Space Flight Mechanics
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Lecture - 05
Gravitational Central Force Motion

Welcome to the 5th lecture. So this lecture is concerned with the gravitational central field motion or the gravitational central force motion. In the earlier one, the last lecture what we have discussed that the central force motion, so it may be directed toward the center or either it may be directed away from the center, but here in this case the force will be always directed toward the center and that is called the gravitational central force motion and moreover the gravitational force, it is an inverse square force or the inverse square field.

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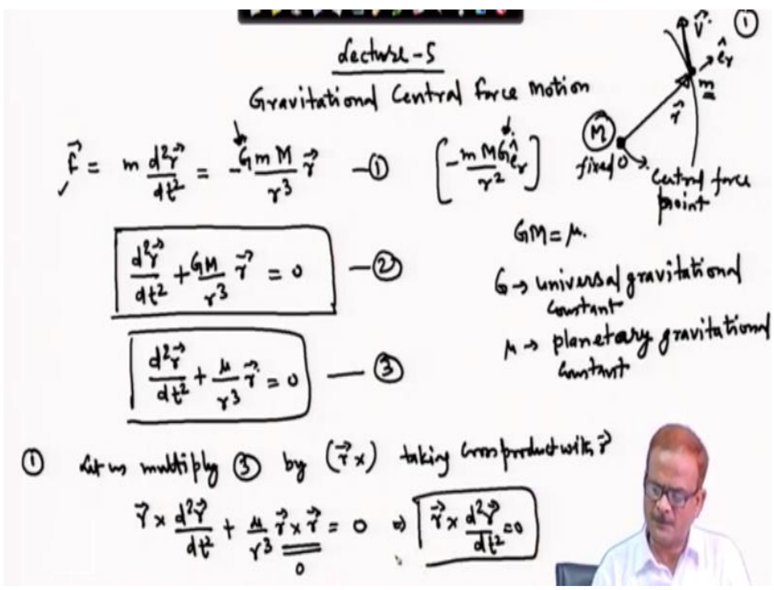

So therefore, accordingly we will be able to solve this for a particular condition. So let us start with, here in this case, the force acting on particle. So we have the center here and some particle is moving here in this place. This is *v* tangent to this curve and this is *r*. In this direction $\hat{e_r}$, we take as a unit vector, *m* is the mass of the particle and here the central mass is assumed to be fixed. This is central force point. It may be considered as an infinite mass, which is fixed.

The motion of the particle m can then be written as

$$\text{from Newton's law } \vec{F} = m\frac{d^2\vec{r}}{dt^2}$$

. Now we know that force here in this case, this will be given by m times M, here this is the mass of this particle, which is fixed. This is fixed. This is not moving.

$$\vec{F} = -\frac{GmM}{r^3}\vec{r}$$

(1)

$$= -\frac{GmM}{r^2}\hat{e_r}$$

Both are the same thing.

$$\frac{d^2\vec{r}}{dt^2} + \frac{GM}{r^3}\vec{r} = 0$$

(2)

and whatever now we have discussed earlier about the central force motion, the same thing all those conclusions are applicable here in this case also. So let us write as equation number (1), this as (2). Here one part is missing, which is G, which is the gravitational constant. So we introduce it here in this place and G times M. we will write as μ.

$$\frac{d^2\vec{r}}{dt^2} + \frac{\mu}{r^3}\vec{r} = 0$$

(3)

This you are aware of. We missed it. This G is always present in the gravitational force, which is universal. G is called universal gravitational constant and μ is called planetary gravitational constant. Then this equation can solved in numerous ways and whatever we have discussed for the central force motion, all those properties we have derived, so we will get all those properties if we apply those principles here. So the first one, we do here.

Let us multiply (3) by $\vec{r} \times$ that means taking cross product with $\vec{r}$. So that gives you r cross and this part will be vanishing, because this is the cross product here of the same vector. So we get here

$$\vec{r} \times \frac{d^2\vec{r}}{dt^2} + \frac{\mu}{r^3}\vec{r} \times \vec{r} = 0$$

This is the same thing as we have obtained earlier.

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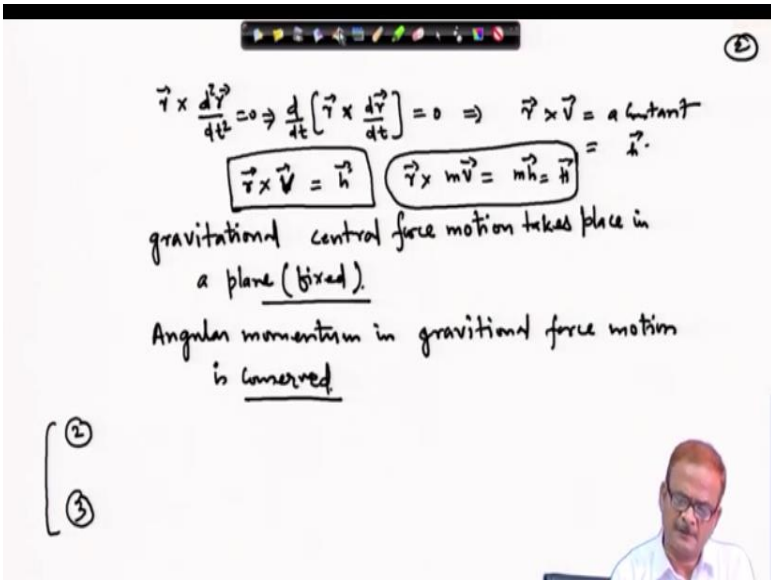

A constant equal to *h* as per our notation. So because it is central force motion, therefore $\vec{r} \times \vec{v}$, this equal to h. So we have derived the same property as we did in the last lecture. So this implies gravitational central force motion takes place in a plane, which is fixed and as we recognized that this is of angular momentum per unit mass. If we multiply both side here *mv*. So this becomes *m* times *h*. We can write this as $\vec{h}$.

So this implies that angular momentum is conserved in gravitational, which is the central force basically. Gravitational force motion is conserved. This is the first property. So all the three properties what we have worked out earlier, so the same thing can also be worked out here in this place.

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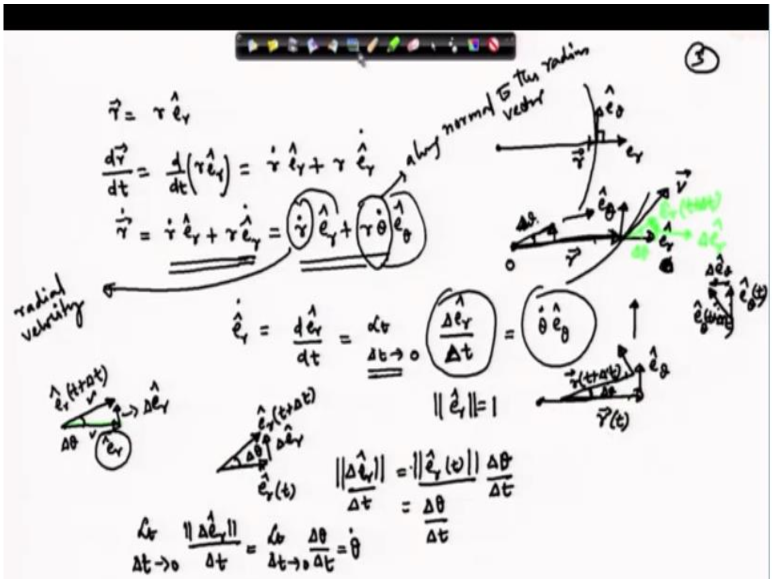

So the next one will be; the energy is conserved. So this is like a repetition of the previous thing what we have done. So now therefore, we will not repeat those things. You can look back and work out all those things. So the rest, the second and third, you can work out using the principle what we have written here. Now we go into, how we can describe the planetary motion. So this gravitational force motion, this is responsible for our planetary or the planetary behavior.

What you see, a planet moving in the orbit, it is because of the gravitational force of the sun acting on that particular planet and obviously there are many other planets. So those other planets, they perturb the orbit. The orbit get perturbed, but those forces may be taken as small for our purpose if we have to look into great details. So we can also formulate all those things, which will come only later on, where the special perturbation general perturbation method of solving certain problems or restricted free body problem, where the free bodies are there.

So how do we solve it, so it falls under that category. So let us continue now with, now I am not going to do those properties rather I am going into the equation of motion.

$$\vec{r} = r\hat{e_r}$$

r is a vector.

So we can write this as

$$\dot{\vec{r}} = \dot{r}(e_r)\hat{} \quad + r\,\dot{\hat{e_r}}$$

So e_r vector is here, this is your $e_r$vector and e_θ a vector perpendicular to this is written as $\widehat{e_\theta}$. See here in this case, if your orbit is looking like this. This is origin.

Velocity vector will be along this direction, $\widehat{e_r}$ will be along this direction. This is your $\vec{r}$and e_θ will be along this direction. So here we need to find out what is the value for the $\dot{\widehat{e_r}}$. So we need to evaluate this. So $\widehat{e_r}$ is a vector as you can see here $\widehat{e_r}$, this is an unit vector and therefore magnitude is 1. If this vector rotates, we will write this as $\widehat{e_r}$t and the vector once rotates by $\Delta\theta$, we write here as $\widehat{e_r}$t + Δt.

So this is the change here, $\Delta\widehat{e_r}$. So magnitude wise what we can see that Δ $\widehat{e_r}$ magnitude is nothing but $\widehat{e_r}$t magnitude times $\Delta\theta$. So this quantity is 1 and therefore we get this as $\Delta\theta$ and if we divided it by Δt on both sides and take the limit, Δt tends to 0, this become

$$\Delta\theta\ /\Delta t\ =\ \dot{\theta}$$

So magnitude wise rate of change of this particular expression is $\dot{\theta}$.

Now the question is, in which direction this takes place. So as you can see that here if I show it by some other colour. Say, if I show this, the change vector $\widehat{e_r}$t + Δt and this is the change angle Δθ. So you can see that this change is taking place here along this direction. So this is your $\Delta\widehat{e_r}$, means this is $\overrightarrow{e_r}$ and the change has taken place along this direction, but as this $\Delta\theta$ becomes small and small it tends toward 0.

So what will happen, that this vector and the vector, they will coincide. This is your $\widehat{e_r}$t + Δt. So this vector will coincide. At that time, this change $\Delta\widehat{e_r}$, this will be perpendicular to this $\widehat{e_r}$. So that implies this change in the limit Δt t tends to 0 is perpendicular to $\overrightarrow{e_r}$, which is nothing but e_θ. This is $\widehat{e_\theta}$.. So this implies that this quantity should be written as $\dot{\theta}$ times $\widehat{e_\theta}$.

And therefore this equation then gets reduced to r× $\widehat{e_r}$+ r × $\dot{\theta}\,\widehat{e_\theta}$and this technique is very useful for working out any problem, wherever you are dealing with the equation of the motion, quite often you may require this technique to expand the expression and work out.

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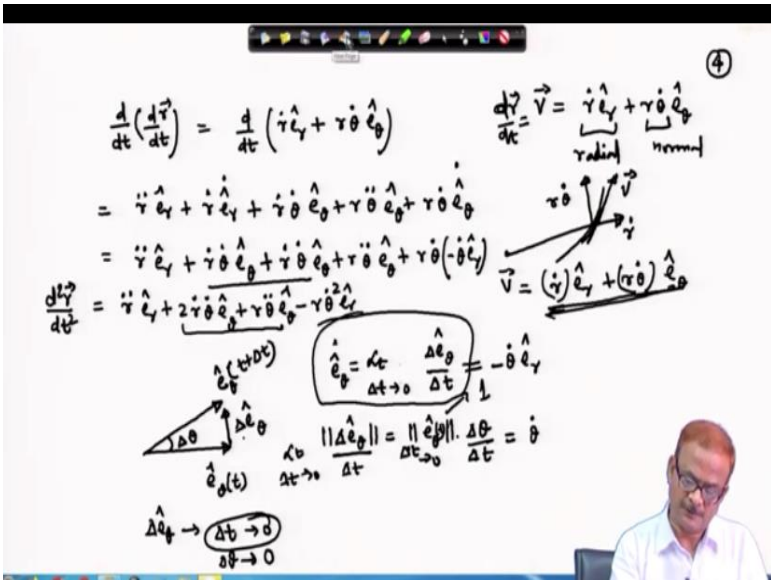

Similarly, we have d/dt. This will be equal to

$$\frac{d}{dt}\left(\frac{d\overrightarrow{r}}{dt}\right) = \frac{d}{dt}\left(\dot{r} \times \widehat{e_r} + r\dot{\theta}\widehat{e_\theta}\right)$$

Now also we need to recognize here see we have missed out here a dot, so we put the dot here, dot is here. So this part, it is called radial velocity. Here in this case if you take this as a scalar, the whole thing will be your radial velocity, including this, so radial velocity and this along including this part along normal to the radius vector. So we go on the next page.

What we are observing that this is the velocity vector, dr/dt is nothing but v dr/dt, this equal to v is $\dot{r}\widehat{e_r}$+ r × $\dot{\theta}\widehat{e_\theta}$. So this is your radial part and this is the normal part. So therefore, if some particle is describing a motion, this is the tangential velocity. This velocity is tangential, so it is component along this direction. This is $\dot{r}$ and component along this direction. This then becomes r × $\dot{\theta}$.

So these are the components. That v is composed of two parts $\dot{r}\widehat{e_r}$. This is one part, the radial part and add to that r$\dot{\theta}$and this is along the theta direction. So therefore, the velocity is this and the velocity is itself tangent to the curve. These are from the basic mechanics. I am not covering it here, you can look into the book by Beer Johnston or Shames Engineering Mechanics. There it is given in great details.

So coming to this, once we expand it, these are self obvious $\hat{\dot{r}}$ $\widehat{e_r}\dot{r}$ times, here dot is missing $\widehat{e_r}$. Now $\widehat{e_{\dot{r}}}$ already we have written here, this is this quantity. So we replace it from here, $\dot{\theta}$ $\widehat{e_\theta}$. times here $\dot{r}$. You can see here $\dot{\theta}$times $\widehat{e_\theta}$ and this one we need to evaluate r$\dot{\theta}$and here this is, dot is missing again. So $\widehat{e_\theta}$, how much this quantity will be, so we have to evaluate.

So following the earlier notation what we have been working with, so if this is $\widehat{e_\theta}$. at time t. This is unit vectors, so the magnitude is not going to change. This is $\Delta\theta$. You can see here, this is your $\overrightarrow{e_r}$. This part is $\overrightarrow{e_r}$and this is the $\widehat{e_\theta}$vector. So if this rotates by $\Delta\theta$ here in this place. If this radius vector goes from this place to this place, from here to here if it moves, as you can see I am showing it by this arrow.

If it moves from this place to place, so it will be along this direction. So your e_θ vector and e_r vector both are moving by the same amount $\Delta\theta$. So they are rotating out by the same $\Delta\theta$. So here this is your $\Delta\widehat{e_\theta}$ and we need to evaluate in the limit $\Delta\theta$ tends to 0, to find out $\widehat{e_\theta}$. So this is nothing but d by dt or you can this as limit delta t tends to 0 delta theta e cap by delta t.

So following the same notation, what we see that $\Delta\widehat{e_\theta}$ magnitude, this will be equal to $\widehat{e_\theta}$ $\Delta\theta$. So if we divide by Δton both sides and take the limit, so limit Δt tends to 0, this is of unit magnitude. This is 1 and therefore this becomes $\dot{\theta}$, but the question is along which direction this change will be. That means Δe_θvaried points in the limit Δt tends to 0 or $\Delta\theta$equivalently tends to 0.

So that we need to work out. So here if you look in this part, this is your e_θ vector. So e_θ vector is pointing up. So as it rotates, as your radius vector rotates from, say the radius vector right now it is here. This is your *r* radius vector; e_θ is directed along this direction. So if your radius vector rotates from this place to this place by delta theta, so your e_θ vector will rotate like this. So you see that this is $\widehat{e_\theta}$and this is your $\widehat{e_\theta}$t + Δt and this is e_t.

This is the change. So the change you can seen that once this *r* vector, this is r_t and this is r_t + Δt So in the limit $\Delta\theta$ it tends to 0 means rt + Δt will coincide with r_t almost. So in that case where this delta $\widehat{e_\theta}$. will point. So if it coincides with this, you will see that this vector will come closer to this. This will be very closer to this. So in that case, it will be just pointing toward the opposite to the r vector. We go to next page.

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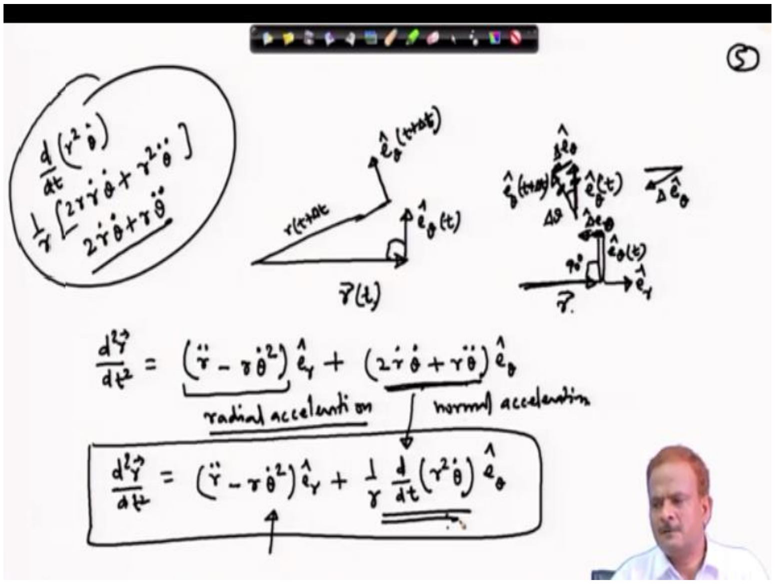

This is your r vector, $e_\theta$vector at time t. This is at time t, r vector goes here rt + Δt. So e_θ vector will be perpendicular to this one, so this is your $\widehat{e_\theta}$. Magnitude does not change, because this is a unit vector. So what are the changes looking into this? This is $\widehat{e_\theta}$t and this vector, you see that I can write it like this and this is of unit magnitude. So therefore, it will appear little shorter, because now it has tilted toward the left. This is Δθ.

This is $\widehat{e_\theta}$. t + Δt. So this is your change and this change you can see that this is inclined below this horizontal here in this place, if I am assume these to be vertical. So this is $\Delta\widehat{e_\theta}$. So in the limit, this one goes and coincides with this one, where I am putting this one coincides with this one. Is it visual on the screen? So in that case both are becoming almost parallel. They are becoming almost parallel and Δe_θ then in that case, it is just pointing along this direction.

And what is that direction? This direction is along this direction; this is nothing but a direction opposite to r vector, r is here and this is perpendicular here, θ = 90° here in this case. We will not write it byθ, we will just indicate that this is 90°, because this is perpendicular to the r vector $\widehat{e_r}$ is here. So this is 90°and this is your $\widehat{e_\theta}$t. so $\Delta\widehat{e_\theta}$.

So this is just opposite to the r vector and therefore, this part then we can write as this equal to $\dot{\theta}$times $\widehat{e_r}$ with minus sign, because this change has taken place along the negative direction. So using this property, then we can complete it. So here this will be $\dot{\theta}$ times $\widehat{e_r}$ and with minus sign. So let us remove it and put it with proper sign $-\dot{\theta}\widehat{e_r}$. If we complete it, so $\ddot{r}\widehat{e_r}$ + this can be added to $\dot{r}\ \dot{\theta}\widehat{e_\theta}$+ r $\ddot{\theta}\widehat{e_\theta}$– r $\dot{\theta}^2\widehat{e_r}$.

So this is your $\frac{d^2r}{dt^2}$ and we can reorganize it on the next page. So $\frac{d^2r}{dt^2}$, then it appears as $\ddot{r}$- r^2, r $\dot{\theta}^2$ is ω^2 r basically.

So this is your radial acceleration and this part is your tangential normal acceleration, because this is in the theta direction.

$$\frac{d^2r}{dt^2} = (\ddot{r} - r\dot{\theta}^2)\widehat{e_r} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\widehat{e_\theta}$$

So this is the normal acceleration. Let us check this term, what I am writing $r^2\ \dot{\theta}/$dt, this becomes 2r $\dot{r}\dot{\theta}$+ r^2 times $\ddot{\theta}$ and this we are going to divide 1/r. This quantity is not 2, this is 1/r. So if we simplify it, this gets reduced to $2\dot{r}\ \dot{\theta}$+ r$\ddot{\theta}$. So we can see that whatever we have written here, this is okay. This gets reduced to this one.

Now if only gravitational force is acting, in that case only this part will be present, because the gravitational force is always radial in nature; this part will be absent. So this we will set to 0. So next we are going to solve this part. So whatever we have derived, this is a very standard way of doing solving the gravitational force equation. Other ways are also there, this is not the thing that we can do, but this is one of most elegant way, we can solve this problem. Next lecture, we will continue. For the time being thank you very much.