Space Flight Mechanics
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Module No # 10
Lecture No # 50
General Perturbation Theory (Contd.)

Welcome to lecture number 50 we have discussed about initial part of the general perturbation theory.

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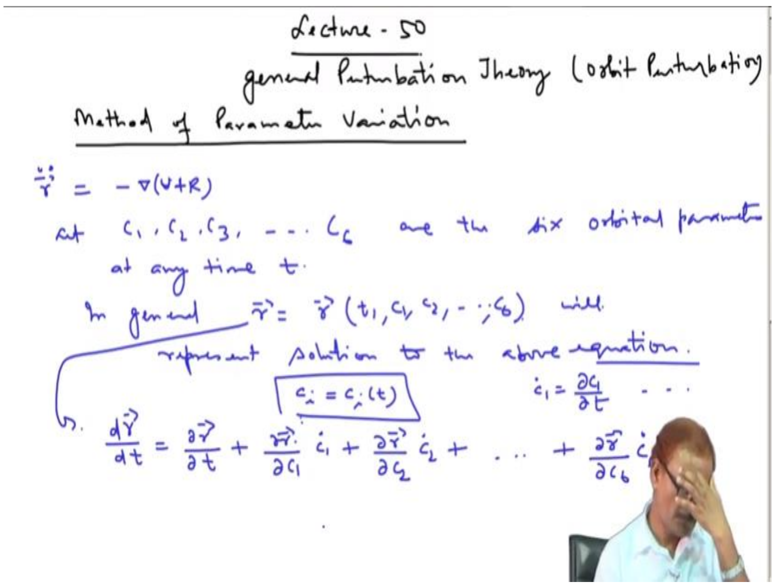

And in that method of parameter variation we are going to apply to solve that problem. So our problem was $\ddot{\vec{r}} = -\nabla\,(U + R)$ this is in bracket and this we need to solve. So this is our basic equation let c_1, c_2 to c_6 are the 6 orbital elements or parameters at any time t. So in general r = r t c_1, c_2 to c_6 will represent solution to the above equation and this we are going to apply on a simple problem to understand how the parameter of variation method works.

And here the ci we can write as ci t so therefore from immediately this place we can observe that dr/dt this can be written as $\partial r/\partial t + \partial r/\partial c_1 \times \dot{c}_1 + \partial r/\partial c_2 \times \dot{c}_2$ where $\dot{c}_1$ is nothing but $\partial c_1/\partial$ c and so on $\partial c_6 \times \dot{c}_6$.

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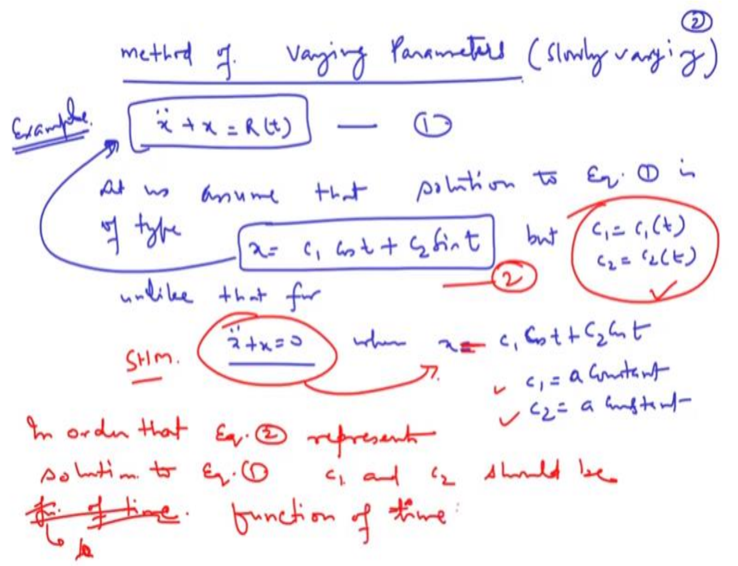

Method of varying parameters slowly varying so here we have example problem

$$\ddot{x} + x = R(t)$$

and this we want to solve. Let us say this is our equation number 1 solution 2 equation 1 type

$$x = c_1 \times Cos\ t + c_2 \times Sin\ t$$

but $c_1 = c_1(t)$ and $c_2 = c_2(t)$. Unlike that for $\ddot{x} + x = 0$ where $x = c_1 \times Cos\ t + c_2 \times Sin\ t$ and c_1 is a constant and c_2 is also a constant. So this is a difference so if this equation fits into this if this is a solution of this, so it must be true that c_1 and c_2 they must be function of time ay.

Here in this case because this is simple harmonic motion equation so we know that solution to this will be of this type were c_1 and c_2 will appear as a constant. This is our equation number 2 should be function of time as I have indicated here. So should be function of this is function should be function of time .

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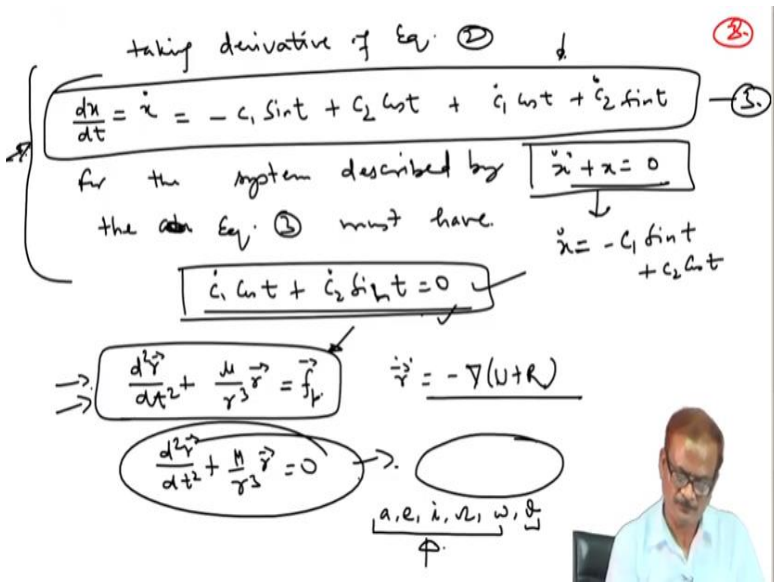

So this implies taking derivative of equation 2 which is this equation this gives us

$$\dot{x} = dx/dt = -c_1 \times Sin\ t + c_2 \times Cos\ t + \dot{c}_1 \times Cos\ t + \dot{c}_2 * Sin\ t$$

for the system described by

$$\ddot{x} + x = 0\ .$$

The equation 3 must have

$$\dot{c}_1 \times Cos\ t + \dot{c}_2 * Sin\ t = 0$$

so what does it mean? That for this system if we want to apply this solution then we must take this quantity to be 0 as we are aware of that solution to this system is only x dot equal to the first integral this is - $c_1 \times Sin\ t + c_2 \times Cos\ t$ this is the first integral and therefore this must be 0 for this.

$$\ddot{\vec{r}} + \frac{\mu}{r^3} * \vec{r} = 0.$$

So our intention is to apply this kind of formulation to a system where say this part also we removed and here we write Rt. We want to apply the kind of problem we are solving to this problem which we have written in the form $\ddot{\vec{r}} = -\nabla\ (U + R)$.

Here then this will be represented by some other function let us say if ft is the perturbation function so we will write it by ft. So here if we have the system like this $\ddot{\vec{r}} + \frac{\mu}{r^3} * \vec{r} = 0.$

so for this the solution will exist where the already we have done that so we have seen that for the that solution that trajectory it turns out to be either an ellipse or an hyperbola or a parabola and here the parameters involved a, e, i, Ω ,ω and Θ so out of this 5 parameters are constant only this turns out to be a function of time.

so that gives us the solution trajectory but if we are looking for this system so in that case all this parameter should vary as it is indicated here. If we are differentiating so, x is the solution where the c_1 also dependent on time so $\dot{c}_1$ will appear here in this fashion.

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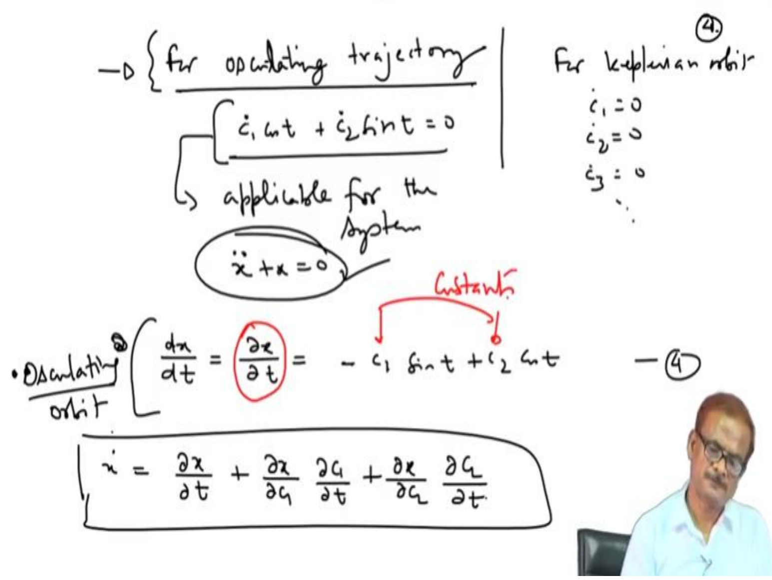

For osculating trajectory we will sit

$$\dot{c}_1 \times Cos\ t + \dot{c}_2 * Sin\ t = 0.$$

For Keplerian orbit $\dot{c}_1=0$,$\dot{c}_2=0$, $\dot{c}_3=0$ and so on here in this case we are osculating trajectory means I am comparing this problem with the orbit problem. So this is applicable for the system $\ddot{x} + x = 0$ this must be true. Therefore then we can write here dx/dt = $\partial x/\partial$t because for the osculating orbit osculating for the osculating orbit we can write like this dx/dt = $\partial x/\partial$t because it is a only function of the states not for the parameter.

The states are x here in this case x and x dot this are 2 states of this system so therefore for the osculating orbit dx/dt = $\partial x/\partial$t and the other part we write here - $c_1 \times Sin\ t + c_2 \times Cos\ t$ this as equation number 4 as in this part we have written here only this part will be represent and this part will be 0. So we are utilizing this information

$$dx/dt = \partial x/\partial t + + \partial x/\partial c_1 \times \dot{c}_1 + \partial x/\partial c_2 \times \dot{c}_2$$

This is true for osculating orbit for which remember we are writing here in this only fashion it is only a function of time. The states are only the function of time not the parameters it is turn out to be constants here.

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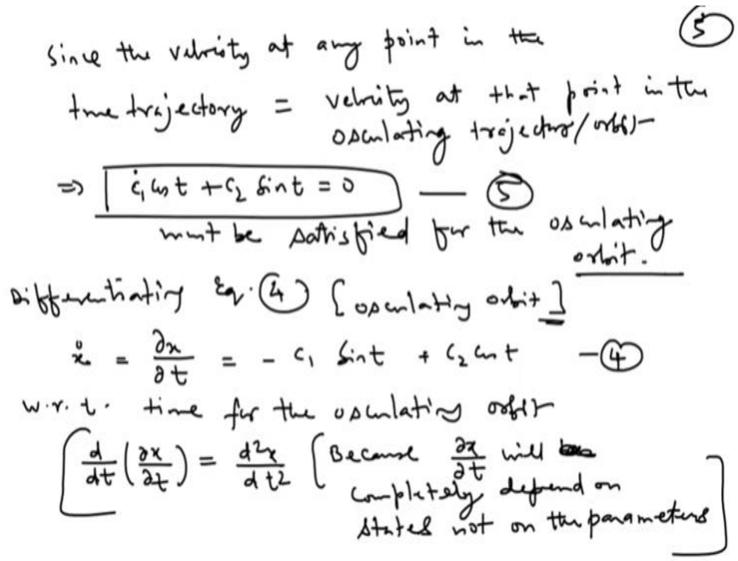

as we have discussed within the last lecture as the velocity at any point in the true trajectory is equal to the velocity at that point in the osculating trajectory slash orbit. So this implies that for osculating orbit we must have

$$\dot{c}_1 \times Cos\ t + \dot{c}_2 \times Sin\ t = 0$$

This we number as the equation number 5 which we have written here for osculating trajectory $\dot{c}_1 \times Cos\ t$ + $\dot{c}_2 \times Sin\ t = 0$. This is our equation so in this equation this part must be 0 if we are considering only osculating orbit solution.

So this will represent osculating orbit and this part is equal to 0 then satisfied for the next differentiating equation 4 which we have written as $\partial x/\partial t = -\, c_1 \times Sin\ t + c_2 \times Cos\ t$ this was our equation 4. So we differentiate this with respect to time next differentiating equation 4 with respect to time and for the osculating orbit we are representing this is for osculating orbit. So if you are differentiating for the osculating orbit we write it this way dt square because $\partial x/\partial t$ will completely depend on states not on the parameters for the.

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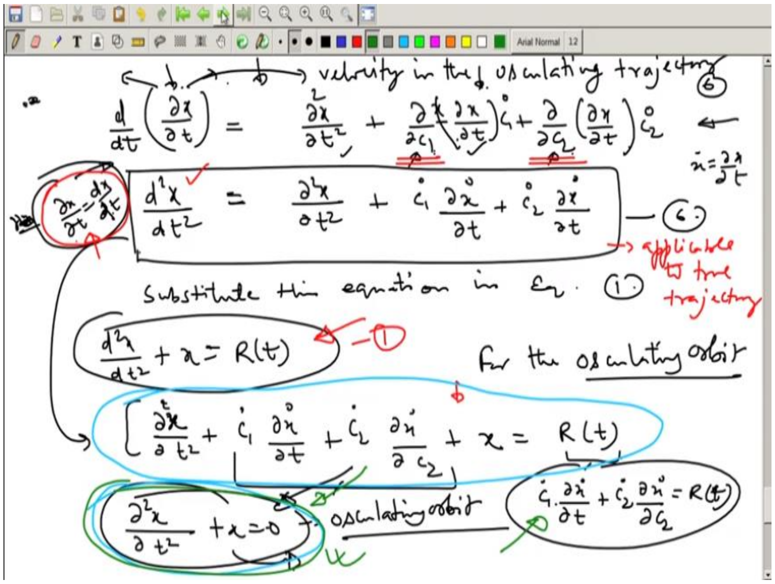

Thus we have this quantity we differentiate so this get as

$$\partial^2 x/\partial t^2 = \partial x/\partial t \times \partial/\partial c_1 * \dot{c}_1 + \partial/\partial c_2 \times \partial x/\partial t \times \dot{c}_2$$

Now if we are writing like this so this is what I am going to discuss what do I mean here . So let me write this part first this is $\dot{c}_1$ this quantity we can write as $\dot{x}$ where $\dot{x}$represents $\partial x/\partial t$ in the sort we can represent it like this.

So the way we have written here so here we are taking this differentiation so at first we are assuming that this is depending on t and also on the variation of the parameters c_1, c_2 though we are doing for the osculating orbit where c_1 and c_2 for which the c_1 and c_2 are not taken care ahh the differentiation is not taken with respect to the c_1 and c_2. But what is the reason we are taking for this it will be clear in the next step .

Now we number this equation as 6 substitute this equation in equation 1 so what do we get? Their equation 1 was $\ddot{x} + x = R(t)$. so if this substitute this here so we get this as $\partial^2 x/\partial t^2 + \partial x/\partial t + \dot{c}_2 \, \partial\dot{x}/c_2$ and + x = $R(t)$. So for the osculating orbit we must have this quantity equal to this quantity because ultimately this should emerge as see let me write here first then next I will explain here.

This is for the osculating orbit where this solution to this is not parameter dependent so if we insert this by differentiating with respect to the parameter c and in insert here in this equation in this equation and then set this quantity to write this quantity as $\partial\dot{x}/\partial c_2 = R(t)$. So if this is applicable then you can see that this gets reduced to this equation ay it will get reduced to this format.

Means this is the presenting my osculating orbit so the objective was to get only this relationship to solve this problem particular problem. If we write here if make this quantity as 0 that this is not varying with respect to function of time with respect to time so we do not get equation ay. Like if we go back you see here in this point so, for that true trajectory this whole thing is applicable this whole equation.

So first we get this and there after what we do that we make this part equal to 0 for the true trajectory and this we have done in this place writing like this. So we get 1 equation from this place which we have written as equation number 4 somewhere not equation number 5 this we have written as equation number 5. So this thing we have written has equation number 5 here in this place.

So how we are getting we are getting in the true trajectory we are making some part equal to 0 to fit it to the osculating trajectory. In the same way we are looking for the solution for the osculating trajectory and then what we do this is the velocity in the osculating trajectory $\partial x/\partial t$. So if we differentiate this so how it is going to work here so this is the velocity in the osculating trajectory. And if we differentiate it which respect to time so this will appear in this way.

So velocity in the osculating trajectory we are differentiating with respect to time that gives me acceleration ay and this acceleration whatever we have got then this we are because of velocity in the this quantity $\partial x/\partial t$ in the $\partial x/\partial t = dx/dt$ for true trajectory and the this part is very important for the true trajectory and the osculating trajectory $\partial x/\partial t = dx/dt$ this is true ay. Because in that part this is valid and because of this the only one part is remaining which corresponds to the osculating trajectory.

So with this in mind we get this $\partial^2 x/\partial t^2$ so this is applicable to the true trajectory what we are trying to work for the osculating trajectory. So, velocity in the osculating trajectory, if we differentiate with respect to time because at that point the velocity in the true trajectory and osculating trajectory both are the same. So, if we differentiate it we get the acceleration at that pint and there we assume that the c_1 and c_2 they are variable.

And there after whatever the equation we get we insert here in this equation which was equation number 1. Once we insert this so we get this equation now this equation it will represent the osculating trajectory differential equation if and only if that means this is the osculating orbit and differential trajectory. So this it will represent this trajectory if and only if this quantity this equation is satisfied this relationship is satisfied.

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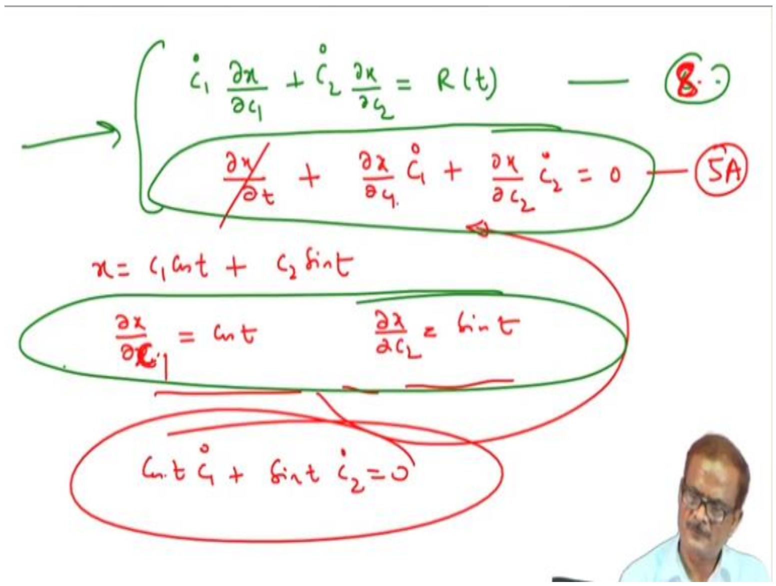

So in the next phase we write that the $\dot{c}_1 \times \partial x/\partial c_1 + \dot{c}_2 * \partial x/\partial c_2 = R(t)$. should be satisfied. And this we write as equation number 6 this is equation number one more equation we have to name here now this we write as equation number 7 and this 6 we have already named and this we name as equation number 8. So equation number 8 and equation number 4 so what was the equation number 4 we have written earlier ∂x/going back this part.

So from where this part is appearing this part is appearing from see this part is appearing from this place what we have indicated here from there we have written equation number 5 is written in this format but I want to just represent in the differential format first. So that part we have ay let me write here itself this $\partial x/\partial t$ this part we have to just remove the other part is $\partial x/\partial c_1 \times \dot{c}_1 + \partial x/\partial c_2 \times \dot{c}_2$.

This equal to 0 the other part is this represents your equation number 5 rather than writing this 5 I will write this as equation number 5A. because this equation we were treating what is your cost immediately we can check here see let us go on this page. So $x = c_1 \, Cos \, t + c_2 \, Sin \, t$ if we differentiate $\partial/\partial x_1$ $\partial/\partial c_1$ this quantity then comes out to be Cos t and $\partial x/\partial c_2$ this turns out to be Sin t.

And if we insert this quantities here in this equation so what we get

$$Cos\ t \times \dot{c}_1 + Sin\ t \times \dot{c}_1 = 0$$

and what this equation is go back now and lo here is this the same equation the fifth equation ay. So this is representing your fifth equation in differential format and we have to lies this 2 equation here to work out and we will remember that this is the relationship involved here.

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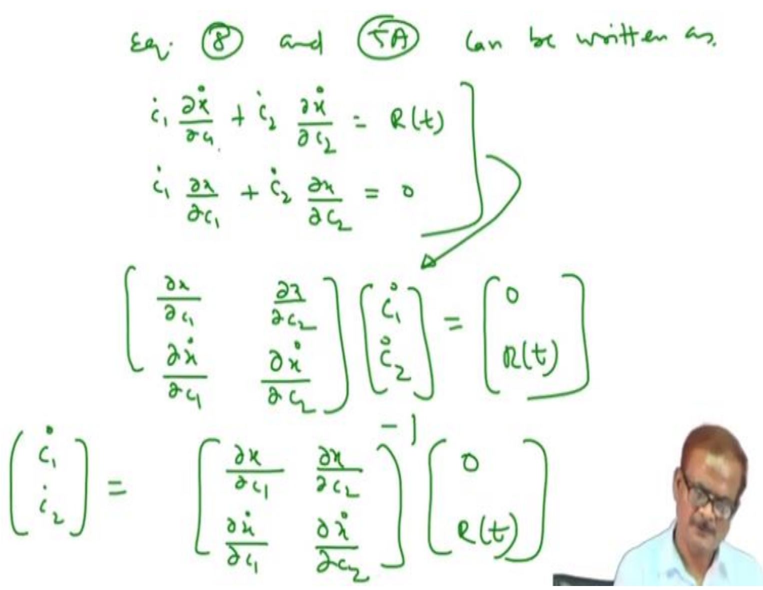

So from equation 8 and 5A and we written as we will take out c1 and c2 this part is 0 this part we are eliminating this part is not present ay this I am removing only this part we are considering this equal to 0. So we are $\dot{c}_1\ \partial x/\partial c_1 + \dot{c}_2\ \partial x/\partial c_2 = R(\ t\ )$here this is x dot this dot is missing here there was dot here so this dot was missing. So this dot is present and here in this equation we have $\dot{c}_1$ $\partial x/\partial c_1 + \dot{c}_2\ \partial x/\partial c_2 = 0.$

And in the matrix format we can write this as $\partial\dot{x}/\partial c_1$ or let us write first the lower equation so this can be written as $\partial x/\partial c_1$ and $\partial x/\partial c_2$ and then the upper one $\partial\dot{x}/\partial c_1 \times \dot{c}_1\dot{c}_2 = \{0\ \ R(t)\}$. The solution to this matrix equation which can be written as

$$\begin{bmatrix}\dot{c}_1\\ \dot{c}_2\end{bmatrix} = inv\begin{bmatrix}\partial x/\partial c_1 & \partial x/\partial c_2\\ \partial\dot{x}/\partial c_1 & \partial\dot{x}/\partial c_2\end{bmatrix}\times\begin{bmatrix}0\\ R(t)\end{bmatrix}$$

So we have to just take the inverse of this part.

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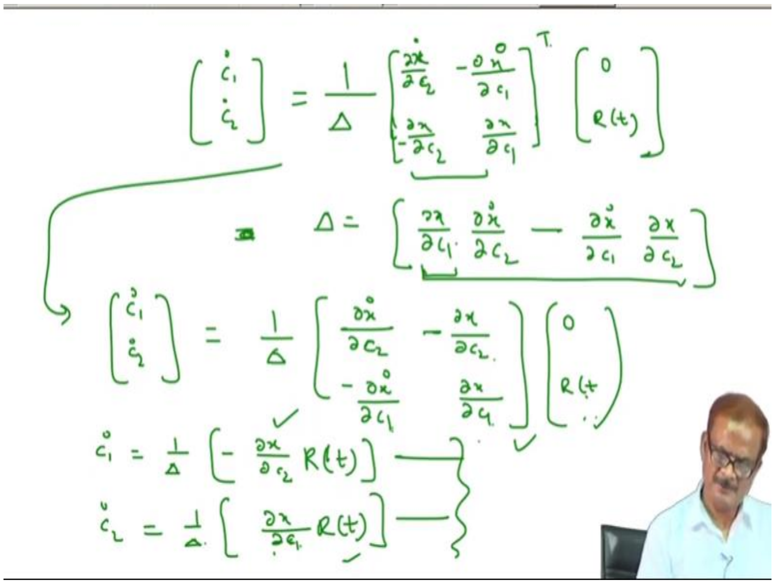

Then therefore $\dot{c_1}$ $\dot{c_2}$= this get reduced to. So this will be $\partial x/\partial c_2$ into for this we will have $\partial x/\partial c_1$ with minus sign here this are the first we are taking the cofactors and then we will take transpose of that. And then for this element this will be $\partial x/\partial c_2$ with minus sign and this will be $\partial x/\partial c_1$ with plus sign and which have to take the transpose of this and {0 R(t)}.

And this divided by the determinant of the matrix where Δ equal to $\partial x/\partial c_1 \times \partial \dot{x}/\partial c_2$ $\partial x/\partial c_1 \times \partial \dot{x}$/- $\partial x/\partial c_2 \times \partial \dot{x}/\partial c_1 \times$ into $\partial x/\partial c_2$. so from this place when can get $\dot{c_1}$ $\dot{c_2}$= 1/Δ taking transpose of this $\partial \dot{x}/\partial c_2 - \partial \dot{x}/\partial c_1$. And then this last row $\partial x/\partial c_2$ with minus sign and $\partial x/\partial c_1$ {0 R(t)}and solve this

So by solving we get $\dot{c_1}$ = 1/ Δ × – $\partial x/\partial c_2$ R t and $\dot{c_2}$= 1/Δ $\partial x/\partial c_1$ × R t. So these are the 2 differential equation we have got for c_1 and c_2 so how these parameters will vary we will know from this point. So we have $\dot{c_1}$ here equal to $\partial x/\partial c_2$ ×R(t)and let us verify this and $\dot{c_2}$= 1/Δ ×$\partial x/\partial c_1$ × R(t) $\partial x/\partial c_2$ this come here and this goes here dx/dt this is ay $\dot{c_1}$ = 1/Δ × R(t) – $\partial x/\partial c_2$ this $\dot{c_2}$ 1/Δ ×$\partial x/\partial c_1$ × R(t) .

And from here now this is ay so the quantity Δ involved $\partial x/\partial c_1$ and $\partial x/\partial c_2$ these are the quantities which are present. So Δ we can get from our previous discussion so I will write here in this place or may be on the next page.

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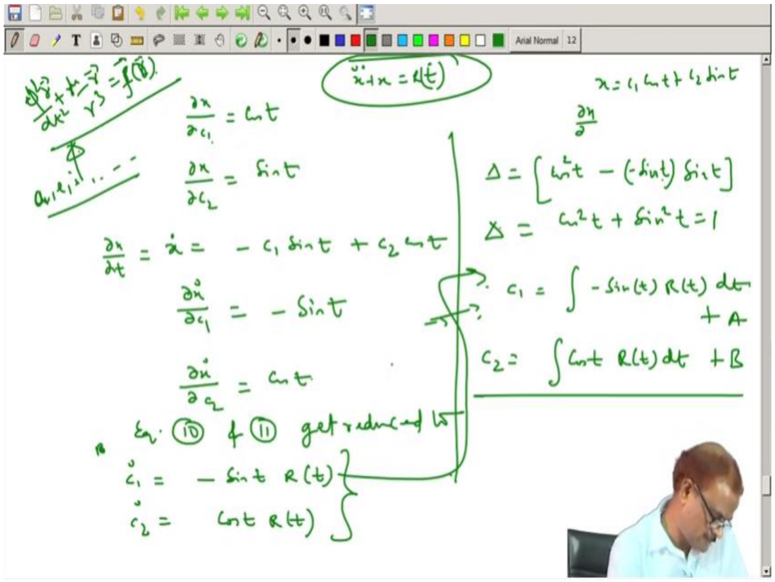

$\partial x/\partial c_1$ where $x = c_1 \, Cos\, t + c_2 \, Sin\, t$ so $\partial x/\partial c_1$ then becomes Cos t and $\partial x/\partial c_2$ = Sin t. Similarly we have $\dot{x} = \partial x/\partial t = \dot{x}$ we have written as this part was $- c_1$ Sin t + c_2 Cos t this was the equation so here from we have get $\partial\dot{x}/\partial c_1$ = - Sin t and $\partial\dot{x}/\partial c_2$ = Cos t. So we utilize this information and insert in the solutions we have got here so this is 10 and this is 11 inserting in 10 and 11 first we calculate Δ.

So $\Delta = \partial x/\partial c_1$ and $\partial\dot{x}/\partial c_2$ and $\partial x/\partial c_1$ from this place we have Cos t and $\partial x/\partial t$ is so this is Cos square t and then $- \partial x/\partial\dot{x}/\partial c_1$ and $- \partial\dot{x}/\partial c_1$ = -Sin t n × the quantity here $\partial x/\partial c_2$ $\partial x/\partial c_2$ is Sin t. So that gets reduced to $Cos^2 t + Sin^2 t$ this equal to 1 so this is your Δ therefore the equation number 10 and 11 2 equations 10 and 11 get reduced to $\dot{c_1} = \partial x$ - $\partial x/\partial c_2$ – Sin t × R t.

And $\dot{c_2}$ similarly get reduced to $\partial x/\partial c_1$ *is Cos t R(t)* and if we integrate it we get this as

$$c_1 = \int - Sin\, t\, R(t)dt + a \text{ and } c_2 = \int Cos\, t\, R(t)dt + b$$

So this is the solution so how the parameter as varying with time we get through our very simple example or taking $\ddot{x} + x = R(t)$. Therefore in our original equation where we have $\ddot{\vec{r}} + \frac{\mu}{r^3} * \vec{r}$ equal to this is the perturbing function which we write as the function of r here r is a function of t.

So therefore in this case also the parameter which are involved the c_1 c_2 are say the a, e, i etc., they will vary with time and they can be estimated using this method. But it is a long procedure and we have to work it out presently so we stop at this stage thank you for listening we will meet again in the next lecture.