Space Flight Mechanics
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Lecture-53
General Orbit Perturbation Theory (Contd.)

Welcome to lecture number 53, we have been discussing about the general orbit perturbation theory. So, in that context, we are looking into the Lagrange bracket and their properties. So, last time we have looked into the 3 properties of the Lagrange bracket. Now today we will start with evaluation of the Lagrange bracket because for solving the orbit perturbation theory, this must be evaluated.

So, that the rate of change of orbital parameters that equation we can get . If you get the equation, so from there what will be the future value of the orbital parameter it can be calculated, .

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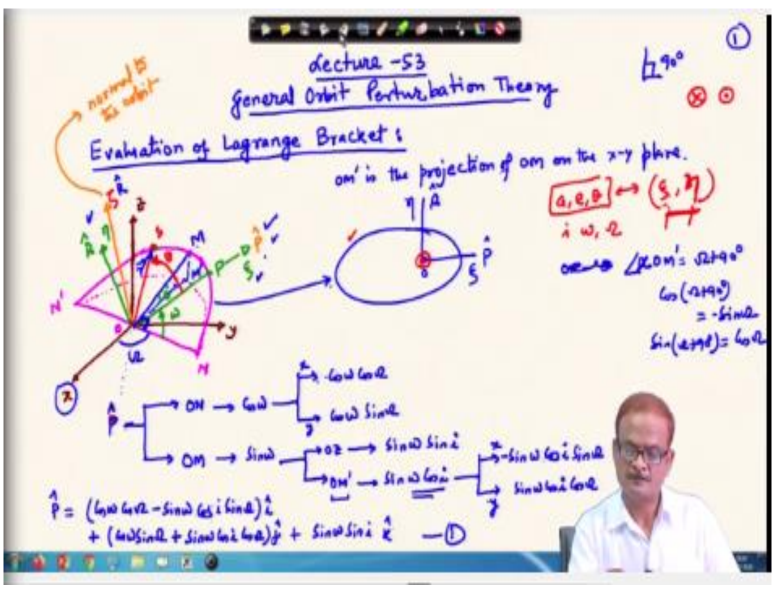

So, let us start, this our x, y and z inertial access and we have got the orbit here, n, n' as in our earlier discussion of the 2 body problem, we have derived orbital parameters. And in that context we have looked into this orbit, so it is the same thing not different. This angle is 90 degree, the dotted line is a projection of the orbit in the x, y plane. And then we define the line or periapsis which we indicate by let us periapsis we indicate by P and this one will indicate by M.

This will write by M because earlier also we have written using M and perpendicular from this point to this point on the orbit projection, this is your M‘. And if we join the line here on this point to this point, so this is projection of the line. So, OM‘ is the projection of OM on the x-y plane. Now here we have start working with what we need to do, so in this direction we define ζ as a unit vector here in this direction, we take one reference frame.

So, this is the green one is a periapsis line, so in that direction the unit vector we take as the ζ cap. And perpendicular to this in the orbital plane, we take unit vector, in this direction the unit vector will be a η cap. So, this is ζ direction and this is η direction and this angle we know this is ζ, this angle is ζ angle of inclination. We have not drawn the orbit normal rate and let us say that the orbit normal we indicate by.

So, we show the orbit normal along this direction, this is the orbit normal. To make it little more convenient what we will do that to make it a little bitter, let us make this as unit vector here in this direction as $\hat{P}$. And unit vector along the η direction as $\hat{Q}$, in this direction as $\hat{Q}$. So, that will make it little better to work with. Now, this is the configuration, now the angle this is the small omega which is the argument of perigee.

And somewhere in the satellite maybe located here in this position, so satellite position it take along the direction, this is a satellite here. So, the angle from green line to displace, this is θthis angle is 90 degrees from blue line to the pink line. This we use to show the 90 degree angle . Between these 2 green lines, this one and this one the angle is 90 degree. So, what we have done here in this orbital plane we have taken $\hat{P}$ and $\hat{Q}$as the reference lines like in your orbit.

If you remember, if this is the focus, so this orbit I show it like this, so this becomes your $\hat{P}$ here in this direction and $\hat{Q}$ here in this direction, this is your ζ direction and this is the η direction and this is your origin here in this place. So, this is exactly the pink line is shown here in this place, this notation is being used. And along this direction, so if ζ and here we write as ζ.

So, orbit normal will be perpendicular to this orbit, this blue orbit which I cannot show yet, we can just show it like. See, once we in your 12th class or earlier you might have learned that if we show it like this. So, this indicates backside of the arrow and this indicates the front side of the arrow, so if I indicate like this, so your z axis is coming outside. Now we need to write the unit vectors $\hat{P}$ in terms of x, y and z.

And why we are doing this, right now it will not be visible to you but there is certain purpose to this which will be toward the end, what exactly we have done. So, what we are trying to do that, we are trying to work with a, e ,I,**Ω** ,ω and Θ. And a, e and Θ these are of line in the orbital plane. So, therefore they can be expressed in terms of ι and this is ζ and η .

This is Xi and η, in terms of Xi and η we can represent and ζ is perpendicular to these two, . So, this is equivalent to x and this is equivalent to y and ζ is equivalent to z. And by doing so, the problem will get little simplified, so let us proceed and lo into this . So, $\hat{P}$ we have to first write, so $\hat{P}$ it consists of this vector $\hat{P}$vector which is shown here in this place. This is a unit vector, therefore it can be written in terms of i, j and k .

So for defining this, we need to break this ON along the ON direction and the OM direction. Along the ON direction we will have the component cos ω using this angle and along the OM direction this will be sinω. Thereafter ON can be bren along the x and y direction. So, here we break it along the x and y direction this angle is **Ω**, so this becomes along the x direction we write here first x maybe we can write here x.

So, along the x direction we have cos ω × cos **Ω**, and along the y direction cos ω × sin**Ω**. So, this is along the ON direction then another component which is along the OM direction, this can be first bren along the OM ‘ and OZ . So, this we break along the OZ and along the OM ‘, so we will have the component along the direction OZ direction, sin ω and the angle is ζ.

So, this becomes sini and this one is becomes sinω, this type of analysis we have done earlier the figure it is visible I need not keep on explaining all these things, so sinω and cos i. So, component along the OM direction, now component along the OM direction can be broken along the x and y direction, so this exercise we have to do. So, z component remains as it is while the component OM it can be broken along the x and y direction.

So, from here we have to break along the x and y direction, so this becomes OM ‘, so angle between the OX and simply we can right angle O, angle M ‘. See if here it may be appearing that the OX and OM ‘ they are collinear but this is not so, this is because of the favor. This line if I draw it here in this place rather than. So, immediately it will be visible that they are not the same thing.

So, the angle O XOM ‘, this angle is Ω + 90 degree. And therefore along the OX direction this we get as cos Ω + 90 degree, this will be - sinΩ. And similarly sinΩ + 90 degree, this will be cos Ω. So, we use this information to write these values here as a component along OX then this becomes sinω, sine, i × sinΩ and this with a - sine, so – sin replace here .

And then the y component this becomes sinω, cos i × cos Ω, so this is the y component. So, similar exercise we can do also further the Q cap and therefore $\hat{P}$ we can write as combining the x and y, components x y and z separately. So, this becomes cos ω, cos Ω, - sinω, sini × sinΩ × $\hat{\imath}$ because this is a along the x direction.

And then + along the y direction then we have cos ω × sinΩ and + sinω, cos i × cos Ω × $\hat{\jmath}$. And finally the z component which is sin ω, sin i $\hat{k}$. So, this is how the $\hat{P}$ los like, oh here this is cos i, so here this would be cos i we have written here. This is cos i, sinω cos i because we are breaking this part, so this is cos i, so we make it here cos i.

$$\hat{P} = (\cos\omega \times \cos\Omega, - \sin\omega \times \sin i \times \sin\Omega)\ \hat{\imath} + (\cos\omega \times \sin\Omega + \sin\omega \times \cos i \times \cos\Omega)\ \hat{\jmath} + \sin\omega \times \sin i \hat{k}$$

These are some of the errors which keeps creeping in but I keep correcting all these things . So, if we have written $\hat{P}$ along the same line we can also write the $\hat{Q}$.

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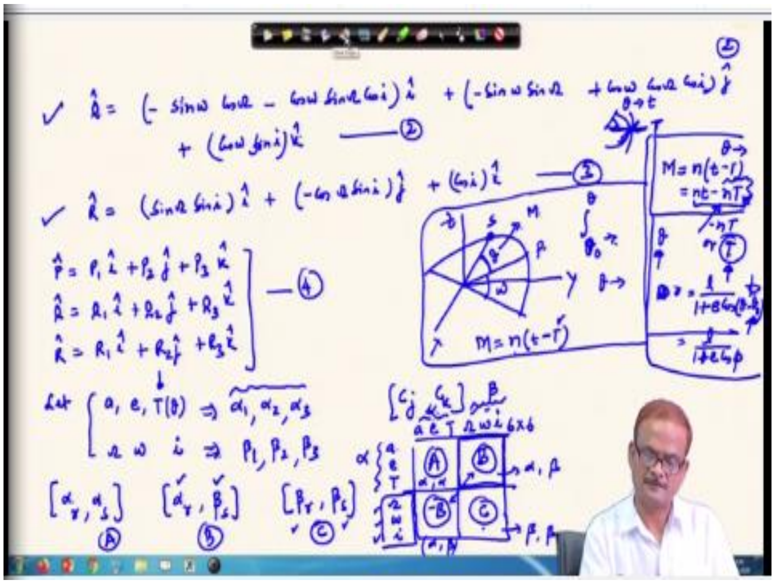

The $\hat{Q}$ the unit vector along the η direction, this can also be written in the same way and I will write directly the values for that. And you can check those values this will be - sine, ω, cos Ω cos i × i cap. So, this $\hat{Q}$ and along this direction along the ζ direction let us write the unit vector as $\hat{R}$. Then the $\hat{R}$ can be written as sinΩ, sin i × i cap and we name them.

$$\hat{Q} = (- \sin \omega \times \cos \Omega - \cos \omega \times \cos i \times \sin\Omega)\ \hat{\imath} + (- \sin \omega \times \sin\Omega + \cos \omega \times \cos i \times \cos \Omega)\ \hat{\jmath} + \cos \omega \times \sin i\ \hat{k}$$

$$\hat{R} = (\sin i \times \sin\Omega)\ \hat{\imath} + (- \sin i \times \cos \Omega)\ \hat{\jmath} + \cos i\ \hat{k}$$

Let us say this is 1, this is 2 and 3, so we have got the unit vectors and therefore now we will express

$$\hat{P} = P_1 \times \hat{\imath} + P_2 \times \hat{\jmath} + P_3 \times \hat{k}.$$

$$\hat{Q} = Q_1 \times \hat{\imath} + Q_2 \times \hat{\jmath} + Q_3 \times \hat{k}$$

The set of equation 4. Let a, e and θ so instead of using this θ we write here in terms of T actually what we are interested in.

In the orbital parameters we are looking for variation of the orbital parameters. So, θ is already a varying thing but this is not of interest rather if you look into the mean anomaly. So, in the mean anomaly we have written as the mean angular rate × t - T, so, nt - n T. So, this is the quantity - n T or simply T this we are interested in. And this T defines the time of perigee passage, if you remember that while we have written R the expression for R we have derive.

So, this was 1 + e cos $\theta - \theta_0$ or later on this we wrote simply as 1 + e cos pi. So, here θ_0 this is referring to the time of perigee passage or which we are taking as the reference point, this may be taken as a reference point even. So, we are not working with this right now, we are working with this particular symbol we will be using it. And what we are going to do, so shortly if while we discuss so these things will be clear.

But, the issue is that θ is already a function of time, so this is varying , it will be varying in any orbit. But what is the spacability that in the true orbit, the time of perigee passage this is the line of periapsis . This periapsis line also recedes , this is your θ here you are indicating this angle as θand reckoning time at this time θ the true anomaly can also be represented in terms of T .

So, what is the time at this point the periapsis, so this time we are representing as T. So, this is the quantity which is entering here in this place from where you are referring to. See here in this place I will make the figure and explain, so this is your θ position , satellite position and somewhere here there is the periapsis. This is the y line, x line and this is the z line, this is the periapsis line.

This angle we have taken as ω and this angle as θ so mean anomaly what this referring to. Mean anomaly we are trying to measure from this θ is being converted in terms of represented in terms of mean anomaly which is purely a mathematical quantity. But if you remember in the integration the lower limit we have set to 0 and the upper limit to θ. If we do not set the lower limit to 0 then what happens .

So, not setting the lower limit to 0 but rather than to θ_0 this gives rise to from you are integrating it, so that gives rise to this term. So, we will return back to this particular issue and I will explain you in some more details. But you should remember that the just at this instance that the mean anomaly we have written in terms of t - T where t is the time of the perigee passage or what we call the this is the time of the perigee passage.

And this we have represented as mean anomaly, so here representing this T, this is important where the θ is already has a changing with time, for any orbit this will change, this is the true anomaly, so we are interested in this value. And I will take up this issue afterwards, right now we should not worry about this particular point. So, this is related to θ and these parameters I will represent as α_1, α_2, α_3

And similarly we will have the parameters Ω, ω and i and this will be represented by β_1, β_2, β_3.. And it depends on the sequence you change, it is not necessarily that i we represented by β_3 it can be represented by β_1. And if we use this, so we need to evaluate the Lagrange bracket α r, α s, α r then βs and βr, βs, these are the Lagrange brackets we need to evaluate.

And from where these are coming if this is the same thing as I have explained to you earlier cj, ck this is your Lagrange bracket. And then I also mentioned that I can break it in the 4 quadrants this bracket and write in this bracket then become the elements of this 6 by 6 matrix. So, this one the first one let us say this is the A, so A is referring to this part and this one is referring to B, so B is referring to this part.

And this becomes - B and this is C which is referred to herein this place. That means you have the a, e and T, so a, e and T here and similarly, you will have Ω, ω and i So, you can see that the A just consists of α R and α s, A just consists of α R and α s. That means it will range over this means only over a, e and T, so this is over a, e and T while this one if you lo for this particular quadrant.

So, if of the matrix, so here you will see that this is a mixture of a, e and T and Ω, ω and i, so this is therefore mix as α R and βs. Similarly this part this is Ω, ω, i and this is mixed with a, e and T. And already we have shown that this is related to these 2 are little related by a - sine, so B and -B, so these are the mixture. And then this finally C, so this is just Ω, ω, i, Ω, ω and i.

And therefore this is ranging over R and βs, so these are the Lagrange brackets we need to evaluate. So, rather than writing in terms of right now at this stage in terms of a, e etc we generalize it in terms of α_1, α_2, α_3, β_1, β_2, β_3. So, this is related to α and this is related to β, so this becomes α, α, so this is also α . So, this becomes α α related to this becomes related to α β.

And this will be related to βand this is related to have been α β but with minus sign. So, this strategy we are going to follow here.

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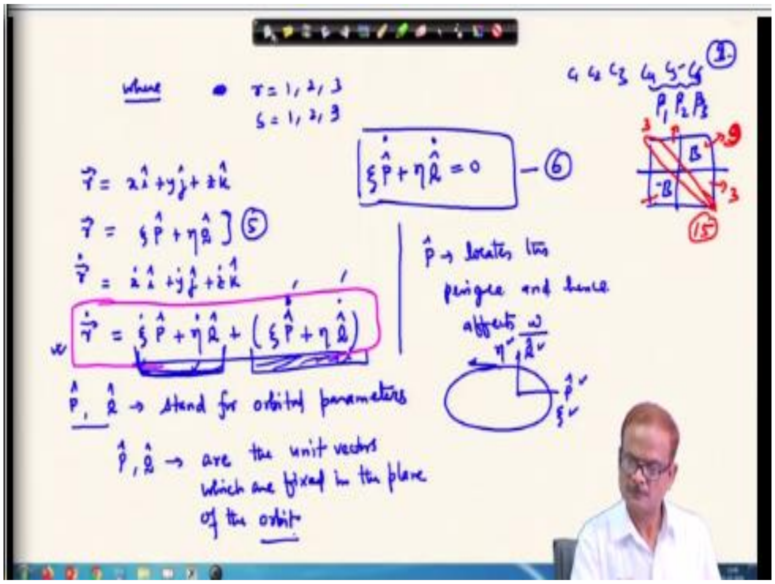

, where, so α, R and s, so here r equal to 1, 2 and 3, and s equal to 1, 2, and 3. So, what we have deduced instead of using the symbol that c_1, c_2, c_3, c_4, c_5, c_6. So, this we are replacing in terms of β, so this becomes β_1, β_2, β_3. So, that our indexing it runs only over 1, 2, 3 rather than running over 1 to 6. And it is a convenient also because in this Lagrange bracket matrix if we are forming.

So, here in this, this part this is B, so this is - B, so I do not have to evaluate this and thereafter also the diagonal elements are 0 , so we get rid of that. And only if the last time we have loed into that we need to evaluate sorry the 3 Lagrange brackets here in this place and 3 here in this place. And in this, we have to do 9 and here we do not have to do anything.

So, that means total we have to evaluate 15 Lagrange brackets this is the strategy we are going to follow. So, therefore our r becomes $x\hat{\imath}$, $y\hat{\jmath}$ and $\widehat{zk}$, this we can write as your r vector. Now going back to this place, so here is your r vector, this is your vector, so this r vector it can be represented in terms of x, y and z components in the initial frame or either in the orbital plane we can write in terms of η component this ξ and η component .

So, this also we can express as $\xi \times \hat{P}$ and $\eta \times \hat{Q}$, this becomes a planar case here in this case while we deal in this manner. And r dot this becomes $\dot{x}\ \hat{\imath}$, $\dot{y}\ \hat{\jmath}$ and $\dot{z}\ \ \hat{k}$. And this also we can write as

$$\dot{\vec{r}} = \dot{\xi}\ \ \hat{P} + \dot{\eta}\ \ \hat{Q} + \xi \times \dot{\hat{P}} + \eta\ \dot{\hat{Q}}$$

So, this equation we will name as equation number 5, now here we have to lo in this particular expression.

In the osculating orbit already we have discussed that r dot is representing the velocity. And in the osculating orbit before this we write this part $\hat{P}$ and $\hat{Q}$ they stand for orbital parameters. As this visual from you can look into $\hat{P}$ stands for orbital parameter. All the orbital parameters are involved except the small θ there you can see that $\hat{Q}$ is also involving ω **Ω** all these parameters.

And $\hat{R}$ also involves the orbital parameters . So, if $\hat{P}$ and $\hat{Q}$they involve orbital parameters which are taken as or simply we can say that $\hat{P}$, $\hat{Q}$are the unit vectors which are fixed in the plane of the orbit and therefore this quantity should vanish. So, we will have here

$$\xi \times \dot{\hat{P}}\ + \eta \times \dot{\hat{Q}}\ = 0.$$

This equation we will name as equation number 6.

, which is very much this is a visible that $\dot{R}$ is the velocity of the. In the osculating frame the r dot this velocity whatever we are taking, so it is lying in the orbital plane . And therefore this will be described only in terms of $\dot{\xi}$ and $\dot{\eta}$ not in terms of the variation of the P, $\dot{\hat{P}}$ and $\dot{\hat{Q}}$ will not come into picture . So, this is our equation number, so what exactly we are doing.

We are proceeding in the same way as we have done earlier, only difference is that we are now dealing with ξ and η . So, with this now we are ready to work further, one more thing we can write here $\hat{P}$ locates the perigee. I hope this part gets clear, this is important to note that here while we are working with only this term will feature in the velocity of the satellite in the orbit.

We are taking this plane, so in this plane we have here $\hat{P}$ and in this direction we have $\hat{Q}$. And in this direction we are writing ξ and this direction η, so this would appear the velocity in this orbit at any point. It should appear only as a function of $\dot{\xi}$ and $\dot{\eta}$ not as a function of P, $\dot{\hat{P}}$ and $\dot{\hat{Q}}$ and therefore this part must be 0 . And we will discuss this issue in further in the coming lecture .

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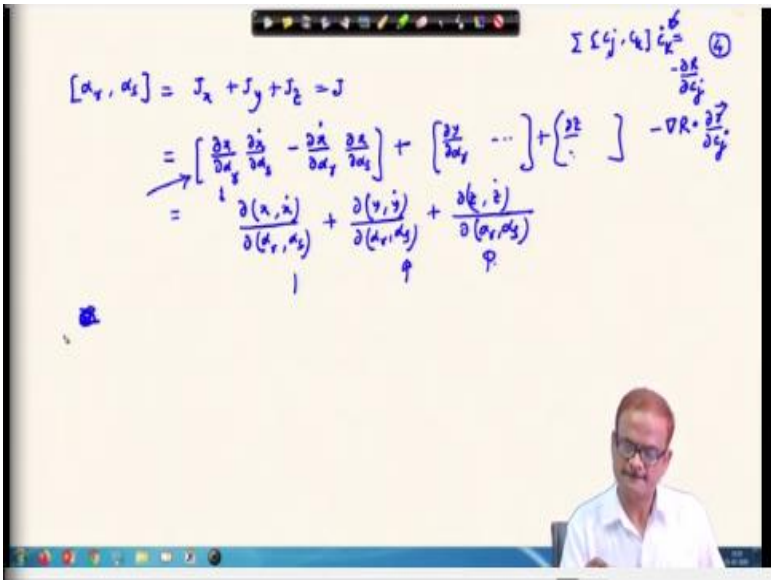

So, therefore α r and α s what we have written as the Lagrange bracket, we need to evaluate them . And what is our ultimate objective, ultimate objective if you remember that it was something c_j, c_k × $\dot{c_k}$. And on the right hand side we have - ∂R by ∂c_j this we got from writing ∇ R dot we had the other part ∂R by ∂c_j. So, our ultimate objective is to get this quantity $\dot{c_k}$ for this we have to solve .

And for solving for that we need to do so much work, so if actually if this parameter variation method if we lo mathematically this is the most complicated part and time taking also. So, therefore I will not be able to completely cover this in our lecture here. So, I will give you the short copy of the written material also to support the derivation in some places .

So, this part at least let us finish it quickly, so this we are writing as J x + J y + J z equal to J, this we have done earlier. And if you remember this we also write as ∂x by ∂in terms of cj, ck instead of writing this we are now writing α_r and α_s. So, we can write this as ∂x by $\partial\alpha_r$, $\partial\dot{x}$ by $\partial\alpha_s$ - ∂x by $\partial\dot{x}$ by $\partial\alpha_r$ × ∂x by $\partial\alpha_s$.

And in the same way we can write for ∂y for $\partial \alpha_r$, so on and for the z also we can write in the same way. So, this quantity in the shorter notation we can write this as ∂x, $\dot{x}$ and those who are familiar with the partial differential. So, you know well how to represent this and this is a determinant, so each of them it is a determinant. So, we have what we have we will go into the next lecture, we will because this becomes longer, so we stop here and we will go into the next one, thank you very much.