$ written here and $\mu^*$

written here on the right hand side. So, on all these surfaces, you C = $C_1$ has been written and this circle is shown larger and on the right hand side the circle is soon smaller.

What is the reason this is all not a circle, but almost a circle. So, we can consider it to be a circle or some oval shape similarly here if you see in the figure b. So, in a figure b also if you look C equal to has been set to $C_2$ and then the size of this surface is grows and then it merges somewhere and here the point has been written as $L_2$, but this point where it appears, we follow this point as $L_1$. We do not write this as $L_2$ but further as $L_1$ $L_2$ we have taken on the right hand side of the mass $\mu$ or $\mu^*$ here $\mu$ recessed into the $L_2$ so, this has got a different notation that book.

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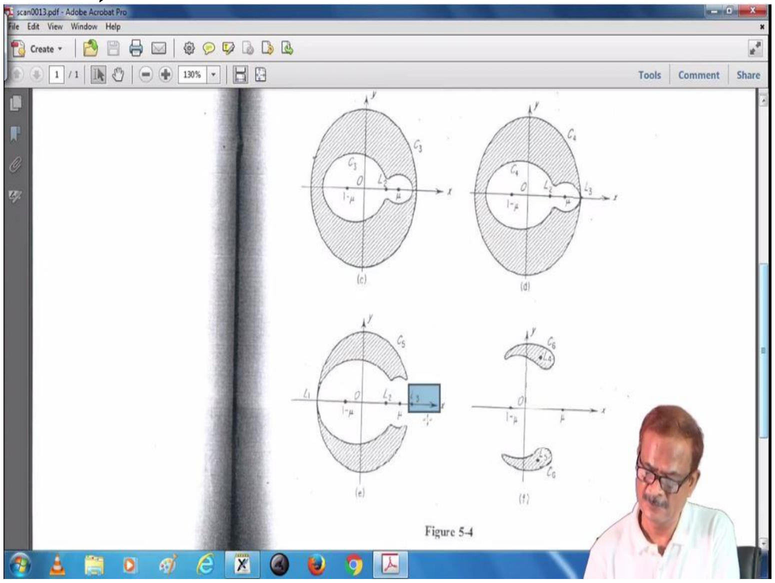


So, what the figure is the only thing that the notation we have to change thereafter you can see that the as the value of the C change so this $Z_2$ inside the white surfaces they merge and here the $L_2$ is appearing. So, $L_2$ is already there, but now, the 2 surfaces merge and you can see that some doll shaped of things it has appeared. Similarly, the $L_3$ which has been shown here, this is actually $L_3$. $L_3$ here in this point as you can see, I am doing it, so, this will create our $L_2$ not $L_3$ here $L_3$ is on the left hand side for us.

And here what has been shown as the $L_1$ this is our $L_3$ and what has been shown as the $L_2$ this is our $L_1$ and what has been shown as the $L_3$ this is our $L_2$ and $L_4$ and $L_5$ similarly, they emerge here in this place. So, we will discuss this how all these points they emerge from this in the equation we are using. So, we have been discussing here about that. So, $C_1 = C_0$ is very large.

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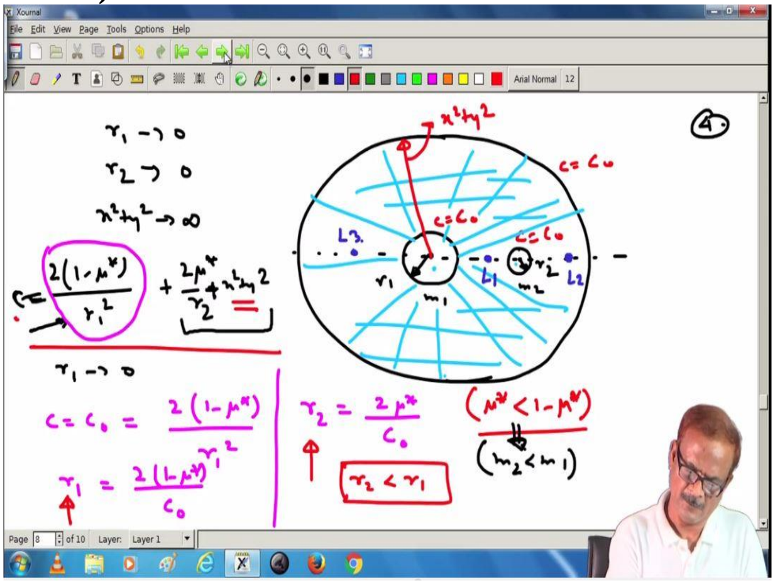

So, as I stated this is possible if $r_1$ tends to 0 or $r_2$ tends to 0 or $x^2 + y^2$ this tends to infinity. So, I will show it like this we point here I am showing it as spherical for convenience. So, this is once surface I am showing here. Another surface I will show by a smaller size $m_2$ is located here $m_1$ is located here, this is your $r_1$ and this is $r_2$. The question is why I am showing this $r_1$ to be larger and $r_2$ to be the smaller. This will be evident from the equation we are using C equal to this quantity we have written let us say.

This $r_1$ tends to settle and $r_2$ is having certain finite value. So, at that time, we can ignore these quantities. Because once $r_1$ tends to 0 you can see that this quantity will become very large. So

$$C = C_0 = 2\frac{1 - \mu^*}{r_1^2}$$

So,

$$r_1 = \frac{2(1 - \mu^*)}{C_0}$$

, $C_0$ is very large you can see that $r_1$ tends to 0. Now, along the same line we can write

$$r_2 = -2\frac{\mu^*}{C_0}$$

So, if we do that, immediately it will be reachable why $r_1$ and $r_2$ are soon with different sides of the sphere or the circles whatever you say mission is $\mu^*$ less than 1 - $\mu^*$ we are taken $m_2$ to be a smaller mass. So, immediately you can see that the $r_2$ will be less than $r_1$ for that region and therefore we show it by the smaller circle. So, here this is C = 0 and region inside itself forbidden in this region no particle can exist.

So, somewhere your $L_3$ is located here $L_2$ is located here in this point and $L_1$ located in middle between somewhere. So, if you have this kind of configuration so these are shown by this blue set it portion so it is not possible to go from this one to the bigger circle to the smaller circle. So, there is no communication you cannot go from sent particle from one circle to the; another circle. Next we so, and this is your $r_1$ hard to find also we should so, this is corresponding to $x^2 + y^2$ becomes very large. So in that case $r_1$ and $r_2$ can be ignored in this equation and directly, you can see that $x^2 + y^2$ it is presented by C.

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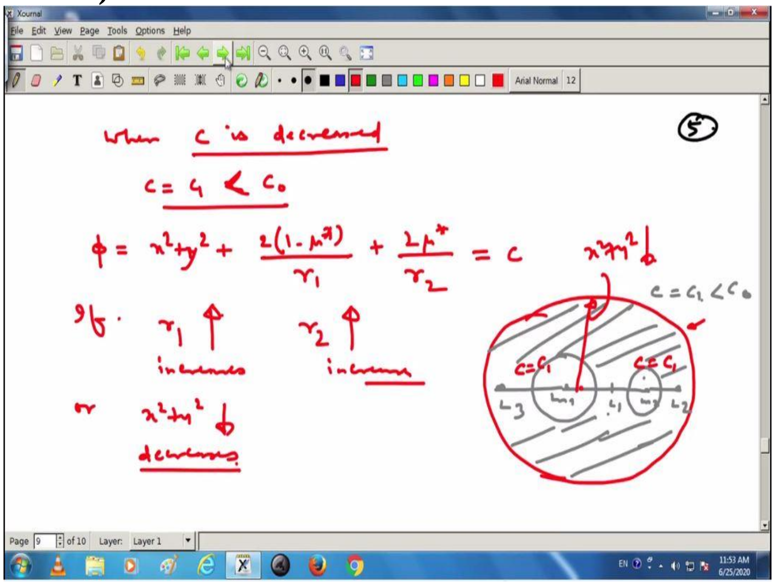

When C is decreased. So, let us say C = $C_1$ which is less than $C_0$. So, this can happen if in your equation for ϕ. This can happen if $r_1$ goes up $r_2$ goes up which increases and therefore, C will decrease or $x^2 + y^2$ goes down. So, under these 3 conditions, you will see this to happen and along the same line we can work out this. So, here in this case this circle becomes large this circle also this becomes large, this is $m_1$, $m_2$ is located here.

$L_1$ is located here on this side you have $L_2$ and on this side you have the $L_3$ located as I have shown in the previous figure that the figure from the Archie Roy or whatever you have shown. So, this is how this comes into picture. So, whatever we are showing here, so, you can consider this to be the cross section of the surface and this is how it is visible. And then here we will write C = $C_1$ which is less than $C_0$. Then this also on this you have C = $C_1$ on the surfaces.

So, here your $x^2$ $y^2$ this as seen. So, from the distances we are taking from the barycenter, barycenter some that will be gives as a bigger mass barycenter data safe this distance we are taking from barycenter because x and y has been written from the refer to the barycenter. So, this is your point B which we have written as the barycenter and this is then $x^2 + y^2$, the same is also applicable here in this case. So, you are the x $y^2$, this $x^2$ + $y^2$ will decreases. This goes down, it stinks. So, the outer surface is shrinking and the inner surfaces are growing up.

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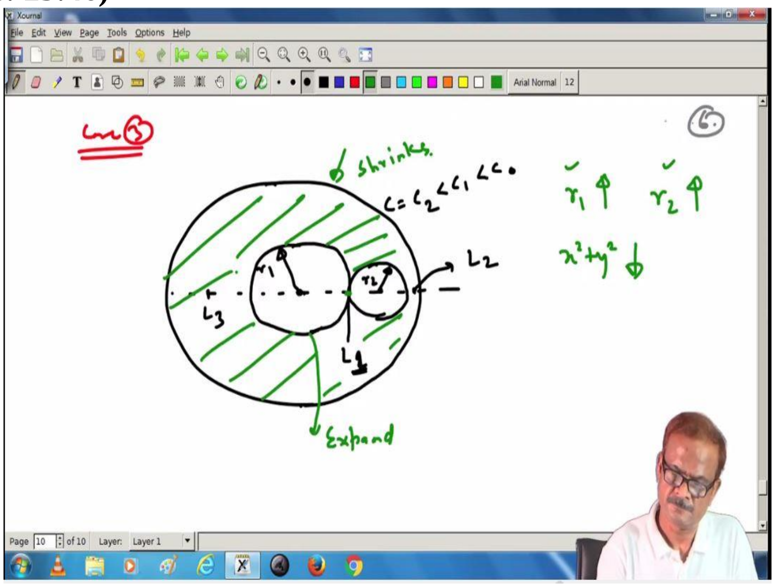


In the next level the case 3 as the value of C is further decreased. So, now these surfaces they touch each other this is the location of the $L_2$ $L_1$ and $L_2$ is located somewhere here and $L_3$ is located somewhere let us say in this point here this is your $r_1$ and this is $r_2$ C = $C_1$ we have taken. So, this is $C_2$ is less than $C_1$ less than $C_0$ and the forbidden region is stated by the green line, this is the forbidden region.

So, once the c value is further decreased, so, $r_1$ and $r_2$ they go up and $x^2$ + $y^2$, this goes down so, these surface strings these strings while these surfaces they expand. So, $r_1$ and $r_2$ this are going up and $x^2$ $y^2$ quantities this goes down and as a result your $L_1$ point manifest.

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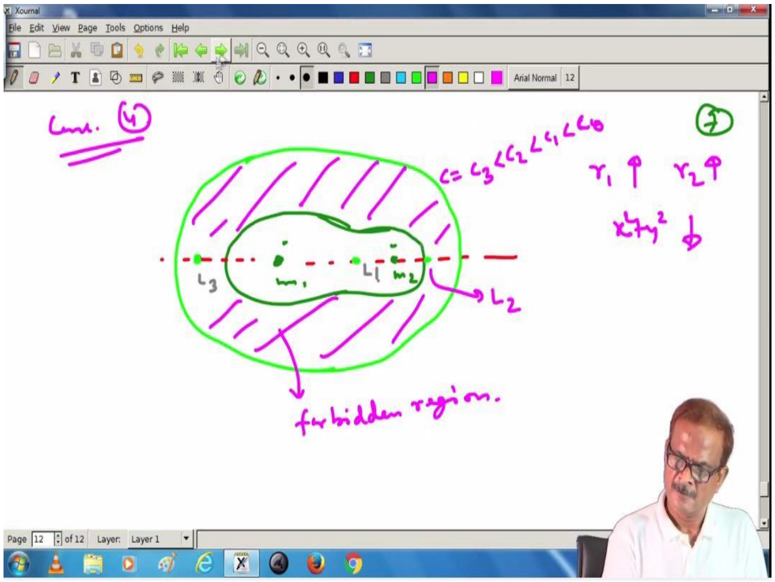

Now thereafter the further decrease the value of the C then the 2 surfaces they merge like this. So, $L_1$ is located somewhere here and then 2 surfaces as merged $L_3$ is located here in this point. This is your $L_3$, $L_1$ is here and $L_2$ is just been, it is above to manifest and outside you have this covering. So, on this you have C = $C_3$ in this case, case 4. So, this is as a result of $r_1$ going up $r_2$ going up and $x^2 + y^2$ this is going down.

So, C = $C_3$ is less than $C_2$, because of this, it has rained and this is the forbidden region. Now, you can see that the masses are here, this is your mass $m_1$ and the mass $m_2$ is located here. So, the movement can take place from mass $m_1$ to $m_2$. This is not forbidden region that shown by the pink color is the forbidden region this is forbidden region this is point 2 we about to manifest. So if we keep on doing this iteration then the all the 3 Lagrange points will manifest one after other.

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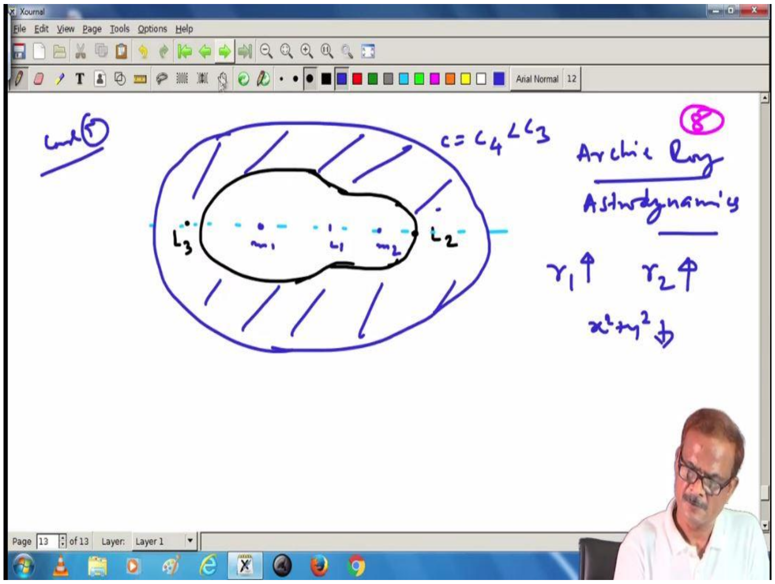


So refer to the figure the graphics I have shown already from the Archie Roy. So, $L_2$ known manifest here in this point and $L_3$ is above to manifest. Now it is on the curve itself or on the surface itself mass $m_1$ is located here, mass $m_2$ is here and $L_1$ is located here in this point. So size of the circle has further diminished. Though I am not showing by the same kind of thing. You can refer to that figure or refer to the book by Archie Roy and astrodynamics. So this is your forbidden region no C equal to here. Earlier we have taken C = $C_3$. So again the $r_1$ has further gone up, $r_2$ has also further gone up and $x^2 + y^2$ this has gone down.

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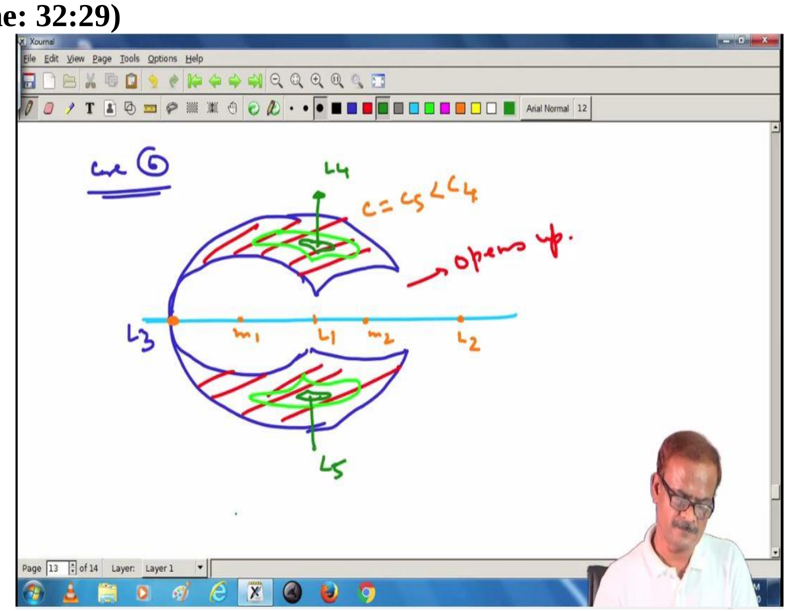


In the case 6 $L_3$ comes here in this point. This is a location of the $L_3$ now it has come over the surface

$$C = C_5 < C_4$$

mass, $m_1$ is here mass $m_2$ is here, $L_1$ is located here $L_3$ has come here and the point $L_2$ this has opened up means a point $L_2$ is also accessible. $L_2$ is just accessible. Now here it is opened up. So in this open area, this is the hatched area is forbidden area. This is the forbidden area rest other area it is accessible. So, this part opens up.

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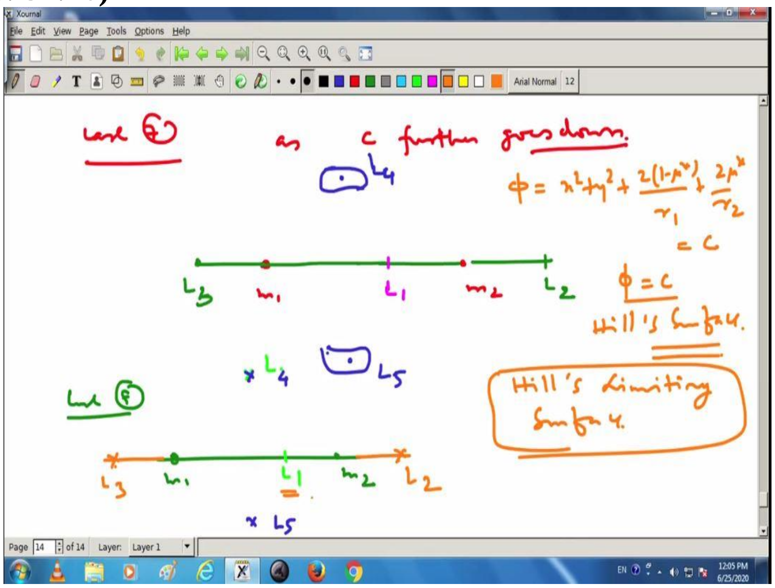


So, as the value of C is further decreased C further goes down. So, then what happens? Your mass $m_1$ is here and mass $m_2$ is located here $L_1$ is located here. So, as you see in the previous figure, this figure will start shrinking instead of like this now, this will shrink like this this part also as you increase the value, decrease the value of the C. So, this will further shrink and thereafter further decrease, this will further shrink.

So, as you keep decreasing the value, so, ultimately the point $L_4$ and the point $L_5$ they will manifest. So, this is what I was trying to show here. So, as you further decrease, so, you have the $L_4$ and $L_5$ points coming into picture on this side we have $L_3$ on this side we have $L_3$ this is $L_4$ has a started to manifest is above to manifest. So, finally in the case 8 we conclude this we have mass $m_1$ and mass $m_2$ located here.

So, your point $L_1$ is located here and obviously, here, if you see, it should have been shown here, but it is not true graphics are located as we know this will be located in at the vertices of the equilateral triangle. So, I will shift it a little bit here in this point and so, like this, this is your $L_4$

and here this is your $L_5$. So, finally your $L_4$ and $L_5$ they will manifest over the vertices of an equilateral triangle on this side you will have $L_2$ $L_1$ is already here in place and on this side, we will have $L_3$.

So, this way, all the points are now available to move from 1 point to other point. So, the ϕ, which we have written

$$\phi = x^2 + y^2 + 2\frac{(1 - \mu^*)}{r_1} + \frac{2\mu^*}{r_2} = C$$

So that means ϕ = C, this is also called the hill's surface are the hill's limiting surface because the surface cannot be penetrated cannot be crossed, so, we will conclude this chapter lecture here and in the next lecture one more lecture I will devote to the issue of Lagrange points.

Because whatever we have discussed it is all about the stationary Lagrange points in the Synoptic frame, they appear to be stationary, these are the stationary solution as you remember we have returned it, but the solution we have got by putting $\ddot{x}$ = $\ddot{y}$= $\ddot{z}$ = 0. So, in the Synodic frame, this is the situation but if we look at dynamic solution which we call as the generalized Lagrange solution. So, the generalized Lagrange solution is also available it was developed by Lagrange.

So, there the shape of the triangle, it will remain the same, but the size of the triangle will change it may contract it may expand it may as linear similarly, the linear solution, we are the linear solution, we have looked and the $L_1$ $L_2$ $L_3$. So, in the stationary solution, these are the fixed points in the Synoptic frame, but in a generalized solution, these points are not fixed rather the ratio of these points they remain in actually they remain in the distances between the points, they remain in a particular ratio which I will describe a little bit in the next lecture.

So, I will devote one more lecture to this and then we will wind up this chapter because time does not permit a very limited number of lectures. We are mainly 5 lectures we are meant for this year discussing the Lagrange points, but we have taken a lot more than that. So, in the next lecture, I will wind up this particular issue giving you a little bit of introduction to that topic and hands out I will supply in which you will get all the details all the mathematical conclusions.

How the points $L_1$ $L_2$ $L_3$ are on this table and $L_4$ and $L_5$ they are stable. So, all those things will be discussed in those handouts. So, wait for the course to a start for that time we will get all those materials. Thank you very much.