Space Flight Mechanics
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Lecture - 46
Restricted 3-Body Problem (Contd.,)

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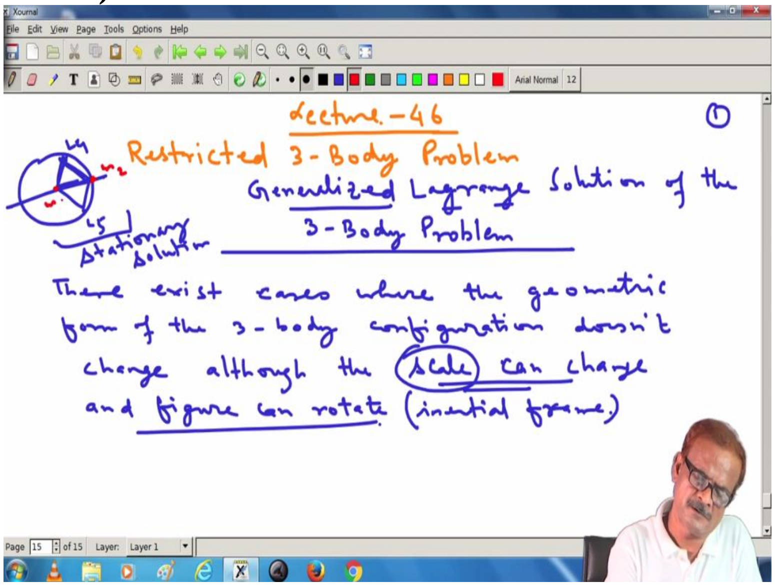

Welcome to lecture number 46. Today in this lecture, we will wind up the issues related to the 3 body problem. So, we are going to discuss about the generalized Lagrange solution, obviously if we try to work out the whole thing, it may take at least 6, 7 hours to discuss the whole content. And we do not have that much time so forth for half an hour whatever I can discuss and disrupt the materials I told you earlier, I will give you as handouts. In the soft copy I will supply for that there exist cases does not change.

There exist cases where the geometric form of the 3 body configuration does not change, although the scale can change and figure can rotate. So figure can rotate this is in the inertial frame not in the synoptic frame the scale change will be visible in the synoptic frame and also in the inertial frame, but finger rotation this is applicable to the inertial frame. So, already we know that the synoptic frame is rotating and therefore, obviously the earlier case also if you have seen that earth moon system in the inertial frame the L_4 and L_5.

And all other points they will be rotating but simultaneously there is a scale change involved in this case, the L_1, L_2, L_3 ,L_4, L_5 we got these were the stationary solution. So, these are the stationary solutions, but what we are discussing here is in the generalized solution, there is change of scale. So, this figure will not remain on the same scale, but rather it will expand. So, if this is mass m_1 and somewhere the mass m_2 is there. So, this figure is not going to be the same, this is bound to change and this is what in the generalized solution we look for.

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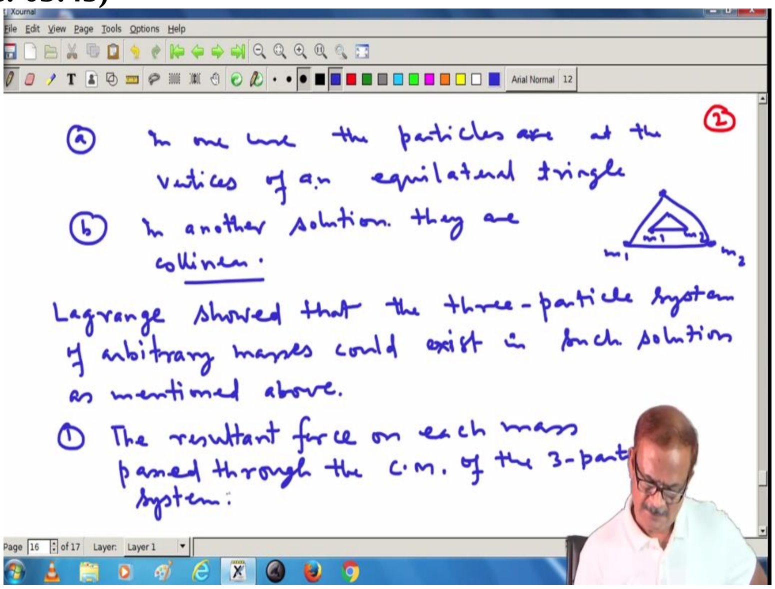

So, here in this case this particle distance between this particles which is fixed here, this also will change, m_1 and m_2 distance that also will change particle, in one case the particle are at the vertices of an equilateral triangle. Still it is a collateral triangle but the triangle is expanding. It is expanding like this. So obviously, if you have mass m_1 here, mass m_2 here, so this is mass 1 this is mass m_2 for distance between these masses also, it is a changing.

Another solution they are collinear Lagrange showed that the 3 particle system as we have been discussing earlier 3 particle system of arbitrary masses here we are not putting restriction on the mass m_3. This is the difference from the previous stationary solution that we are considering Lagrange so, that the 3 particle system of arbitrary masses could exist in can occur could exist in such solution as you mentioned above under what condition so the conditions under which this will happen those conditions.

I will be once I supplied those soft material short copies of the derivation. So, in that you will find I am not going to derive all those things here. The resultant force on each mass, or through the center of mass, of the 3 particle system 3 particle or 3 body system.

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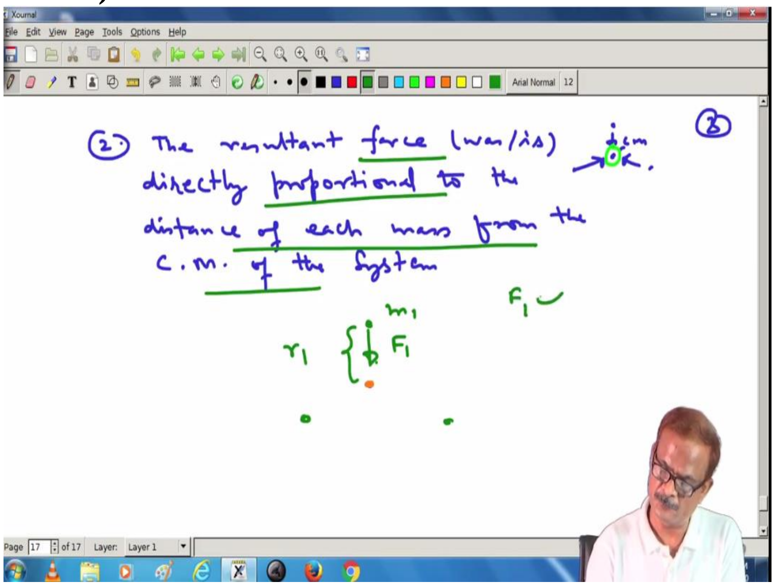

So, what it is telling that resultant force on each mass pass through the; or passes through the center of mass of the 3 particle system. So, if we have 3 particles here, so, on each mass the center of mass may be located here. So, on each mass the force acting, it passes through the center of mass like this, this is your center of mass here, this is center of mass. So, this is what the first statement says.

And all these things can be derived that if we are looking for a solution. So, if we are looking for solution where the geometric form of the 3 body configuration does not changed, but the scale change can take place. So, for that case, this is applicable while considering the stationary case or so, the mass m which was that tertiary mass m or the third mass which mass presumed to be negligible, so, that it does not affect the mass m_1 and m_2.

While here in this case all the 3 masses can be are important and as written here they are of arbitrary 3 particle system of arbitrary masses. So, these are the some of the important points, then the resultant force on each mass should pass through the center of mass of the 3 particle system if this happens, condition 1 then the condition 2 the resultant force was/is force directly proportional to the distance of each mass from the center of mass of the system.

So, that means, if this is the center of mass here, this is the center of mass and there are 3 mass located. So, resultant force, force acting let us say this is mass m so force acting on this is F_1. So, this will be directly proportional to the distance of the mass from the center of mass. And if this distance we write as r_1, so this implies that F_1 will be proportional to r_1 this also can be derived. If you have to maintain the configuration, provided the scale can change.

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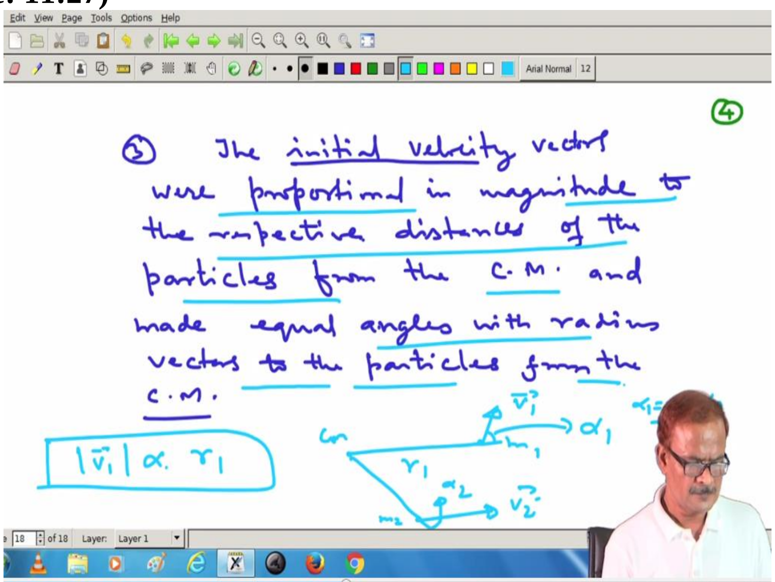

And the third one and this is the initial velocity vectors work proportional in magnitude to the respective the initial velocity vector to the respective distances of the particles from the center of mass and made equal angles with radius vectors to the particles from the center of mass. So it says that Initial vectors were proportional in magnitude to the respective distances of the particles from the center of mass is the center of mass is located here, particle m_1 is here.

So the initial velocity vector we are proportional in magnitude means, if this is the velocity vector, so given magnitude is proportional. We are proportional in magnitude to the respective distance, this distance is r_1. So, this is applicable here and made equal angles with the radius vector to the particles from the center of mass that means, this is m_2 and this is v_2. So, these angles are equal.

If I write this is α_1 and this angle is α_2 similarly α_3 for the third particle. So, all these angles are $\alpha_1 = \alpha_2 = \alpha_3$ these angles are equal which is being met with the reduced factor and with a comprehensive treatment, all these things can be carried out.

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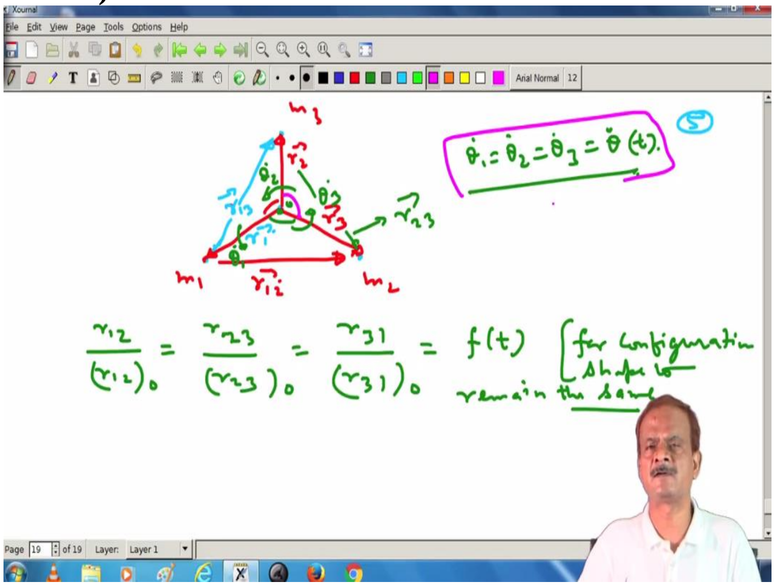

So, shape of the configuration remaining same implies let us say that this is the center of mass, we have particle m_1, m_2 and m_3 here, this is m_1, m_2 m_3 and this is the center of mass this vector we write as r_1 this vector we write as r_2 in this vectors r 3 then this vector will represent by r_{12} and similarly, r_{13} we can represent like this from this place to this place. This is r_{13}. This is your r_1 and here this is r_{13}, this vector is r_{13}.

The same way, r_{23} condition. So r_{23} from this side to this side, going from here to here this green 1 I am showing us r_{23}. So, configuration remaining same it implies that

$$\frac{r_{12}}{(r_{12})_0} = \frac{r_{23}}{(r_{23})_0} = \frac{r_{31}}{(r_{31})_0} = f(t)$$

and r_{31} or r_{13} for configuration shape to remain the same, this is the condition, which is required. And thereafter, another condition will be required, that if the shape also has to be maintained, then the rotation rate of this rotation rate of this about the center of mass. This is the center of mass so say.

So, this is if we write this one is the $\dot{\theta}_1$ this rewriters $\dot{\theta}_3$ and this is $\dot{\theta}_2$. So

$$\dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3$$

this must be equal to this same. If that happens that means, the angle between all of them will be maintained, this angle will remain same and there why configuration will be maintained if this rate becomes different. So, no longer the shape can be maintained. So, this is another condition.

So, using these conditions, we will be able to we can work out all this thing so, in great details, I will supply all these materials once the course runs, but right now, it is not possible to cover this because of the constraint on time. So, using this we can get the linear solution also and triangular solution also and whatever the stationary solution we have got the L_1, L_2, L_3, L_4, L_5 the earlier we have discussed.

It is just a special case of what we I am talking here right now. He said much more generalized case for the 3 body problem. So, in the case of the generalized 3 body problem, so, we have to put certain constraints so, as earlier it was told that under certain constraints only it can be solved otherwise it cannot be solved 3 body problem that to study the generalized property of the system we do not look into the explicit solution.

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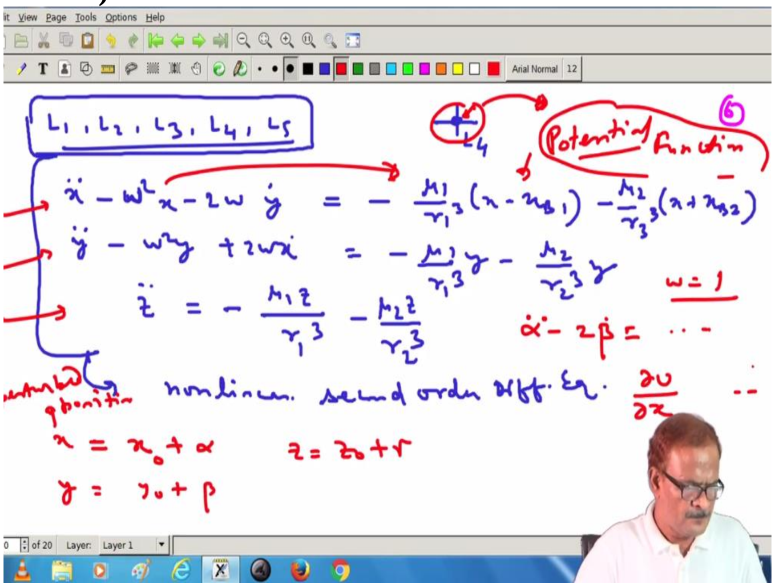

Can I have also stated that stability of these points? So, stability of the; it can be a stability of the generalized point it can be a stability of the stationary points. So, what if we consider here in this case while I supply you will be only discussing about the stability of the points L_1, L_2, L_3, L_4 and L_5 what we have discussed earlier for the stationary solution and in that case we got the solution as $\ddot{x}$ if we recall the equation

$$\ddot{x} \; - \; \omega x^2 - 2\omega\dot{y} \; = \frac{\mu_1}{r_1^3}(x - x_{B1}) - \frac{\mu_2}{r_3^3}(x + x_{B3})$$

and similarly for $\dot{y}$ we have the equation

$$\ddot{y} - \omega^2 y + 2\omega\dot{x} = \frac{\mu_1}{r_1^3} y - \frac{\mu_2}{r_2^3} y$$

and for the z we had.

$$\ddot{z} = \frac{\mu_1}{r_1^3} z - \frac{\mu_2}{r_2^3} z$$

So, in this equation this is nonlinear second order differential equation nonlinear each of them. So, to solve this and to study the stability what we do that we linearize the system we use; we studied the stability about certain operating point.

So, and here in this case our operating points will be L_1, L_2, L_3, L_4 *and* L_5. So, if the system is disturbed from let us say this point is L_4, so, if it is disturbed in the neighborhood of this point. So, whether this will get a tendency to return back to its original position or it will orbit around this point or it will go away from this point. So, this will decide whether the system is a stable or not a stable and for that we need to linearize each of the equations and that representation.

You know that this was written in terms of the potential function also the right hand side was written in terms of potential function. So, we can express this in terms of potential function and thereafter we can expand and say the, if x is the original position. So, x is a deviation from this a few indicated by α. So, or say the x_0 is original position and from deviation from this we write it by α. So, this is the perturbed value perturbed position.

Similarly, the y we can write as $y_0 + \beta$ and z we can write as $z_0 + \gamma$. So, α, β, γ these are the perturbation in x_0 x , y and z. So, if we do this done the linearized equation it can be written in terms of like the $\dot{\alpha}$ - 2 $\dot{\beta}$ assuming $\omega = 1$ and this can be taken on this side and it can be whole thing can be presented in the potential function form, which we have done earlier.

So, if we follow this particular procedure, so, this can be expressed the right hand side can be expressed in terms of the derivatives of the potential function $\partial u/\partial x$ and which are higher order derivatives. So, this part I am not going to exchange any more here I have just given you the introduction or just an idea what can be done to using this procedure, this problem can be solved and once you solve it, so, what are the conditions for stability to exist you can obtain.

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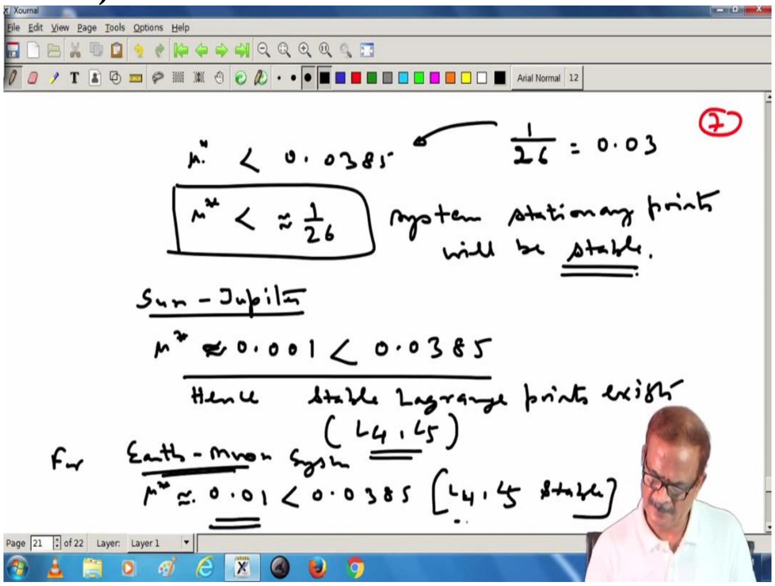

So, for stability to exist mu which we are write as μ^* actually mu we are using all the time μ^* notation for, m_2 normalized value. So if this quantity is less than 0.0385 and perhaps this quantity is 1/26 yes, this is 1 by around 1/26. So, if μ^* is less than around 1/26 the system will be Lagrange point's stationary point's means equilibrium points, by definition I have written that the equilibrium points are stationary points.

That is just a system of stationary points will be stable, if this condition is satisfied and this you will get while I supply the material. So in the case of the say the sun Jupiter system μ^* is around 0.001 which is less than hence the stable Lagrange point exist and what those points will be, this will be L_4 and L_5, L_1, L_2, L_3 they are not a stable for earth moon system μ^* this is nearly equal to 0.01 this is again less than 0.0385 and L_4 and L_5 correspondingly will be stable.

And in fact in the nature around L_4 and L_5 a number of particles they exist so those details I will not be writing here you can get those materials in the soft copy, because it will take time writing all those materials, but this is what we get for the earth moon system this L_4 and L_5 stable for the sun Jupiter system also this is stable and for the sun earth system here.

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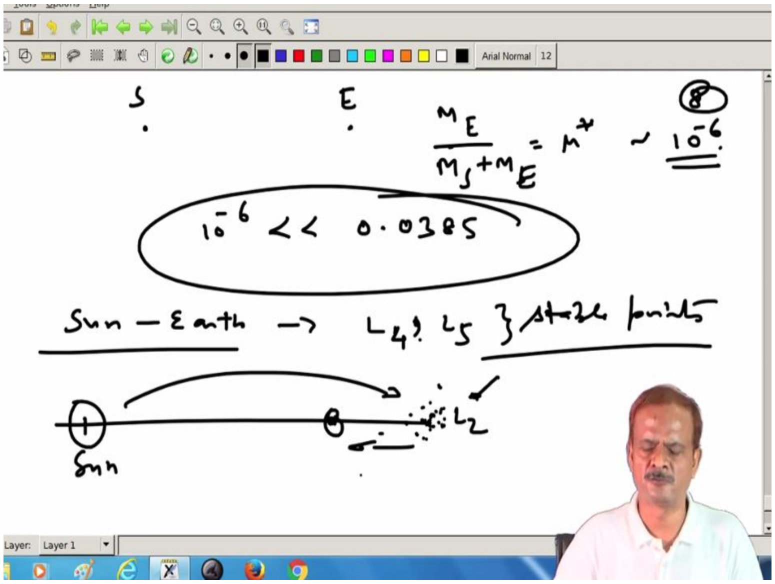

The sun is there and the earth is here. So, earth mass divided by m-sun + m- earth which equal to μ^*, this may be of the order of 10^{-6} that means this is much smaller than 0.0385. Hence, the sonar system also contains L_4 and L_5 we will find a stable points and in the nature I told you the Trojans, they are there under the Jupiter system, the sun Jupiter system around the L_4, L_5 points and also for the sun earth system for the collinear solution also they are of very gaseous particles, various small particles.

They also exist L_2, and these are very small particles. So, can exist around the point L_2 and these points they are visible in the night sky because the ones the sun rays fall from them. Sunrays once they fall on them so it is a visual though it is on a stable. But one stuff gas particles they are trapped here even those unstable these are very small particles. So, they becomes visible so it is considered as indecision.

So, it appears like a mirror image of the sun here in this place because the rays from the sun reflected back to the back and then the on from the earth itself visible. So, this details I will not consider here and I will give you a written message for that. So, I will close this discussion about the Lagrange points it is a stability other issues at this stage and whatever the extra things are required I will supply and those material based on this already.

So much of things we have developed, it a just mathematics you can go through that and some of the written materials will be there you can study and you will get to know the; what is the related reality. Thank you very much.