Space Flight Mechanics
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Lecture - 47
General Perturbation Theory

Welcome to lecture 47 today we are starting with the general perturbation theory. So, first we look into the basic equation from where the perturbation is arising. So, as we have done the 3 body problem already, so, from here we are aware of the equation of motion for the 3 body system or many body systems. So, the same thing first I will recall and thereafter I will start with further processing.

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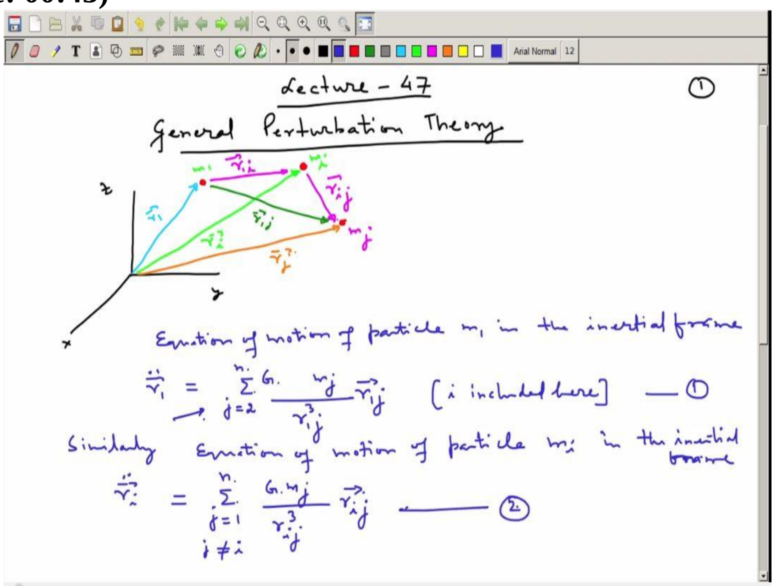

So, first we write the equation of motion of particle 1 in our simply we write as in particle m_1 in the inertial frame $\ddot{r}_1$. So all the forces acting on this particle to this we can write us m_1 times m_j multiplied by G r_{1j}^3 r_{1j} and j spans from except 1, this is from 2 to n so i included here. This minimized equation 1. Similarly of particle m_i in the inertial frame $\ddot{r}_i$. This will be given by

$$\ddot{\vec{r}_i} = \sum_{j=1}^{n} \frac{Gm_j}{r_{ij}^3} \overrightarrow{r_{ij}}$$

but the particle i cannot apply force on itself. So, we have to negate this. In this place particle 1 cannot apply force on itself. So already 1 is not there we ever started from j = 2.

$$\ddot{\vec{r}}_i = \sum_{\substack{j\,=\,1 \\ j \neq i}}^{n} \frac{Gm_j}{r_{ij}^3} \overrightarrow{r_{ij}}$$

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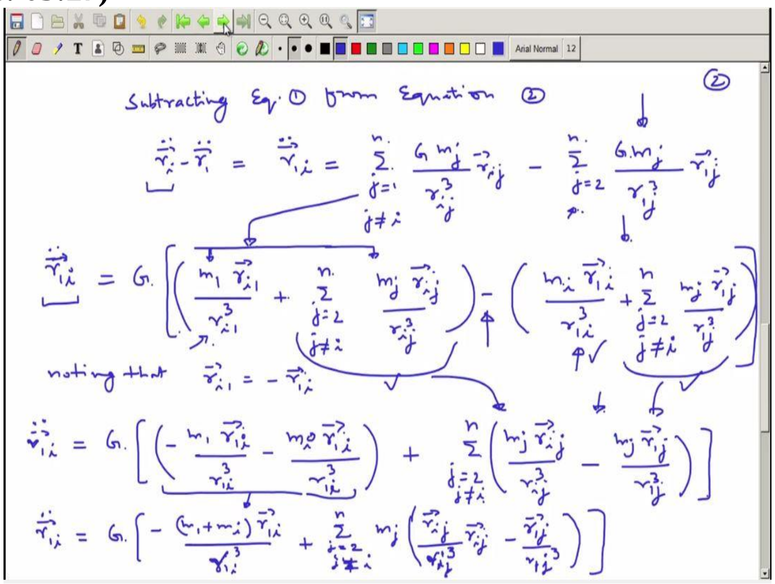

Now subtracting equation (1) from equation (2), $\ddot{r_i}$ - $\ddot{r_1}$ this vector becomes $\ddot{r_{1i}}$so this equal to j = 1 to n G times. Therefore, we can rewrite this equation G we can take it outside, rewriting it, this will break into 2 parts. We will write it like this putting j = 1. So, this is G already we have taken outfit. So, we removed that the first part is putting j = 1. So, this is m_1 times r_{i1} divided by r_{i1}^3, this is the first part and the other part remains set j = 2 to n.

Remember this is the further ith list particle. So, here j not equal to ith is will stand m_j times r_{ij} divided by r_{ij}. So, these are 2 parts we have divided it and similarly we will divide this into 2 parts. So, this is coming because of the force on the first particle due to other ones. So, force on the first particle due to the ith particle here see, we have not written j not equal to i that means i is included here. The first particle is getting affected by the ith particles. So, first of all, we will break that and G already we are taking outside common.

So, that goes and we get here. So, j first we replaced by i. So, this becomes m_i and r_{1i} j we are replacing by i and then r_{1i}^3 cube and plus the other terms for j = 2 to n m_j r_{1j} divided by r_{1j}^3 and here we need to write j not equal to i because the i part we have already taken outside and displace and then we close the bracket here. Rearranging the terms noting that r_{i1} this equal to minus r_{1i}. So, r_{i1} we are going to replace by r_{1i} because here the r_{1i} is available.

So, this every equation gets reduced to m_1 - r_{1i} and this is magnitude and therefore also this we can replaces r_{1i}^3 and from here we pick up this term and put here in this place. So, this is m_i r_{1i} divided by r_{1i}^3 and other terms are combined together whole cube see in both the places we have the same indexes over j. So, we do not have any problem in discussing this part. So, m_j r_{1j} divided by r_{1j}^3; j = 2 to n j not equal to i.

Now, one more step and we can write it more in a concise way - m_1 + m_i times r_{1i} divided by r_{1i}^3. So, this is the term, this one we are using here and thereafter we pick up this term and write here in this place So this term we will be using and this term will be using combining them together and here we should have a minus sign because once we have taken this term in this place. And this term we are bringing here in this place so this term comes with this minus sign.

So we should place a minus sign here j = 2 to n and j not equal to i j = 2 j not equal to i m_j we can take it outside and we can write it as r_{ij}. This is cube r_{ij}^3 r_{1j} divided by r_{1j} because we can recognize if we reorganize it.

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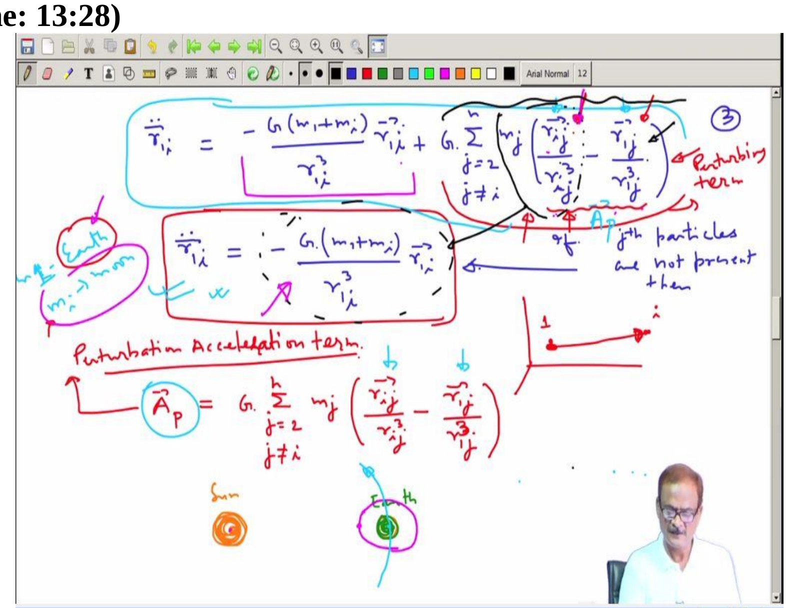

So we have

$$\ddot{\vec{r}}_{1\imath} = -\frac{G(m_1 + m_i)}{r_{1i}^3}\overrightarrow{r_{1\imath}} + G\sum_{\substack{j=2 \\ j\neq i}}^{n} m_j\left(\frac{\overrightarrow{r_{\imath J}}}{r_{ij}^3} - \frac{\overrightarrow{r_{1J}}}{r_{1j}^3}\right)$$

if you can recognize that $\ddot{r}_{1\imath}$ equal to minus if we have suppose,

$$\ddot{\vec{r}}_{1\imath} = -\frac{G(m_1 + m_i)}{r_{1i}^3}\,\overrightarrow{r_{1\imath}}$$

the JTS particles are not present.

Notice right here that k_{th} particles if j^{th} particles are not present, then we get here, the equation in the form $m_1 + m_i$ divided by r_{1i}^3 and this is your 2 body problem. So, this is written the motion of the i^{th} particle about the 1 this is particle number 1 and this is the i^{th} particle. So, you are trying to describe the motion of the i^{th} particle how it will appear about the particle 1. This term this is the extract from which is appearing as a perturbation term.

So, over the 2-particle system this term is trying to disrupt the motion of this 2-particle system and therefore, treating this problem in a general manner. So called a general perturbation theory because of this perturb term. This is the perturbing term. So let us write this perturbing term as for the time being, as A_p because it is a victor so we write this as A_p = G times summation m r_{1j}^3 .

Now we need to recognize what this perturbation term is perturbation acceleration, the first term and the second term these are the 2 terms even in this particular one for this we are writing as A_p this whole thing supporter vision due to some other planet. Just consider this is the A_p for convenience we have written it. Now, what this term is and what this term is so, to understand this I take one problem and then discuss this in the context of the sun earth and moon.

So, let us consider that the sun is here and earth is I show it by green. So, this is earth and moon is going around this moon I will show it by green color or by pink color, so moon is somewhere here in this place. So it is going around in the orbit around the earth and simultaneously earth is going around in the orbit around the sun, if we try to describe the motion in this way. So, if you look for the force on the earth on the moon, so force on the earth, on the moon, this will be given by say, m moon is the mass of the moon. So, let us write just acceleration, acceleration we go on the next page.

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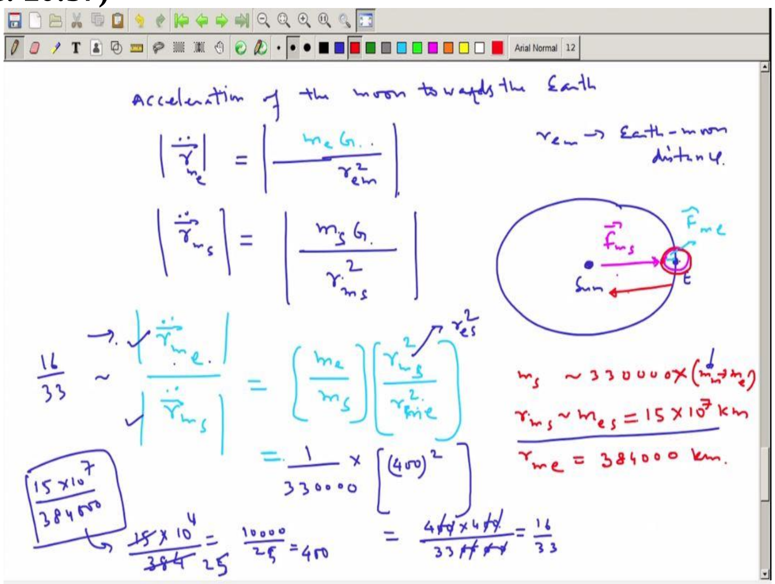

Acceleration of the moon towards the earth. So, how much this will be magnitude wise if we look this will be given by m earth times G divided by r earth to moon the mass of the moon is small as compared to the so we can neglect it for simplicity otherwise if you want to include it so you have to write it the m moon times, m earth plus m earth times G divided by r e-m square where e m r_{em} is the earth moon distance. And this is your acceleration magnitude device or moon about the earth and what about the acceleration of the moon about the sun?

So, if we write that and suppose we write it in this way m sun moon to sun distance, so what I am trying to show here, here your son is located, the earth is located here in this place and around this the moon is going in the orbit. So, what is the force on the moon and what is the force on the moon due to the sun and force due to on the moon due to the sun and force on the moon, which is acting here.

Force on the moon due to the earth what these quantities are. So suppose we neglect this here in this equation this m part the moon part, because it is a small value and simply we write for convenience in earth times G. So, this is the acceleration and multiplied by mass. So you will get the corresponding force. So, which force is greater? This is the question here. So, to look into the corresponding this magnitude of the acceleration, we can compare it by r m this is very important to understand r m s double dot.

This magnitude and if we divide it so this becomes m-e / m-sun G, G cancels out r m-sun square divided by r-moon to earth distance or earth to moon distance. This s square. So, what is the value here that we have to look into what this ratio is and what this ratio is? So this value we can look into so we can write here that m sun this is nearly 330000 times the mass of the moon and plus earth while r moon to sun distance this is will be nearly the same as earth to sun or through sun distance.

Because it is a nearby only on data scale it is not much different and this quantities around 15 crore km so 15 into 10^7 km and what is the value of the what is the distance between the earth and the moon so it is around

$$r_{mc} = 384000 \; km$$

So, 384000 km now calculate these ratios 3 put this value m earth. So here moon is small so m mass by m-sun, we can right here this will be 1 / 33000. And this quantity here will be, it will look into this distance.

So, moon to sun distance, and moon to earth distance so moon to sun distance this is nearly equal to r earth to sun, distance so we can write as r earth to sun square. So, this part r m s we pick up from here, r m s this part 15×10^7 and divided by 384000. So, this turns out to be around 15×10^4 divided by 384 and then we need to square this also. So first we get this ratio and thereafter we will square it. So this distance is if we reduce it so this is 2 and then say around on we approximately write 25.

So, if you do this, so we will be doing little higher estimate of that. So, this will make it easy. So, that way this becomes 10,000 divided by 25. So this is around 400. So this is on higher side. So here, this becomes 400 square and divided by this quantity so 400 into 400 divided by 330000, results in $\frac{16}{33}$. So this way what we see that the acceleration due to on the moon due to the earth, this turns out to be around 16 by this ratio 33, which this is the upper one is less than the lower one. So attraction on the moon due to the sun is larger but it is still it is going down the moon.

So, from where this paradox is appearing, so, this answer is hidden in the equation that we have arrived here in this place this equation so, if we look from the earth and moon point of view, so this is your main equation, or sun moon equation, so we can consider that one is earth m_1 is earth and m i is moon. So this gives you the 2-body problem on that here if you look into these 2 terms present inside the bracket, so it is getting subtracted. The first term is the first one, which I am showing here by this red color.

This is the perturbation on the i^{th} particle here in this case, this is the moon due to the j^{th} particle which is the sun and from there then you are subtracting acceleration of the first particle which is here in this case the earth due to the sun. So, once you subtract it so then what happens then that quantity becomes quite small. So, this quantity therefore, because of this subtraction this quantity becomes too small.

So, what it means? What does it mean that though the force acting on the moon is small as compared to the sun, but simultaneously the earth is also accelerating towards the sun? So earth is in free fall motion toward the sun and because of that reason even though the force on the moon due to the sun is small, but it does not moon does not get out of the earth and go towards the sun. Other way, you can understand that both of them are falling towards the sun going into orbit, we call this the free fall motion and there is a reason for calling this as the free fall motion.

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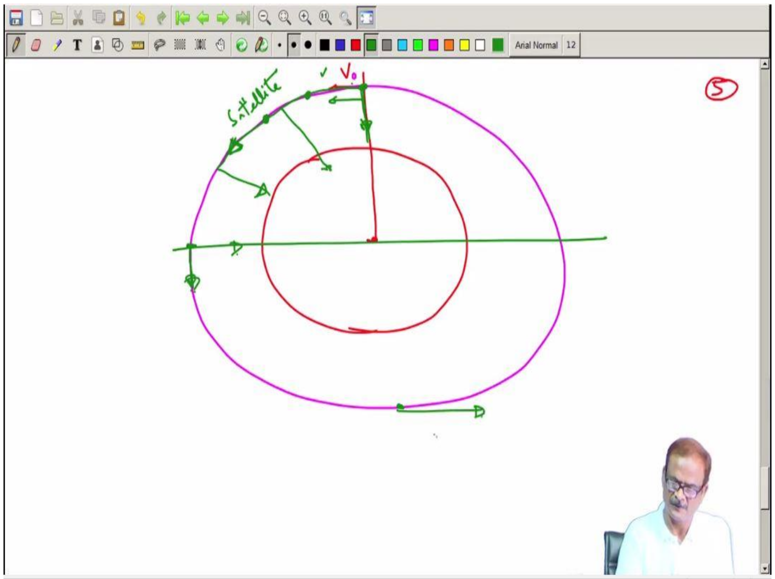

If we take the case of the earth and some particle if you throw it velocity v and we call this as the free fall while this is going in the orbit, it is going here in the orbit and say this velocity is V_0 here. So as the particle starts moving from this place excessively moving under the gravitational acceleration. So, it starts falling down. So, as a result itself falling downward and simultaneously this v is present. So, because of v this is moving here on this side.

So, the net result is that it comes to displace it comes to displace while it comes to this place, this V_0 vanishes and the motion because of this vertical pool only is present here in this point. So, this is a falling trajectory, the particle these are the satellite is falling here satellite is falling like this. So, when gravitational attraction is acting all the way in this direction again it has starts falling from this place.

So, this motion while it comes to this place, this vertical motion will get removed because of the gravitational attraction and only you get a motion here in this place and because of this we call this as a free fall. So, here both are in free fall motion the earth also and the moon also. So moon actual trajectory around the sun it appears like this. So, the main term the conclusion is that this is our main term of the which is representing the motion i is in this case the i^{th} particle is the moon.

So, motion of the moon around the earth, this is the main term which is written here and this is the perturbation term so perturbation term is subtraction of 2 terms, the first term is directly perturbing the moon and obviously, this quantity is large. So, this term I will use another color to indicate this term along with G this term is much larger than this term but from this term, then we are subtracting this term, this particular one, and because of that this part the whole term the perturbation term it becomes quite small.

And that is the reason that moon does not get off from the orbit of the earth and goes toward the sun, it just rather sun acts just as a fertility force on the moon's motion. So, both of them, this is what is called this is the first is the direct acceleration of the moon towards the sun. And this is, this part has to indirect acceleration of the moon towards the sun, because this appears, this term x, this particular term, it is acting on the earth and in turn through the earth to the moon is getting affected. It is not a direct. So, this is an indirect term and this term is the direct term. So, we wind up here, this particular lecture and move to the next lecture. Thank you very much.