Space Flight Mechanics
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Lecture-57
General Orbit Perturbation Theory (Contd.)

Welcome to lecture number 57, so we have been trying to evaluate the Lagrange bracket. So, in that context we wrote the equation for the ξ, $\dot{\xi}$, η and $\dot{\eta}$.

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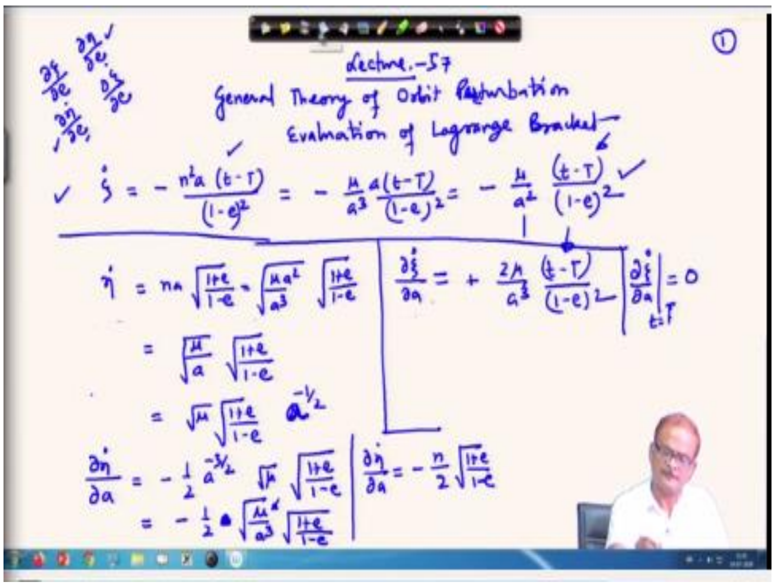

So, we have the equation for the ξ I have kept it ready for you, so $\dot{\xi}$ we have written like this and this can be reduced here in this format. So, finally we get $\dot{\xi}$in this format and we are looking for $\partial\dot{\xi}/\partial$a. So, we just differentiate this quantity, this is not dependent on a. So, if therefore once we differentiate it this gets a + sign and this is

$$\partial\dot{\xi}/\partial a= 2\ \mu/a^3 \times(t - T)/(1 - e^2)$$

And once we evaluate this at the point say if $\partial\dot{\xi}/\partial$a if we evaluate at T, t = T. So, once we evaluate immediately you can see that this quantity will be 0 because of presence of this term. Similarly we have the $\dot{\eta}$ here given, so $\partial\dot{\eta}/\partial$a this quantity, here this is a

$$\partial\dot{\eta}/ \partial a = - 1 /2 \times a^{-3/2}\times(\mu \times(1 + e) /(1 - e))^{1/2}$$

So, this gets reduced to the format - 1 /2 a to the power r this will come in the denominator with a to the power 3 /2. So, we can take it inside and write it as $\mu a^{3 \times}$ (1 + e)/(1 – e). And this is nothing but then this quantity $\partial\dot{\eta}/\partial a$ this will get reduced to - n /2, this quantity is nothing but n $^{\times}$(1 + e)/(1 – e). So, there is not T appearing, so this way this case is simplified .

So, this way we will be able to evaluate all the terms like we have to evaluate now $\partial\xi/\partial e$, $\partial\eta/\partial e$. And also we need to evaluate $\partial\dot{\eta}/ \partial e$ and $\partial\dot{\xi}/ \partial e$. This is for the second bracket we require this quantity η because there we are working with respect to see here a, e a, e we have done.

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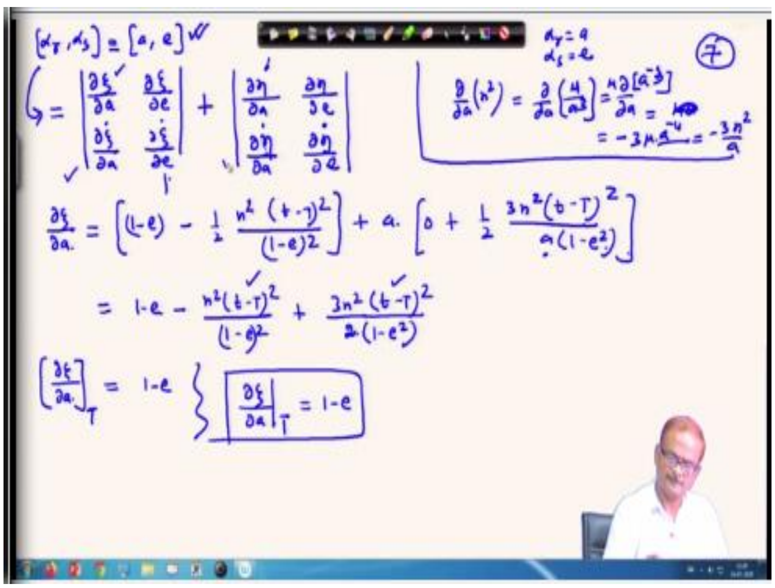

So here η is appearing with respect to E, so this we have to supply and also here in this place we have to supply the term .

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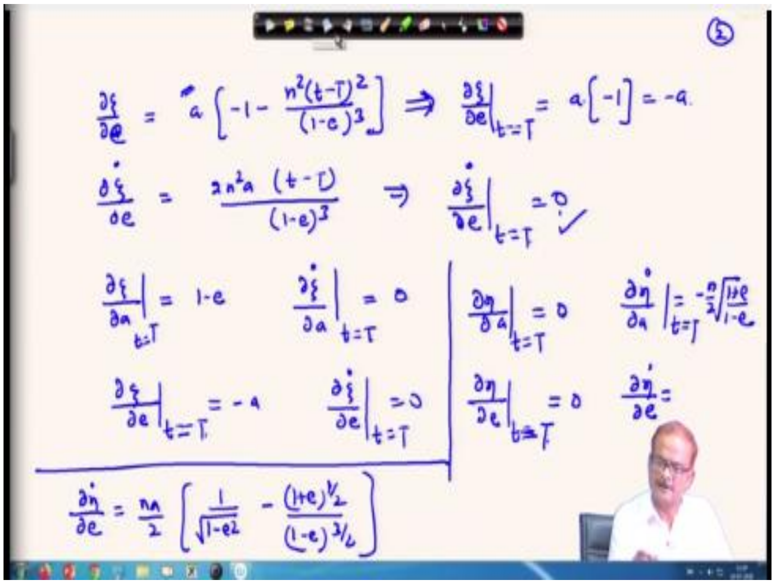

So the other terms, we can evaluate following the same trend

$$\partial\xi/\partial e = -1 - n^{2}(t - T)^{2}/(1 - e)^{3}$$

Similarly

$$\partial\dot{\xi}/\partial e = 2n^{2}a(t - T)/(1 - e)^{3}$$

As the evaluation we are doing at this implies that $\partial\xi/\partial e$ at t = T that is at the perigee this quantity gets reduced to a - 1, this is - a.

Similarly from here we get $\partial\dot{\xi}/\partial e$ at t = T = 0 . So, this way you will be able to evaluate those quantities. So I summarizes the results here at t = T this quantity is(1 – e) $\partial\xi/\partial e$ at t = T this = also 0, t = T from here this is – a, t = T from this place this is 0. So, this way we can complete this all the terms which are required similarly for $\partial\eta/\partial a$ at t = T, this will be = 0.

This will be = - n /2,×(1 + e) /(1 – e) = t this turns out to be 0 $\partial\dot{\eta}/\partial e$. So, this quantity we have to evaluate and the following the same scheme you can evaluate it and this will turn out to be we can write it here in this place . Once all these things are available, so it remains to evaluate the Lagrange bracket.

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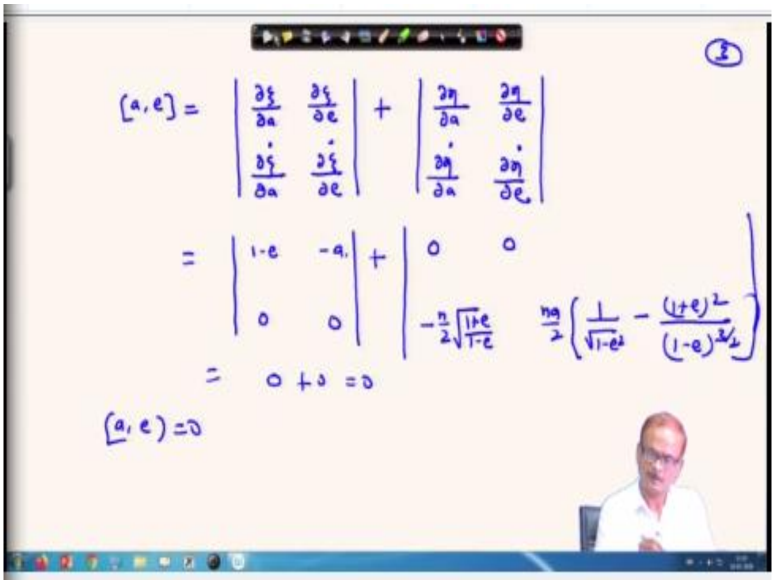

And therefore the first thing we are looking for this is a, so this quantity how much this will be a, e is $\partial\xi/\partial a$ and $\partial\xi/\partial e$ $\partial\dot{\xi}/\partial a$ $\partial\dot{\xi}$ taught/∂e and plus we have written here $\partial\eta/\partial a$. Now insert all the values that we have worked out, so $\partial\xi/\partial a$ this quantity is $1 - e$ $\partial\xi/\partial e$ we have written as - a worked out and this is 0 and this quantity is also 0 and + $\partial\eta/\partial a$.

This quantity was 0 $\partial\eta/\partial e$ this quantity also be found out to be a 0, $\partial\eta/\partial e$ we have . And finally for this quantity $\partial\dot{\eta}/\partial a$ we have got as $- n/2^{\times}(1 + e)/(1 - e)$ and $\partial\eta/\partial e$, this we have got as na /2. See immediately it is a visual we need not write so much for working out this quantities 3 /2. So, what we observe that this quantity is 0 0, so this turns out to be 0, so a, e this quantity is 0 .

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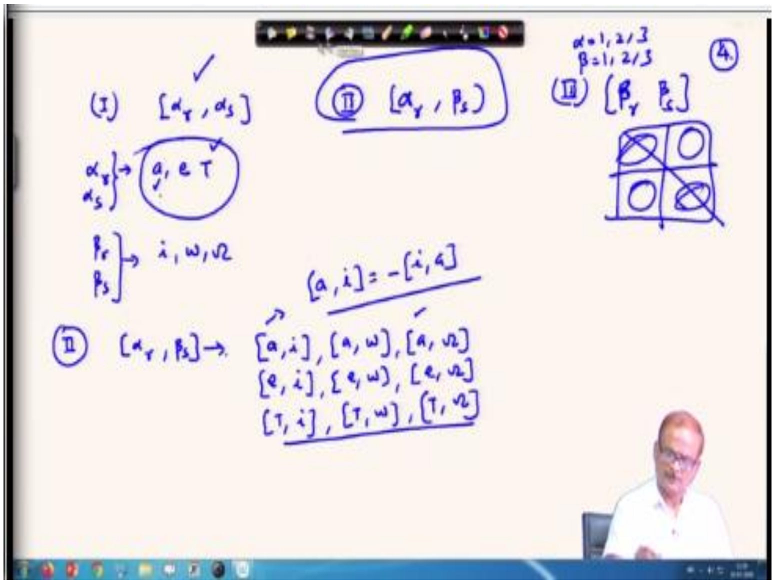

So, this way we can evaluate all the terms, now we divided our Lagrange bracket in 3 types. The first type we wrote as α_r and α_s, the second type we wrote as α_r, β_s and the third type was β_r and β_s where α_r and α_s they are wearing over 1, 2, 3 and β also from 1 to 3. And this I have explained earlier, why we are not writing 1 to 6 but rather using 1 to 3 this is for the ease of processing.

So, our α_r, this is concerned with a α_r or α_s, this is concerned with a, e and T . And β_r, β_s it is related to i ω and Ω. So, and using them we can then form this Lagrange brackets. So, from here if we look into the let us first lo into the second bracket. So, this is the second type, so here α_r, β_s that means we have to form the brackets like what we are looking for.

So, here αif we take from say here a and then from the β we choose, so this will be a, i then a, ω and a, Ω. Similarly we choose the second one e, so then we will have e, i, e, ω and e, Ω similarly we choose the T. So, we will have T, I, T, ω and T, Ω. So, these are the 9 brackets which are corresponding to the off diagonal terms which as I have explained you for this one or either this one.

So, opposite of this strength, if we need here we already know from the Lagrange bracket property that this quantity will be equal to i, a. So, for the other things we need not do, so overall we have to evaluate this 9 bracket here in this place, 3 from this place and 3 from this place, total 15 we have to evaluate. So, out of that already we have looked into this is a, e is a, e, a, e is belonging to this group , it is over the cell over the first row. So, that we have computed to be 0 but first we completed this type 2 and thereafter the rest will follow.

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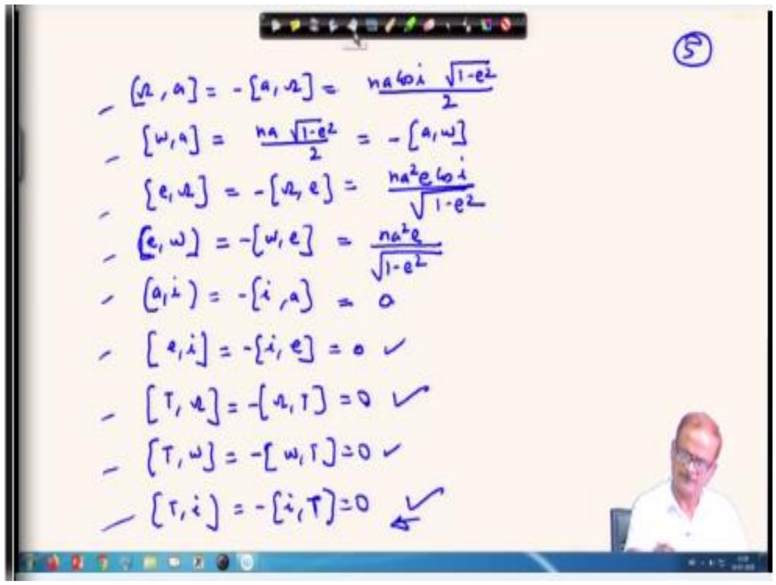

So, for this exercise for can be done we cannot do so many things here in this place. So, Ω, a this = - a × Ω, so, this quantity will be

$$[\Omega, a] = na \cos i \,(1 - e^{2})^{0.5} /2$$

So, four of them let us say listed here, then we have a, i = - i, a this quantity will be 0, then e, i this quantity also = 0. So, how many of you have 1, 2, 3, 4, 5, 6, 7, 8, 9, so 9 are listed here in this place .

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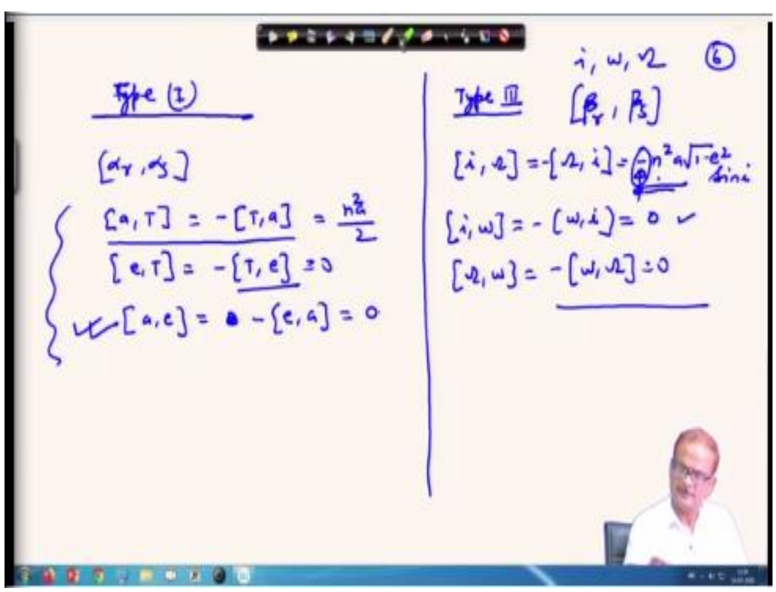

From the type 1 then we have αr, αs type, so here in this case we will have a, t = - T, a, already this part we have evaluated. So, this way is the type 1 is listed here, so you can see that only 3 are here , we need to evaluate only 3 the rest 3 are just with opposite sign and type 3 which is β_r, β_s and this is related to i, ω, Ω. So, these brackets can be written as i $^{\times}$ Ω minus.

So, when number of brackets they are turn out to be 0, so our case is simplified a lot . Thereafter we can once we have evaluated this Lagrange bracket, now we can set to solve for the Lagrange equation that we have written.

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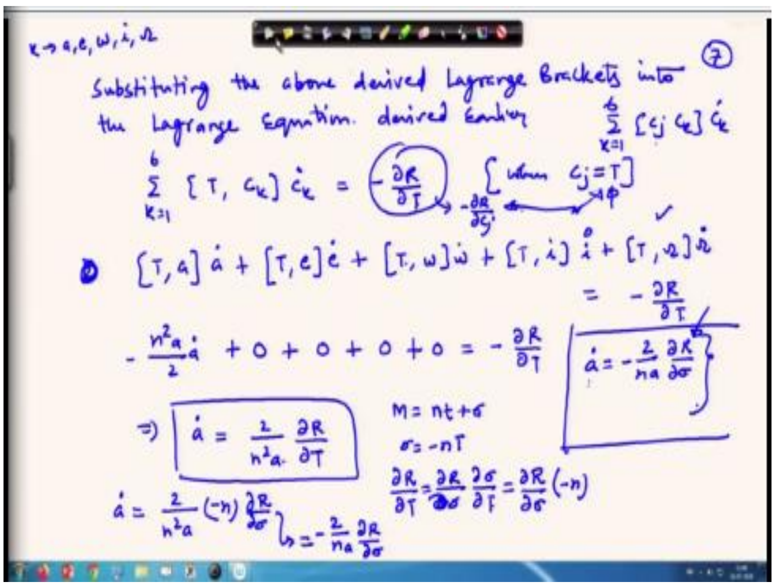

So, substituting brackets into the Lagrange equation we have derived earlier c_j or c_i whatever we have written earlier. So, in this case this is T, actually this was like c_j and c_k $^{\times}$ $\dot{c}_k$ and summation goes over k = 1 to 6. So, c_j we choose as T and then k varies from 1 to 6. So, if we expand it immediately we can see from this place this gets reduced to T and here rest of the things we have to follow.

So, c_k the first one let us say this is a $^{\times}$ $\dot{a}$ + T, e $^{\times}$ $\dot{e}$, then we have a total of the other things are the cell 1 s_0. So, that we obviously we are not counting while here just /putting and then varying it from k from the only thing that T here will be spare. So, this k will which present from a, e ω, i and Ω, so this 5 will be used here.

So, the other one will be T ω and $\dot{\omega}$ + T, i × $\dot{i}$ and then t Ω × $\dot{\Omega}$ and on the right hand side then this is - ∂r/∂T, remember this term was actually - ∂r/∂c_j. So, c_j we are replacing each year by T and therefore this is coming here in this format. Now all the Lagrange brackets we are aware of, so from the previous one we can go and lo into T, a how much this is.

So T, a this quantity is $n^2 a/2 \times \dot{a}$ T, a, T, a this is T, a with and that will come with a - sign. So, here is the - sign will appear, then + t, so T, e this is 0. So, this is set to 0 the whole thing will then get reduced to 0, t ω, so t ω maybe on the previous page T, ω is here, this is also 0. So, this quantity also turns out to be 0, then T, i, T, i is here in this place this is also 0, T, i is 0.

And then finally this term T, Ω, so we will lo for T, Ω this is also 0. So, this is - ∂r/∂T and from there then we can write

$$\dot{a} = 2/n^2 a \times \partial R/\partial T$$

And the same thing if we express in terms of σ there will be a little bit of change. So we have M = nt + σ we have written and where σ = - n T.

So, immediately we can see from this place that if we are trying to write in terms of ∂R/∂T, so this will be ∂R/∂σ × ∂σ/∂T, so this quantity ∂R/∂σ. And ∂σ/∂T from this place this place this will be - n therefore in this equation $\dot{a}$ we can replace T in terms of σ if we replace it, so, this is - n × ∂R/∂σ and this gives as

$$\dot{a} = -2/na \times \partial R/\partial \sigma.$$

So, if writing the equation in terms of σ, what we have got the result $\dot{a}$ = - 2 /na ∂R/∂σ, so this is the first step we have achieved. Now, we have to express R in terms of all these parameters and thereafter we have to take the derivative with respect to this σ and then only we are getting we will get this $\dot{a}$. So, $\dot{a}$ along with the other terms, so it forms the; what we call as the planetary equation.

The Lagrange planetary equation it was derived by Lagrange, so what is the limitation here that R we have to express in terms of potential and every time we cannot express in terms of potential.

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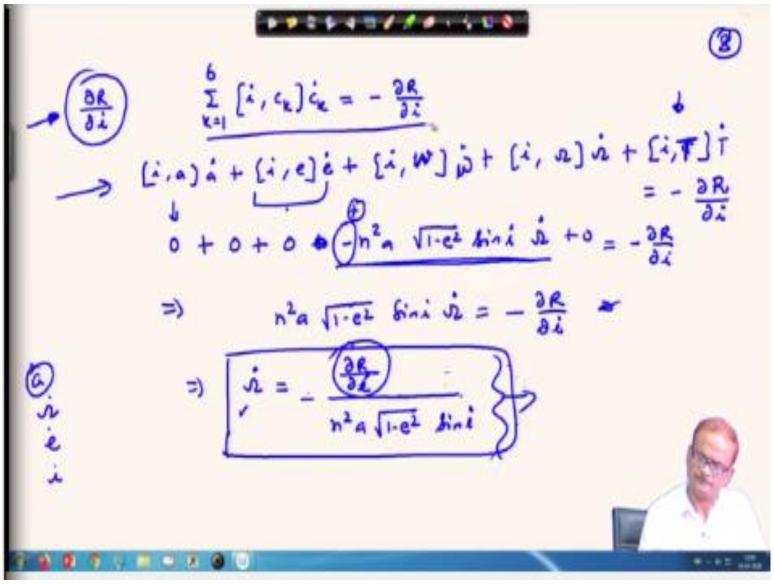

And therefore some of the times like once we have the aerodynamic like things available. So at that time we rather than expressing in terms of potential which is R here, we express in terms of the force . Similarly we will have ∂R for calculating working with the ∂R/∂R i term. So we will have here we will write i and here c_k, $\dot{c_k}$ summation over k = 1 to 6 and this = - ∂R/∂i.

In expanded, expanding in the same way we can write it i, a, $\dot{a}$ + all of them I am not going to do here rest in the supplementary material you can lo or either you can derive it yourself also. Then a ω × $\dot{\omega}$ i σ or i, T × $\dot{T}$ and this = - ∂R/∂i and now insert all these values. So, i, a this value is 0, i, i, a is we go back and lo here into this i, e this is also 0, i ω, i ω term is i, ω is in the same group.

So, we have to look somewhere else i ω we have written here this is also 0 i, Ω also here let us say coming here - n^2 a then this is - n^2 a and this is - n^2 a (1 - e^2)sini then $\dot{\Omega}$ and then i, t T, i, T term where in the first type i, T is 0, **.** We worked with T so there also the T, i term has appeared and here also the T, i or i, T has appeared.

$$\dot{\Omega} = - \partial R/\partial i /\{n^2 a (1 - e^2)^{0.5} \sin i\}$$

So, well this term is this is 0 T, i term and therefore we put here 0 and - ∂R/∂i. And this implies - of n^2 a, e^2 - +1 - e^2. We have to divide it $\dot{\omega}$ = ∂R/∂i /n^2 a, this is $\dot{\omega}$. So, this way you can get all the derivative terms related to a, Ω, $\dot{a}$, $\dot{\Omega}$, $\dot{e}$, i dot but some other terms may be combined together which will involve.

And now here you can see that only $\partial R/\partial i$ is appearing, in some of them they will be mixing up. So, if they are mixing up, so whatever way if sometimes you will see that here only one term fortunately only one term is appearing but suppose this term was also present. If this term were present, so in that case what we have to do that decision mixture of $\dot{e}$ then and $\dot{\Omega}$, s then we have to separate it out the equation.

So, this exercise I have done in the supplementary material, so all of these things I am not going to write here. Those equations I will not solve means wherever the mixing has taken place that I will not solve because it is a merely a solving exercise for solving some linear equation in Ω $\dot{a}$ i, a etc and that can be done very easily by you. So, I will left this all those things as an exercise .

So, this way we have got here $\partial R/\partial i$ and here we have i, ω we have used as i, Ω = this appearing with, let me check the sign, i, Ω this appears with minus or plus sign we will check it once. Otherwise I will have to work out this whole thing . So, finally, so whatever the procedure we are following this is the procedure of working with this and there is nothing wrong in that.

Only thing that sometimes the sign may get dropped out in one or the other place and whenever that happens I will always supply those terms. So, Ω here perhaps this would come with a plus sign in the next class I will verify it . So, once we do this, so if then the let me write here this would be with a plus sign perhaps this is I am going to verify in the next class. So, in that case then there will be a minus sign here in this place.

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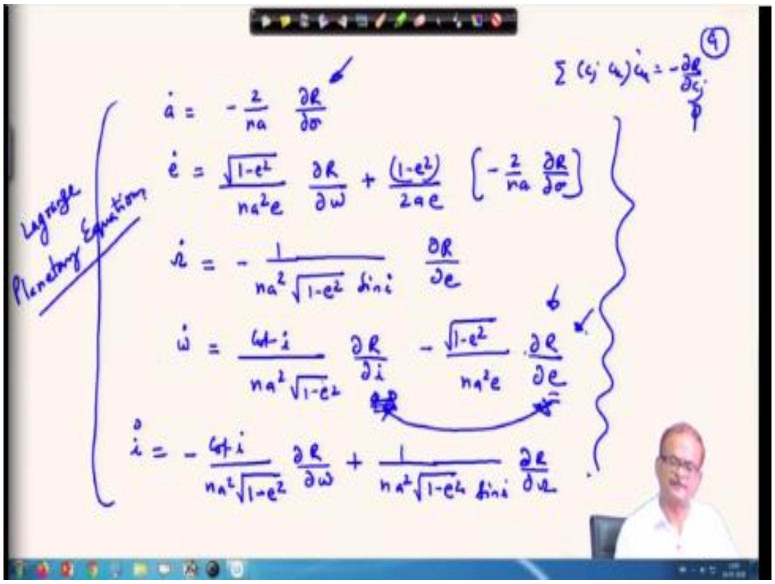

So, this way we can conclude that $\dot{a}$ = - 2, the same process we have to do with all the terms, e dot will be =(1 - e ^{2})$^{0.5}$. I am summarizing the results, so all the things I have not written here cot i /na ^{2} (1 - e ^{2}) ∂R/∂i. You can see that the terms are mixing up, so you have to obviously earlier some of the terms were mixed up by if this is the solved part I have written here Ω and $\dot{\omega}$ say they were mixed up.

So, those are separated out and written separately only then the terms are mixing up here. Because on the right hand side in this equation of the Lagrange bracket, c_j, c_k $^{\times}$ $\dot{c_k}$ - ∂R/∂c_j is there. So, only one e will appear, so that is simply implies that somewhere the ω Ω they were mixing up and from there we have solved it. And therefore this has come here in this format i .

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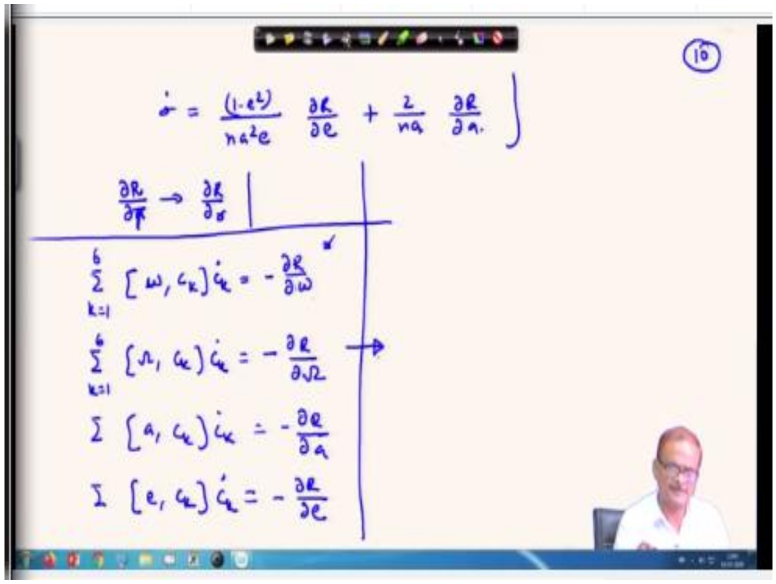

So, this way how many equations we have written 4, 5 year and 1, 6, so total 6 equation and this together they are called Lagrange planetary equations. And these need to be solved but it can be solved only if you note the R in terms of these parameters . Once you inserted then these equations can be integrated and it can be solved. So, the way of general perturbation equation it is valued over a small span of time not very long period of time.

Using that concept the Lagrange did it , he was a great mathematician, so he did it and now if we use this equation, so it will be valid for a short time not very long time. So, the general perturbation method deals with simplifying the equation of motion such that it is a valid over a short period of time and also then it can be integrated to get the solution, the general solution and this is called a general perturbation general orbit perturbation theory.

Otherwise the other term which I have used earlier, this is a special perturbation, so that deals with the numerical integration. So, there if you have to get the what will be the future value of a and all other things. So, then you need to integrate the equation of motion directly integrate it and from there whatever the value of x, y, z, $\dot{x}$,$\dot{y}$, $\dot{z}$ you are getting, that you can utilize to get these parameters at different instant of time , this is the difference.

But there how to integrate those equations that it involves a lot of a number of techniques and for very precise propagation where you are concerned with sending your satellite from one planet to another planet. So, you need to be very precise, so in that case especially the challenge is there, that if your system is your integration is not precise. So, you are thinking that your satellite has reached to this point but it is not at that point , there the problem will be occurring .

So, rest of the things it follows the same step what I have done here, only thing here this sign we need to verify like here as I have told you that this may be plus sign. I might have done a mistake in writing the things, so here on this side this can be a minus sign. So, Ω is = - $\partial R/\partial i$ this quantity and in the next class I will verify all these things.

Now the thing is that, once we have done this the same way all other terms can be evaluated like the terms related to what we have done right till if that extent is ∂R by the first term/∂a perhaps this first $\partial R/\partial T$ first was which we have written in terms of σ. This we converted into $\partial R/\partial \sigma$ and thereafter we have taken $\partial R/\partial i$. So, similarly you can use k = 1 to 6 Lagrange bracket say if then i we have used right now.

Yes, i we have use so ω let us say, ω and c_k, $\dot{c_k}$= - $\partial R/\partial c_k$. So, we need to insert in all the places , similarly we will have Ω c_k $^{\times}$ $\dot{c_k}$ c_k this - $\partial R/\partial c_k$ sorry this is c_j here in this place. So, c_j this is this will be replaced by ω and this will be replaced by ω here. Because this term is c_j not c_k, so this is $\partial\Omega$ as we have written here k = 1 to 6.

Similarly we will have ω T we have already used 5 1, 2 and 2 to 4 we have used, so a c_k, $\dot{c_k}$ is =- $\partial R/\partial a$. And then e c_k $\dot{c_k}$= - $\partial R/\partial e$, so this way to already we have worked out and this 4 needs to be done. Once you do the same process what I have done here, so you get these equations and these are our Lagrange planetary equation which needs to be solved in order to get how the orbit is getting for perturbed . So, we stop here and then we will continue in the next lecture, thank you very much.