Space Flight Mechanics
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Module No # 10
Lecture No # 48
General Perturbation Theory (Contd.)

Welcome to lecture 48 so we have been discussing about the general perturbation theory so in that context we go the basic equation.

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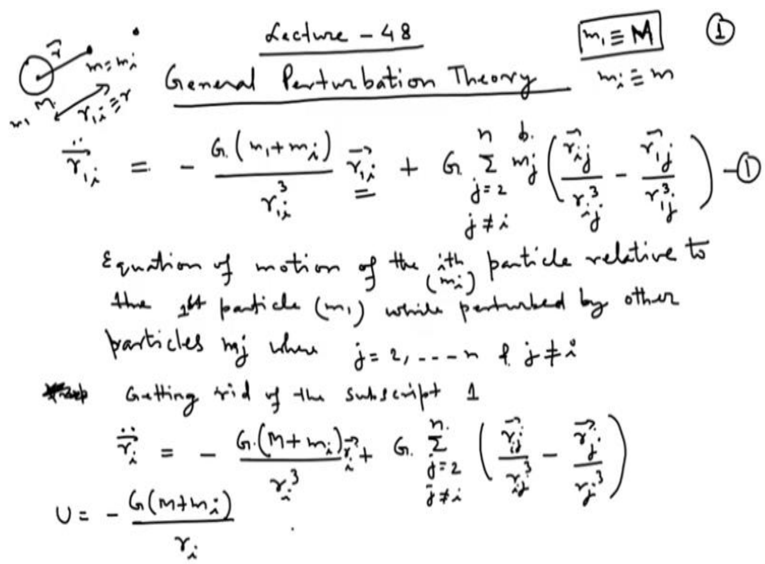

So for this basic equation it shown; here what we have derived in the last lecture so we name this as equation 1. Now we let us write U as minus as we have written in the central force motion or the 2 body problem everywhere we define U as $G \times m_1 +$ before this we do one more exercises. We simplify or we write in the write this equation in a more centralized then so that we get rid of some of the subscript.

So dropping the or removing or say the removing the subscripts getting rid of the subscript 1 we get this equation as $\ddot{\vec{r}}_i$. So that means we are considering m1 to be m and we are replacing it by m or maybe we can write it as m. So we will replace this by m so in that case gets reduce to $G \times (m + m_i)$ and one we are dropping out to indicate that this is from the like here in the case of this is the sun and then there is another planet and this planet is being perturbed by another plant.

So this mass is m and other mass will be considered this as this m which this will be perturbed by other masses which will be the represented by m_j this we will not change. So m_i we will replace by the n because there is no summation over i here so it is easy to get rid of that symbol also. So let us remove this at this stage itself and then the corresponding this distance then becomes r.

So distance from m_1 this is your m_1 and this is your m_i and this distance was r_{1i} so this we are just writing as r. So if we do that so this subscript also we will get rid of and then the equation los little simpler to work with. And on the right hand side in the bracket then we will have j = 2n obviously here j not equal to y which already we have drawn from this place and then this get reduced to $\vec{r_j}$ / r_j^3 -1 we are dropping out so only j remains there.

there is a difference now for the clarification let us keep for the time being we should keep it there later on we will remove it r_{ij} and $\vec{r_j}$ / r_j^3. Now looking at this term if we write here U = - G and more over there is a term here r this sum as to be written here in this place for that reason let us return this part also r_i it is . Now with this only the one we have dropped out so with this notation if we define U equal to this quantity which is

$$U = -\frac{G(m + m_i)}{\vec{r_i}}$$

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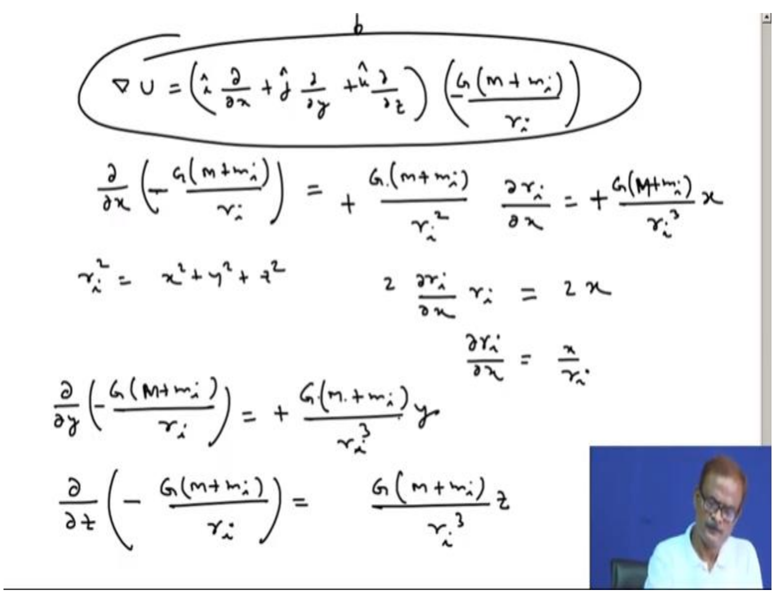

So we can see that ∇U will be $\hat{\imath} \times \frac{\partial}{\partial x} + \hat{\jmath} \times \frac{\partial}{\partial y}$ and U we are defining as here this is scalar quantity so $G \times (m + m_i) / r_i$. So we take the first quantity $\frac{\partial}{\partial x}$ so this quantity can be written as r_i^2 which is minus sign $\partial r_i / \partial x$ and r_i if I define in terms of x^2, y^2, z^2 so taking it is a derivative we see that ∂ ri by this is x / r_i.

And therefore this gets reduced to $-G \times (m + m_i) / ri^3$ times x same $\partial / \partial y$ this quantity can be written at – G U we have defined with and minus sign here. So U this minus sign also we should include here in this place this minus sign will appear here. So minus that will make it plus and this will make it plus this is $-m_j$. Same way the other terms we have to work out so this is $-G \times (M + m_i) / r_i$ this will get $+ G \times (m + m_i) / r_i^3 \times y$.

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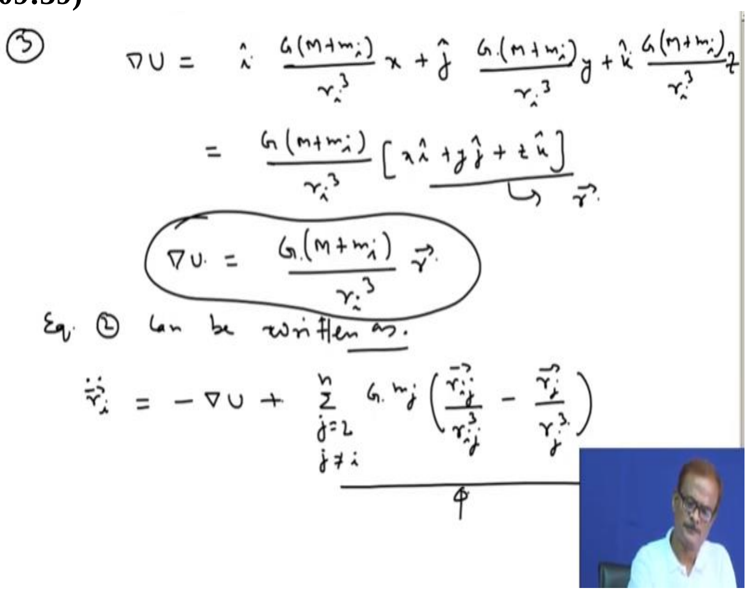

If we insert all this results here in this place so then we get ∇U as $\hat{\imath} \times r_i^3 \times \frac{\partial}{\partial x} + \hat{\jmath} \times r_i^3 \times \frac{\partial}{\partial y}$ we have to write like this taking common $G (m + m_i) / r_i^3$ and this quantity is nothing but we have written earlier as r. This is your r is $\widehat{x\imath} + \widehat{y\jmath} + \widehat{zk}$ and as a result of this if you take the magnitude of this so you get it like this is z square so this is ∇U.

$$\nabla U = -\frac{G(m + m_i)}{r_i^3} * \vec{r}$$

And therefore this is equation 2 so, therefore this equation 2 we will get reduced in terms of this particular derivation we have done. So equation 2 can be written as so what we do this quantity already we have got this is ∇U and then it comes within negative sign. So this is our ∇U so $- \nabla U$

and then the other term is there. So that term also we can reduce in the if this is the term due to gravitation we will always be able to reduce in terms of the using the ∇ operator.

So the other term is j = 2 to n j not equal to i m_j G and this also we need to reduce to get into the potential format. See if the reality is that there are certain forces the conservative forces you can reduce them in terms of the potential function. But if non conservative forces are there present their just like say of aerodynamics drag while the satellite is going in the orbit around the earth. So aerodynamic is present in the low earth orbit the aerodynamic drag is present that drag cannot be represented in terms of potential function.

So aerodynamic drag cannot be represented in terms of potential function so whenever it is possible and generally in the case of the gravitational forces will be able to reduce the terms in terms of potential function. But the non-conservative forces wherever involved so they cannot be reduced in terms of the potential function. the next we reduce this term why we are doing this because it is a useful in working out the mathematics related to general perturbation theory.

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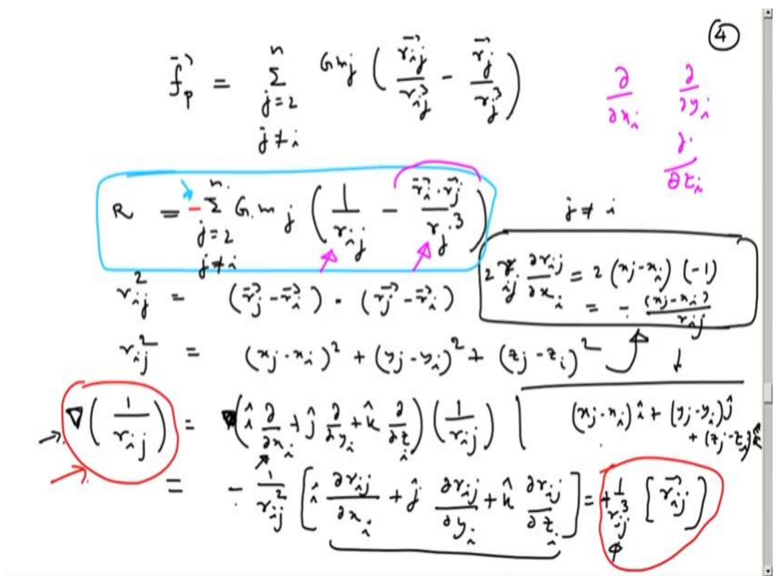

So for considering this term let us write here if perturbation and now let us write R = so if I keep this as 1 / operate on *1 / r_{ij}* this del operator that was same way we have $\hat{\imath} \times \frac{\partial}{\partial x} + \hat{\jmath} \times \frac{\partial}{\partial y} + \hat{k} \times \frac{\partial}{\partial z} 1 /$ r_{ij} this will be and operating on this that gives you *-1 / r_{ij}* square times this can be taken common for all of the them all the term $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$.Hence,

$$\nabla\left(\frac{1}{r_{ij}}\right)=\frac{\overrightarrow{r_{ij}}}{r_{ij}^{3}}$$

So this will come as common and rest of the terms I have to treat it here so rest of the term will be like this $\partial\ r_{ij}\ /\ \partial x\ \hat{\imath}\ +\ \hat{\jmath}$ and from here immediately we can see 2 times $\partial\ r_{ij}\ /\ \partial x$ and we take derivative with respect to the i term. if we do this then this gets reduced to here we will have one more term $r_{ij} \times 2 \times (x_j - x_i)$. And if we are differentiating with respect to xi term so this will be with -1 so that gives you from here immediately we can see that this gets reduced to $(x_j - x_i)/\ r_{ij}$

And therefore the term here can be reduced in the format $-1\ /\ r_{ij}^{\ 2}$ and other terms taking all this terms here in this place. So r_{ij} will go as common so therefore we make this as cube and rest other terms we take from this place this minus sign and this minus sign that will make it plus so we put a plus sign here in this place. So this gets reduced in the format xj so finally this term will lo like r_{ij} already we have taken common here in this place this term is going to lo like $(x_j - x_i)\times\hat{\imath}\ +(y_j - y_i)\times\hat{\jmath}\ +(z_j - z_i)\times\hat{k}$.

So with that this is nothing but your r_{ij} this comes with a plus sign and if this sum over all of them so this is for your del operator only over you have done it for differentiated with respect to i not with respect to the j . So this operator then we can write this as summation j = 2 we go on the next page and write there. So one term we have worked out the another term we will be looking into.

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$$\nabla\left(\frac{\vec{r}_i\cdot\vec{r}_j}{r_j^3}\right)=\left(\hat{i}\frac{\partial}{\partial x_i}+\hat{j}\frac{\partial}{\partial y_i}+\hat{k}\frac{\partial}{\partial z_i}\right)\left(\frac{\vec{r}_i\cdot\vec{r}_j}{r_j^3}\right)$$

$$\frac{\partial}{\partial x_i}\left(\frac{\vec{r}_i\cdot\vec{r}_j}{r_j^3}\right)=\frac{\partial}{\partial x_i}\left[\frac{(x_i\hat{i}+y_i\hat{j}+z_i\hat{k})\cdot\vec{r}_j}{r_j^3}\right]$$

$$=\frac{\hat{i}\cdot\vec{r}_j}{r_j^3}=\frac{x_j}{r_j^3}$$

$$\nabla\left(\frac{\vec{r}_i\cdot\vec{r}_j}{r_j^3}\right)=\frac{x_j\hat{i}}{r_j^3}+\frac{y_j\hat{j}}{r_j^3}+\frac{z_j\hat{k}}{r_j^3}$$

$$=\frac{\vec{r}_j}{r_j^3}$$

So the next term is we want to look into $(r_i \,.\, r_j\,)/\,r_j^3$ so here we have as earlier this 3 terms are present. So we treat the first one and we differentiate with respect to x_i, y_i and z_i. so, now as we see here in this place we have the i is present here and j is present here and we are differentiating with respect to y. So accordingly we have to take care of the here we need to put a q this is r_{jq}.

So in the definition of the r this is the definition of the r we are just now writing and I will modify it little bit for the sign once we get the final result. So here rjq now looking into this part this is $\partial / \partial x_i$ and r_i is your $(x)\times\hat{\imath} + (y)\times\hat{\jmath} + (z)\times\hat{k}\,.\, rj / r_j^3$. And once we differentiate this so we get this as $\hat{\imath}\,.\, r_j$ and as we see the upper quantity is the scalar quantity.

So upper one will get reduce to xj because rj again it is a function of $(x)\times\hat{\imath} + (y)\times\hat{\jmath} + (z)\times\hat{k}$. So this is xj / r_j^3 and here this is dot product. So therefore your the terms which are written as terms here $(r_i \,.\, r_j\,)/\,r_j^3$ this can be written in terms of all this quantities by $x_j \times \hat{\imath}\ / r_j^3 +$ yz $\times \hat{\jmath} \times r_j^3$. And what this quantity is this is simply $r_j/\, r_j^3$.Hence,

$$\nabla\left(\frac{\vec{r}_i\cdot\vec{r}_j}{r_j^{\,3}}\right) = \frac{\vec{r}_j}{r_j^{\,3}}$$

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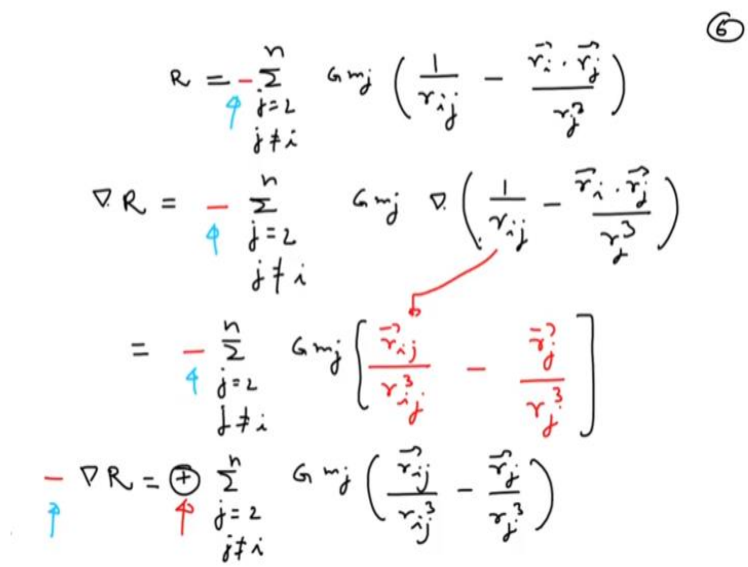

Now we can combine the results together so then therefore the R we defined as j = 2 to n j not equal to i, G$\times$ mj $\times$ $1 / r_{ij} - (r_i \,.\, r_j\,)/\,r_j^3$. So ∇R then gives me I am using the previous results $\nabla\ 1 / r_{ij}$ we have written as $\nabla\ 1 / r_{ij}$ this quantity ultimately we have got this like this. So r_{ij} / r_{ij}^3 now this quantity then gets reduced to and minus now looking into this quantity.

Remember we have differentiated with respect to xi because you are considering the ITS particles if we go back we are discussing about the ITS particle it will be soon clear what why we have done with respect to i. So this term already we have taken care of next we have this term where no minus sign is appearing ay. So therefore whatever we have got here in this place 3 directly applies ∇ operating on this term it is giving me r_j / r_j^3.

Thus we have ∇R = summation j = 2 j_0 = yn and this comes with a plus sign here while for the first turn not this one here we have got with a minus sign. So what this can be reduced to $\ddot{\vec{r}}_i$ this equal to – ∇U + ∇R but it does not lo good this way. So what this way we would like that this sign also becomes minus sign and then what we need to do so that this gets n minus sign ay.

So for that suppose this r we defined with if we put a minus sign here in this place so what will happen? So with this minus sign is stays outside and the whole thing goes there it remains like this here then you have a or you are operating on this on that before the R you have put a minus sign and then you are differentiating so ∇R this will come with a minus sign. And therefore this minus sign which is appearing here if we take it here on this side so then you will get a plus sign in this place.

And then going back here in the original equation this part already we have done so this part we have already written there. And this part this plus then can be reduced with a -∇R sign so if we reduce it with -∇R sign so we have to write it like this. So in that case this gets reduced to $-\nabla (U + R)$. So $\ddot{\vec{r}}_i$ this equal to $-\nabla (U + R)$. where the ∇ operator has been defined as $\hat{\imath} \times \frac{\partial}{\partial x} + \hat{\jmath} \times \frac{\partial}{\partial y} + \hat{k} \times \frac{\partial}{\partial z}$

This is the way we have defined so ultimately we wanted to express the whole system in this format which we have got and for that we required to modify the equation by putting a minus sign here in this place ay. And with this introduction the whole thing we have got into the right away so here then we have ultimately modify the things in all the places and finally this minus sign we carried on the left hand side.

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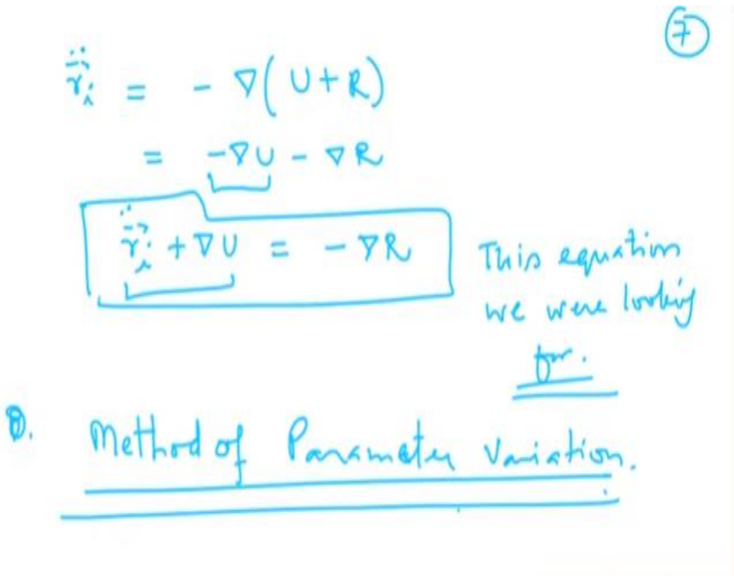

So as a conclusion what we are getting here that $\ddot{\vec{r}}_i= -\nabla (U + R)$. or either simply we can write as $-\nabla U - \nabla R$.. So this is your normal $\ddot{\vec{r}}_i+ \nabla U$ this is a normal 2 body term while on the right hand side you get the perturbation term.

$$\ddot{\vec{r}}_i + \nabla U=- \nabla R$$

So this equation we were looking for now we are ready to dip into the perturbation theory in the next class we will start with the method of parameter variation.

So we need to be particular about writing this term this is $r_i . r_j / r_{jq}$ here this is not q is not present on this but q is present here. Because we are differentiating with respect to like $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}, \frac{\partial}{\partial z_i}$ so with this basic introduction I will stop and the next time we are start with. I will take one example and then I will show you what does mean by the parameter variation method ay, and that will illustrate that will basically present a way of looking for solving this equation which we have written right in the beginning.

How to solve this problem equation? So for the time being thank you very much and will again meet in the next class.