Space Flight Mechanics
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Lecture-56
General Orbit Perturbation Theory (Contd.)

Welcome to lecture 56. So, we have been discussing about the general theory of orbit perturbations, we will continue with that. But today before we are start working with the evaluation of the Lagrange bracket. So, before that we will review the concepts we have been discussing in the previous few lectures.

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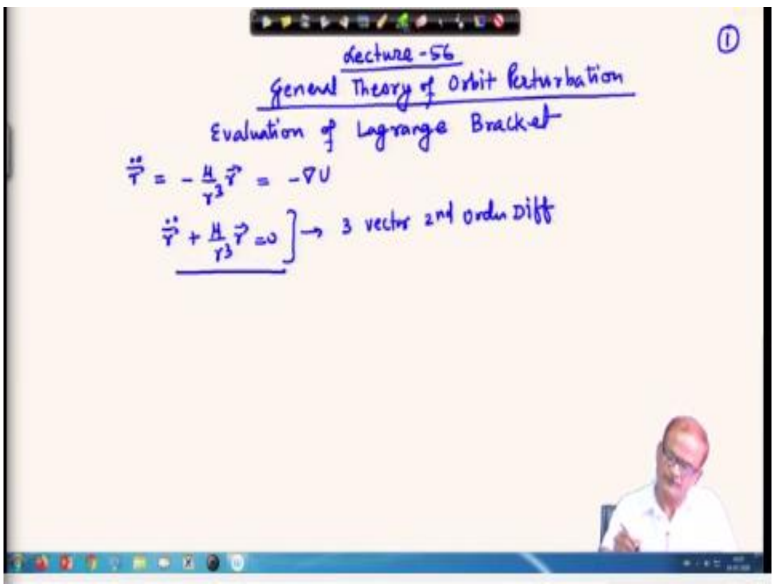

So, what we have done that orbit this moment of a heavenly body, about another heavenly body in the relative terms we have written it like this μ/r^3 r. And this also we have written in terms of - ∇U. Now if the solution to this problem μ/r^3 r equal to 0. So, because this consists of 3 vector second order or 3 scalar.

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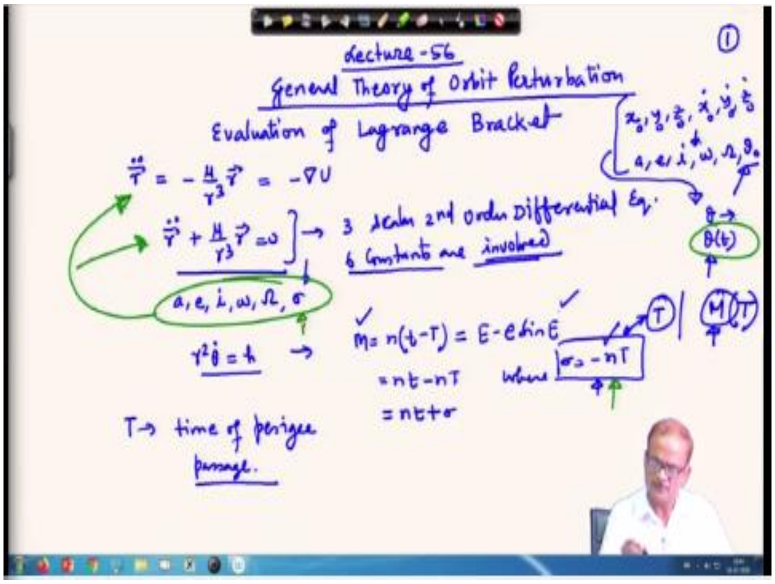

This is a vector equation, so 3 scalar, second order differential equation, so total 6 constants are involved in this. And already we have identified those constants a, e then i, ω and Ω. The another constant see we will make little difference here that we are given x, y, z and $\dot{x}$ $\dot{y}$ $\dot{z}$ at any instant of time say at t_0 which we indicate here putting the subscript 0.

So, corresponding orbital parameters you would say a, e, I, ω, Ω and θ_0 whatever the θ value at this is the corresponding orbital parameters. But here in this case once we are dealing that there are 6 constants involved. So, the last constant will identify with what we write as σ from where it is appearing. So if you look into this Kepler's equation we have written.

So, we solve this equation and while integrating, so we got it in the form M equal to nt - T equal to E - E sinE where E is the eccentric anomaly, this is the mean anomaly. And the quantity here nT minus this quantity we can write as nt + σ where σ equal to - nT. And T is called the time of perigee passage. So, rather than taking this θ as it was once you convert it into the orbital parameters, so you can see that the θ is varying with time .

If you take it for a particular instant then it is ok but if you take it for as a general variable. So, if θ is varying with time , you are taking note at a particular instant of time but in a general manner you are writing, so θ will keep varying with time, so this will not appear as a constant. So, rather than taking that which is written as σ equal to - nT, so this is taken as a constant.

So, either - nT or either T, so either of them can be chosen as the constant of integration . Sometimes instead of T or - nT also you would say usual to use M at a particular instant of time which is a mean anomaly at a particular instant of time.

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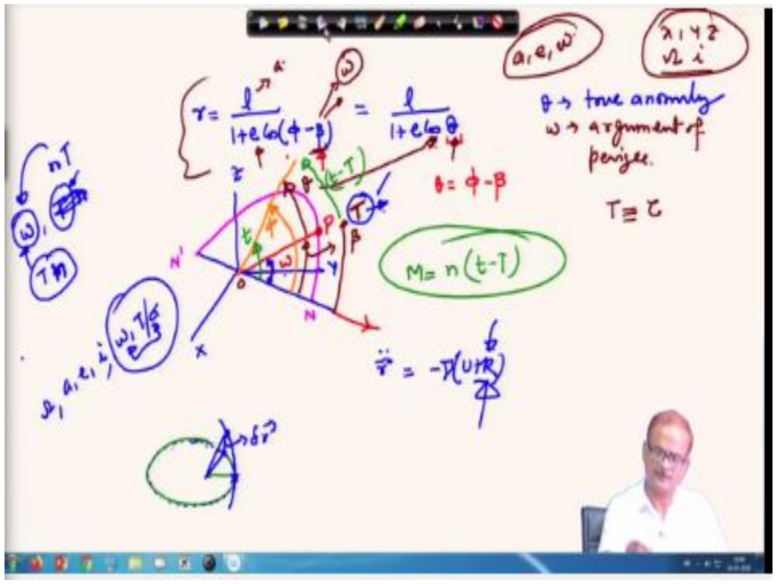

So, now if we look back and the equation we derived for the conic section e cos say if this was something like pi - β which we wrote as 1 + e cos θ, where θ we have written as the true anomaly. This is x, y, z the inertial frame and then we have the orbit here, this point was in M, N' and perigee position is lying somewhere here , so this angle we have written as omega.

So, what appears here in this place, so actually what we have written here θ, so here say we are getting θ as ∅- β. So, pi is being measured from certain reference, so say the pi if we measure from this reference, if we measure pi here in this direction , let us say this is pi . And then β will become this angle from here to here, so here in this case whatever the β is appearing it is nothing but your ω which is the argument of perigee.

But the argument of perigee, we have derived in some other way but the fact is that the β appearing here it was nothing but the argument of perigee it is a being referred to certain reference lines. So pi is being referred to the line N', O, N and then β is to displace, so therefore this angle is your θ which is shown here in this place. So, as a whole here how many parameters were involved in this A and E involved in the both A and E are involved here also E is there another parameter β.

So, in this equation itself you have 3 parameters involved which are a, e and a ω but we wrote it in a way that it was not visible . Now it becomes more purposeful to discuss all these things. So, here of course we have derived ω in some other way which we got in terms of x, y, z $\dot{x}$ $\dot{y}$ $\dot{z}$ etc. So, if x, y, z and then we had also this Ω and terms of i we have got it.

Now, the time of perigee passage means at this point this how much time it is taken to this place this we write as T sometimes it is also written as instead of writing this is also written as tau. So, either of them can be used, so time of perigee passage and from there then if the let me use another color time taken to this place to this place this we write as T . So, you can see that from here to here then this time is t - T.

So, t - T this is directly related to your mean anomaly by this equation. So, what we have been discussing here that instead of using θ because this is not appearing as a constant results are varying with time, so we use this σ equal to - nT as the constant of the orbit . And these 6 constant, they are nothing but the constant of integration in this equation or either here in this equation.

So, if this part is important here considering this σ because once we are considering the orbit perturbation. So, immediately we consider in the case of the orbit perturbation, if this is the osculating orbit and the true orbit is going like this. So, if the disturbance is not present your satellite maybe here. But if disturbance is present your satellite maybe located here in this place or either say if your satellite is here and we move it little bit from this place.

So, this is the delta r perturbation produced because of your the term we have written $\ddot{r} = -\nabla(U + R)$, so this was the perturbation potential. So, because of this, this is the perturbation produced in the orbit otherwise it would have followed the osculating orbit which in this case it becomes a Keplerian orbit if the disturbance is not present . So, if the disturbance is present, so this node also this will rotate .

Here in this case, you can see that what we have written here the this time of perigee passage we have indicated by your T this position this T will also change , remember ω and T they are not the same thing this is the time taken from this place to this place from the reference to this place which we are calling as the time of perigee passage.

And here this part is not angle ω. So, this angle and if you multiply this T by the mean anomaly which is we are writing as n, so this does not give you this ω . So, we have to be careful about this because the angular velocity is changing over only over a whole one period of time. So, from where in the stability and orbit from wherever it is a starting it will get back to the same place.

Now once the θ is varying, so whatever the angle here indicated this ω angle, it will not be covered in the same time. If you multiply it with the mean anomaly then what I mean that if you multiply mean angular rate which the time of perigee passage you are not going to get this. Otherwise, there is no meaning to writing this a, e, i, ω and T, T or either σ what we are using here and Ω.

If both are the same, so there is no meaning to writing that these are 2 different constants . So, this part is should remember and perhaps I have not discuss this till now but this is the right time to get into this and I have described this, this is exactly what the things are taking place. So, this T will also vary if the perturbation term this R is present , now we can start with the evaluation of the Lagrange bracket what I have written here.

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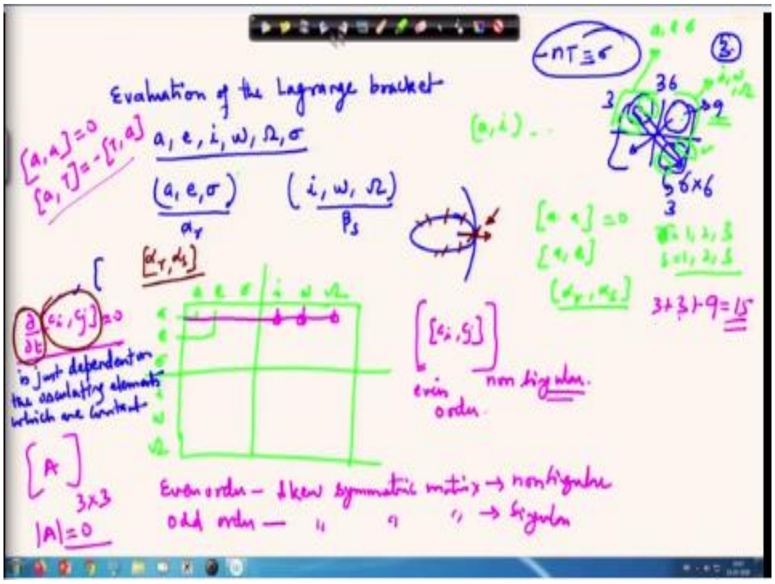

So, earlier as I have told you that we will divide it into the total all the elements a, e, i, ω, Ω and σ instead of this - nT we are writing this as the σ . So this can be divided into 3 parts and we can write it as a, e, a is σ and i, ω and Ω. So, these 2 groups we divided and this also we have discuss earlier this will indicate by αr and this by β s.

So, number of the Lagrange brackets already we have discussed that out of total 36 because this forms a Lagrange bracket element they form 6 into 6 matrix. So, out of that the diagonal one is 0 and thereafter here we have a total of 9. In this place we get a total of 3, in this place we get a total of 3 and here just with a opposite sign . Here also on this side as we have discussed we will get 3, so on the left side of the diagonal here it will be minus of this.

Similarly here in this place whatever the sign it comes with on this side this will be negative or otherwise. So, if then we divided it into 2 parts, so one part this part will refer to from here to here, this part is corresponding to a, e, σ and this part is corresponding to i, ω and Ω. So, you can see that Lagrange bracket then it can be written as a, a which of course is 0 according to the property of the Lagrange bracket we have written.

Similarly a, e, so these are just referring to your αr, αs , where αr varies over 1, 2, 3 and s also is 1, 2, 3. While the mixture one, this one where I have shown this 9, so that corresponds to terms like a, i etc. So, you can see here immediately if I right here a, e, σ and here i, ω, Ω, a, e σ, i, ω, Ω, so here corresponding elements are like this.

In this place similarly for related to this you will get a, i then a ω and then a σ the terms will appear like this. So total of 3 + 3 and plus here we are getting in this place 3 into 3, 9, so 9 + 3 trial + 3 total 15. Because of the skew symmetry here the another thing that you should notice that here the matrix form by this skew symmetric matrix , this is non singular .

So this matrix form by Lagrange bracket ci, cj this is non similar, non singular means it is a determinant is not 0. Although this is a skew symmetry this is because this is of even order . So even order is skew symmetric matrix let us say non singular while odd order skew symmetric matrix will be singular. That means if I have 3 to 3 skew symmetric matrix, so this is bound to be singular means if this is matrix A and it is a skew symmetric, so determinant of this will be 0.

So, we have to evaluate our purpose is to evaluate the Lagrange bracket. So using the first 2 properties that is like the a, a this bracket will be 0 . And also a T or a σ this will be equal to - T, a, so this was a second property , we have already written this. And the third property that we have written it was $\partial/\partial\partial t$ c_i, c_j this equal to 0. That means the Lagrange bracket if we are taking the time derivative of the Lagrange bracket, so this quantity is going to be 0 and this we have proved earlier.

So, if let me just rewrite that is a statement. So, this we have written as Lagrange bracket is just dependent on the osculating elements and not on it is derivatives which are constant. So, if we use this property that the osculating this Lagrange bracket it is a independent of this time. So, therefore we can utilize it to find the evaluate the Lagrange bracket say what I mean here in this place.

That if this quantity is independent of time, so that means it is a derivative either you that means this Lagrange bracket you either evaluate at this point or either at this point or this point, this point osculating orbit it is going to be the same. Because it is independent of time, it is not depending on time . That means I can also evaluate at this perigee location and this property we are going to use next to evaluate the Lagrange bracket corresponding to your αr αs .

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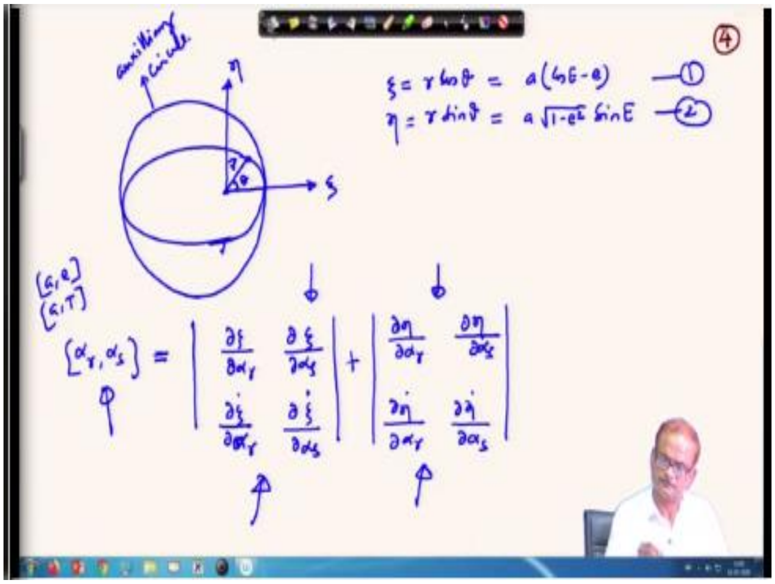

So with this we have, so this is our osculating ellipse and we draw the axis here, this is ξ and η, this is the focus and this is our auxiliary circle. If this point this is r and this angle is θ, so ξ we can write as r cos θ and η is r sinθ, the x and y component in the plane of the orbit. And from here the orbit will get perturbed and already the various relations we have derived in the case of the while we were discussing this Kepler's equation.

So, at that time we have derived all the relations, so I am not going to again write it here and simply I will write the result, this is

$$\xi = a \times (\cos E - e)$$

$$\eta = a \times (1 - e^2)^{0.5} \sin E$$

So, these 2 basic equations and in the Lagrange vector while we are looking for so there we have to the αr, αs this we have to evaluate. And there what we have looked into that this will be $\partial\xi/\partial\alpha r$ and $\partial\xi/\partial\alpha s$ $\partial\alpha r$.

So, we need to evaluate this Lagrange bracket this is a Lagrange bracket and we have expanded it, so we need to evaluate this determinants or the Jacobean. Now for evaluating each of them and where we are going to use we are going to use this in finding out the quantities like a, e, a, T etc.

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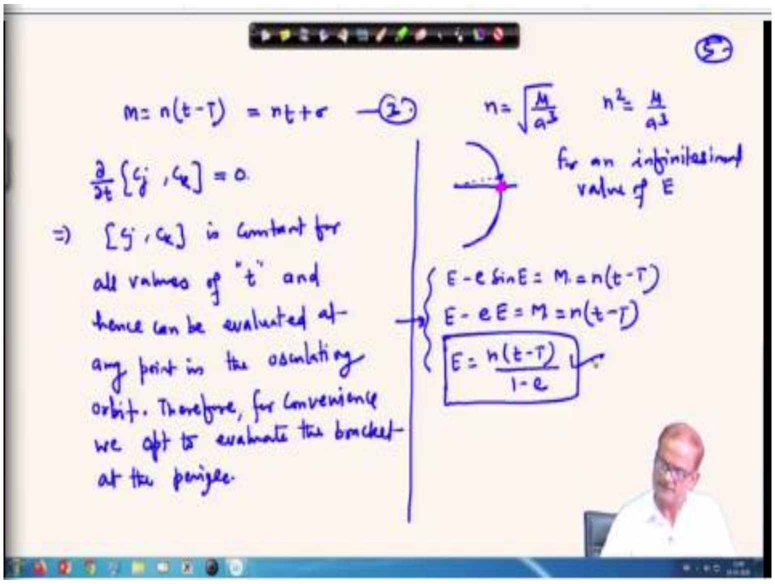

And also we have

$$M = n(t - T) = nt + \sigma$$

as we have written earlier. So, this is our third equation and n =(μ/a^3)$^{0.5}$ under root or n^2 equal to μ/a^3. So, these are some of the information we can use to solve this problem. So $\partial/\partial t$ c_j, c_k or c_i, c_j whatever the term we have used, so this quantity we have written perhaps in terms of c_j and c_k, so we will continue with writing this.

This quantity is going to 0 and what does this mean, this implies that c_j, c_k is constant for all values of t. And hence can be evaluated at any point in the osculating orbit, therefore for convenience we opt to evaluate the bracket, Lagrange bracket at the perigee. So, for this evaluation the about the perigee point we consider an infinitesimal value or E. So for an infinitesimal value or E, we consider this.

So, what does this mean that say here we want to evaluate at perigee, so we just take a near/ location which is infinitesimally close to this and expand our equation which are given there. So, for E - e sine this equal to M and if we expand it this can be simply written as E - e sinE equal to M equal to n t - T. And therefore from this place

$$E = n \times (t - T)/(1 - e)$$

This is the whole process we are trying to evaluate at this point.

So, we take a nearby point, expand it, so basically we have linearized it and then try to evaluate the Lagrange bracket at this point.

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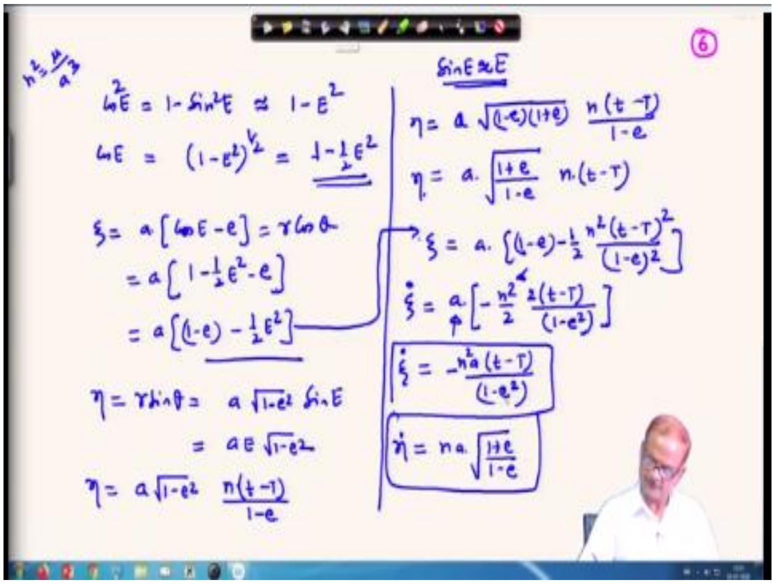

Now for the cos E write as 1 – $\sin^2$ E $\cos^2$E equal to and this becomes then 1 - E^2, this is approximately. Because sinE for a small value of a this will be equal to E and therefore cos E will be equal to(1 - E^2)$^{0.5}$ which we can approximately write as 1 - 1/2 a^2 which in binomial expansion . So, if we utilize this information in our work. So, if this is the equation for the ξ we have written.

That becomes a times cos E - e, so the cos E is then 1 - 1/2 E^2- e which we can write as. So, I will work 1 or 2 Lagrange bracket all of them it is not possible to do here in this lecture because it is everyone is very time taking. And obviously I am not going to derive all the equations for that I will give you some supplementary material which can be used for looking into those derivations.

I will after covering the basics I will be writing the results and the derivation of those results you will find in the supplementary material provided. Similarly the η then becomes this is equal to this is r cos θ, r sinθ is equal to a times 1 - e^2 and sinE we have written, the sinE equal to E so this becomes aE - 1 - e^2. So η then can be written as E already we have written, E we have written here somewhere in this place.

So, you will use this equation to an insert here in this place, so η we can write as this was n t - T/(1 – e), a times 1 - a times 1 + e and t - T/. Similarly this ξ this can also be written in terms of n and T, now this will become a 1 - e and - 1/2 n^2t - T^2/ 1 - e^2. And now we are ready to derive this $\dot{\eta}$ and $\dot{\xi}$.

So, first we work the $\dot{\xi}$ term, $\dot{\xi}$will be a times if we differentiate this, so this quantity we are going to get this as. Now n^2 term is there, so n^2 already we have written as μ/a^3. So, once we differentiate, so this gives as a is there and then - $n^2/2$ times 2 t - T/1 - e^2, - a and

$$\dot{\xi} = -n^2 a\,(t - T)/(1 - e^2)$$

$$\dot{\eta} = na \times \{(1 + e)/(1 - e)\}^{0.5}$$

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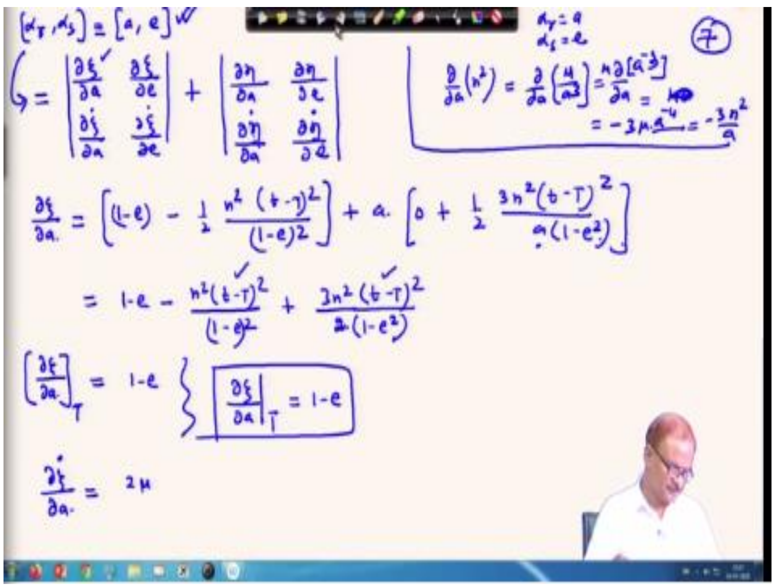

So, this way we have got this terms, now we need to evaluate the terms appearing in the Lagrange bracket. So, there we have the first determinant ∂ξ by let us say that we are looking for αr, αs. So, αr we replaced by a and αs replaced by e and then what will be the value of this Lagrange bracket this is what we are looking for. That means we have written αr equal to a and αs equal to e.

So, we need to replace this αr/a and αs/e, so here then the derivative becomes $\partial\xi/\partial a$ $\partial\xi/\partial e$. And then $\partial\dot{\xi}/\partial a$ and $\dot{\xi}/\partial e$ this we need to evaluate and the other bracket. So, this quantity ∂η/∂a ∂η/∂e, $\partial\dot{\xi}/\partial a$ and $\partial\dot{\xi}$ sorry this is $\dot{\eta}$ ∂e, this is the quantity we have to find out.

So, each of them then we need to evaluate, so $\partial\xi/\partial a$, now ξ equation already we have written. So, directly from there a is appearing, so immediately we can see this can be written as $(1/2\ n^2\ t - T)^2/(1 - e^2)$. This is the first term a was appearing here in this place where this place. So, this we have differentiated, next n is also a function of a, so therefore we need to differentiate that 2 .

So, the next term will be a times 0 + 1/2, 3 n^2 t - T whole square and/a × $(1 - e^2)$. That means this is $\partial/\partial a$, μ we can take it outside a to the power - 3 and this will turn out to be μ times a to the - 3 μ× a to the power - 4. And this is nothing but then you can write as - 3 n^2/a because μ/a^3 is n^2. So this is the thing we have used here in this place and this can be reduced finally to $(1 - e - n^2\ t – T)^2/(1 – e)^2$.

And then + 3 $n^2/(2\ t – T)^2$ and 1 - e^2. So, this - sign which was appearing here dot got plus and a is appearing here, so this a, will cancel out, so this is what we get. And if we try to evaluate at the perigee, that means we are evaluating at the point where t equal to T. So, at that point we get this as (1 – e), this quantity will be 0, this quantity will be 0, so, we will get this as (1 - e).

So, basically this term we write as

$$(\partial\xi/\partial a)|_T = (1 - e)$$

So, this way we have to evaluate all the terms. Similarly, we have $\partial\dot{\xi}/\partial a$ this equal to 2 μ as we have written here $\partial\dot{\xi}$, $\dot{\xi}$ is here. So, once we differentiate this quantity $\dot{\xi}$ a is there and n^2 is also present. So, differentiating this let me copy it here in this place n^2 - n^2 a maybe this time is over. Otherwise lecture will get longer, so we will continue this in the next lecture, thank you very much.