Space Flight Mechanics
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Lecture-58
General Orbit Perturbation Theory (Contd.)

Okay, we have looked into the gentle perturbation theory and evaluated the Lagrange bracket and then we got the Lagrange planetary equation. So, in that case, the perturbing potential was given which we have written as r. Now, we consider the case where instead of perturbing potential which is a perturbing force is given. So, this will be much more generalized because, if you have the drag and other cases, you have the potential is not available. So, in that case you can utilize this equation.

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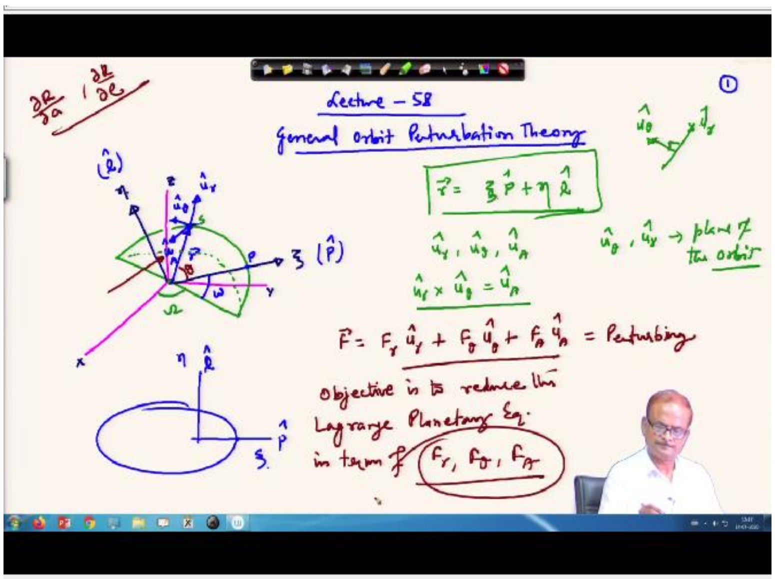

So, let us start with and so whatever the green one is shown here with green line, this is our orbit, projection of this orbit is shown by dotted line in the x y plane, x y z is the initial print, which we are using and as you remember that we have written the orbit in terms of r we have written in terms of ξ $\hat{P}$ and $\eta\hat{Q}$. So, this way we were just working with 2 terms ξ and η, so, in the plane of the orbit.

$$\vec{r} = \xi\,\hat{P} + \eta\hat{Q}$$

Now, if we consider the vectors $\hat{u}_r$, $\hat{u}_\theta$ and $\hat{u}_A$,

$$\text{Here, } \hat{u}_r\ X\ \hat{u}_\theta = \hat{u}_A$$

These are the unit vectors, $\hat{u}_A$ and $\hat{u}_r$ they are the usual definition and they lie in the plane of the orbit r direction and this is r direction and perpendicular to this, this is the θ direction and here S is the corresponding satellite. This is Ω, ω, now ω will, what we are doing that once we describe it in this way that see we are looking into the orbit here in this way.

So, along the periapsis we have the $\hat{P}$ and ξ is along this direction, η is along this direction and $\hat{Q}$ is the unit vector along this direction. The same thing is appearing here. So, this point we can write as here P maybe P we can write here. This is the periapsis position and then from here to here of course, then this becomes theta. So, with all these informations the Lagrange planetary equation we have written.

Now we want to express in terms of the force. So, what are those forces. So, this F we can write as F_r times $\hat{u}_r$ component of the force along the u_r direction, then F_θ times $\hat{u}_\theta$ and then F_A times $\hat{u}_A$. There $\hat{u}_A$ is perpendicular to the orbit which is shown here in this place. So, this is perpendicular to the orbit. Now, we have to work out this particular one. So, F is this is the perturbing force.

So, what we are interested in. So, objective is to reduce the Lagrange planetary equation in terms of F_r, F_θ and F_A. So, that implies that the terms like $\frac{\partial R}{\partial a}, \frac{\partial R}{\partial e}$ we need to replace in terms of F_r, F_θ and F_A.

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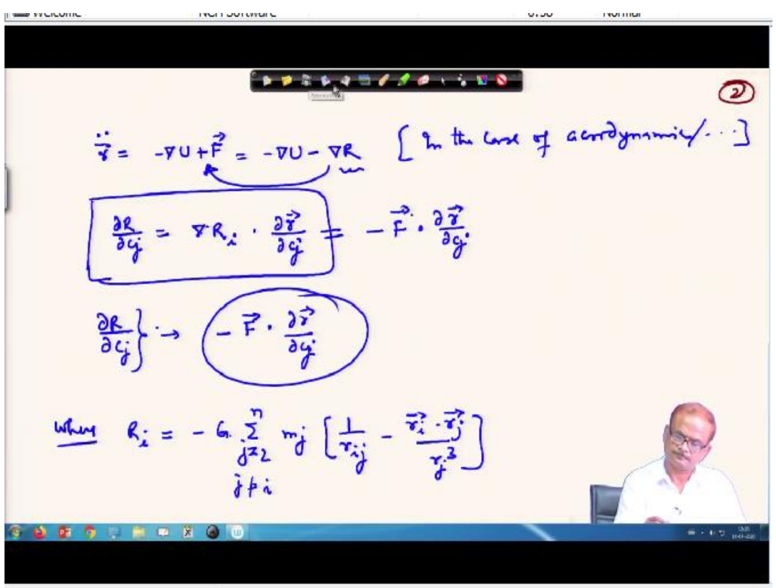

So, instead of writing this as $-\nabla U - \nabla R$ we are writing here in this way. So, F is the force acting on the satellite or in the region even liberty for which we have written this equation. So, in the case of aerodynamic drag or other non conservative forces we can use it okay, this we have done earlier by expanding it you can immediately check that this quantity will be $\frac{\partial R}{\partial c_j}$.

So, this quantity then ∂R is nothing but your F. So, this comes with a minus sign F. $\frac{\partial R}{\partial c_j}$. This simply implies that in the Lagrange from the planetary equation, wherever this term on the right hand side $\frac{\partial R}{\partial c_j}$ is appearing. So, this you need to replace in terms of - F, if this is a vector dot $\frac{\partial R}{\partial c_j}$. So, once you do this, so this will get converted in terms of F_r, F_θ and F_A, this is what our objective is. Where R_i has been written as

$$R_i = -G \sum_{\substack{j=2 \\ j\neq i}}^{n} m_j \left[\frac{1}{r_{ij}} - \frac{\vec{r}_i \cdot \vec{r}_j}{r_j^3}\right]$$

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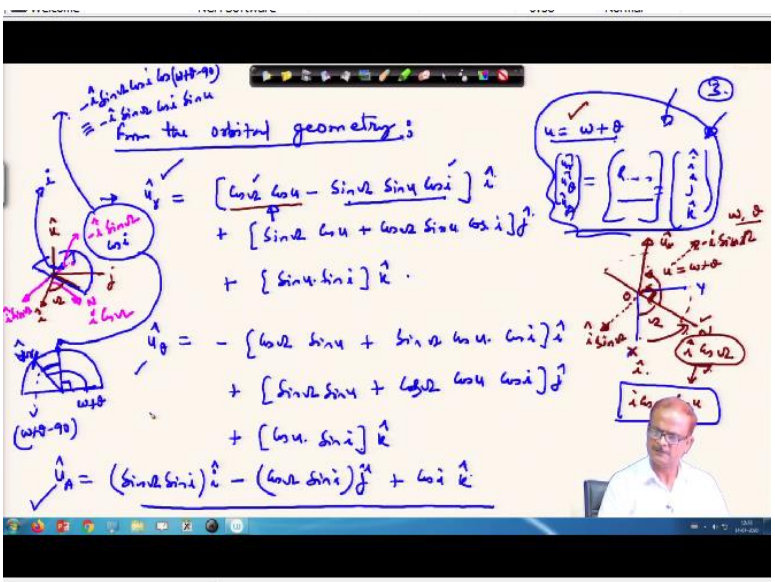

From the orbital geometry, we can write the expression for $\hat{u}_r$, $\hat{u}_\theta$ and $\hat{u}_A$. It is very easy. Say if I use the rotation matrix if and again explaining all this thing say we do not have much time in this course, because already what I used to cover in 45 lectures, we are reduced to 30 lectures. So, it is so much more accomplished. What I will suggest that for if you want to use this matrix method for transferring or finding out this $\hat{u}_r$.

And other things, we just go into the especially this satellite attitude control and dynamics course on notes of the NPTEL. And look into that in the first few 10 lectures, all those things have been explained. So, using this you can immediately get to this point. So, I will simply write all these terms. So, what here we are trying to do that $\hat{u}_r$. So, in this direction, you have to write in terms of $\hat{\imath}$ here so $\hat{\jmath}$ the unit vector and in this direction $\hat{k}$.

So, in terms of this unit vectors, you have to describe the vector $\hat{u}_r$. Then $\hat{u}_\theta$ and $\hat{u}_A$ and this is easy which I am not doing to do here, you can do this as an exercise, what you are supposed to do that convert this i to this u_r vector and for this what you need to do, that you have to think of okay. So, I will just give you some hints for doing this, say i is a vector whose component you required along this direction okay.

So, I can write this i vector as $\hat{\imath}$ 0 0 and then this I need to convert into this P frame and in terms of what we have written here that u_r and sorry this $\hat{u}_r, \hat{u}_\theta$ and this $\hat{u}_A$, here it is $\hat{u}_A$. So, there are many ways of doing this, you can take component along this direction, the first component of i you can take along this direction okay, then this component can be broken along the u_r direction.

So, from here to because they are not in the same plane, so we have to do sequentially, otherwise the rotation matrix if you have to use. So, you have to first rotate about this z axis by Ω and so on. So, this concept is explained in that satellite attitude dynamics course. So, there you can go and look into that. So, here I am writing the final result cosΩ, cosu, here u equal to small omega + theta.

So true anomaly plus argument of perigee, times $\hat{\imath}$. So, you have vector here, which you can represent us $\hat{\imath}$ $\hat{\jmath}$, $\hat{k}$ and this vector then you are expressing as which is operated on by certain rotations here. So, as you can see from here the Ω is appearing, i is appearing for some of the vectors you will require working with that. So, how those vectors see, how you can from i j k by how many rotation you can reach to displace okay, this is the question.

$$\hat{u}_r = [\cos\Omega \cos u - \sin\Omega \sin u \cos i]\hat{\imath} + [\sin\Omega \cos u + \cos\Omega \sin u \cos i]\hat{\jmath} + [\sin u \sin i]\hat{k}$$

So, for that you have to give all the proper rotations which will be indicated here and then on the left hand side you will have the $\hat{u}_r, \hat{u}_\theta$ and this $\hat{u}_A$, these are the 3 things we are looking for. So, for that you require proper sequencing and other things. So, look into this concept. This will be very useful for you for working out this current thing. It takes a little shorter time to work out this concept cos Ω, sinu, cosΩ.

$$\hat{u}_\theta = -[\cos\Omega\, sin\, u + sin\,\Omega\, cos\, u\, cos\, i]\hat{\imath} + [sin\,\Omega\, sin\, u + cos\,\Omega\, cos\, u\, cos\, i\,]\hat{\jmath} + [cos\, u\, sin\, i]\hat{k}$$

$$\hat{u}_A = [sin\,\Omega\, sin\, i]\hat{\imath} - [cos\,\Omega\;\; cos\, i\,]\hat{\jmath} + [cos\, i]\hat{k}$$

So, I will give you some hints about this, what exactly we have done here in this place, say this is your x and y axis and then this is a Ω and then you are going to draw the orbit in which plane in that plane then you are measuring this ω and θ and which you have added together to get this u, here you have the $\hat{\imath}$. So, first you take the component of this along this nodal line, this is O and here N.

So, this will be $\hat{\imath}$ cos Ω and the $\vec{r}$ is located somewhere here. So, this is your $\hat{u}_r$ and the angle from this place to this place, this is u, which is nothing but ω + θ and then you will be taking the component of this vector along this direction. So, component of this then becomes iType equation here.cos Ω times cos u. So, this is the term which is appearing here.

Thereafter, because this i component you are taking here in this place. Now this can be broken along one common component of i is here. So, another component will be perpendicular to this, perpendicular to this O N and here on the opposite side, then as we have done in the orbit parameter estimation problem, if you go on the opposite side, so here this will come as - i. So, here this will be i sin Ω.

So, here on this side it will come as - $\hat{\imath}$ sin Ω okay and thereafter then we need to. So, this is making 90º with this, okay see here with this line this is making this is 90º, okay this component we need to take in the orbit. This is u_r orbit and this is your $\hat{\imath}$. So $\hat{\imath}$ then u_r breaking along this direction and breaking perpendicular to here it is appearing little more skewed.

So, I will make it a little better look better. This is the $\hat{\imath}$ and this is $\hat{\jmath}$ and here this is $\hat{k}$. So, this angle is your capital omega. So, this $\hat{\imath}$ here, this will break along 2 directions, one along this nodal line N and another perpendicular to this, which is, so this component we are writing as $\hat{\imath}\cos\Omega$ and here $\hat{\imath}\sin\Omega$. And on this site this will be going somewhere here and this component will be - i $sin\Omega$.

And thereafter this is cap okay, remember this is not the angle of inclination. Thereafter we need to take a projection of this in the orbit okay. So, orbit is going up. So, here this orbit is then y_i angle okay. So, this angle is i, from here to here this angle is i, so, then we need to multiply it by cos i okay. Now, look here in this place this orbit r is located here in this place okay. So, u_r is here. So, you have got this component already we have got here in this place.

Already we have written the component which is appearing, then this part is written here, this part is here, this total angle is $\omega + \theta$. So, if we subtract 90º, so this angle from this place to this place, this angle will be $\omega + \theta$ - 90º. So, you need to take then component of this along this direction okay. So, if I take this component along the u r direction, so, this will be - $\hat{\imath}$ sin Ω cos i.

And then cos $\omega + \theta$ - 90 degree. So, that mix it – $\hat{\imath}$ sin Ω cos i and this becomes sin u is not it with the same thing appearing here cos i sin Ω. So, this way the component of i has been taken along the u_r direction. And the same way you can do for the j and then the k. So, this u_r is completed the same exercise you have to do for the θ and for the A also.

So, this way you will be able to get this u_r, u_θ and u_A component, because I have already done numerous times in the orbit problem. So, I need not repeat the same thing again and again. But better way of doing the same thing is using the matrix rotation. This is much simpler than doing it this way what I am showing you here. But for that you need to know the rotation matrix, how it is represented and other things which I have not done of course here in this course.

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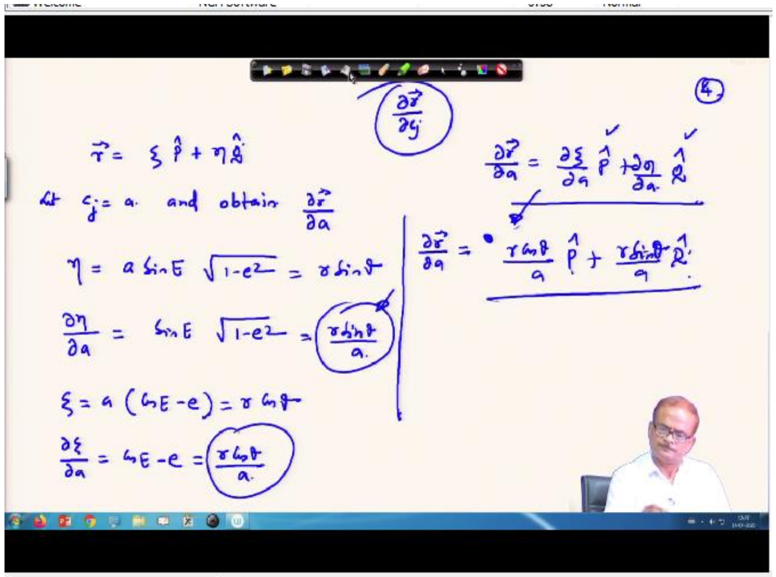

We require terms like $\frac{\partial R}{\partial c_j}$ the terms which is present here, where j will vary over 1 to 6 3 4 5 and 6. This corresponds to a, e, i not necessarily in the same sequence is ω, Ω, T or either σ okay. So, these are just 6 terms we have to get here in this place. So, this we need to work out. So, if you choose j to be c_j to be a, so then we need to work out this quantity in order to complete this particular term finding out $\frac{\partial R}{\partial c_j}$.

This we can express what this we need to find out, okay derivative is not for the $\widehat{P}$ and $\widehat{Q}$, because they are the fixed vectors in the plane of the orbit in the plane of the escalating orbit. So $\frac{\partial \eta}{\partial a}$ and already we have determined that this quantities are sin θ. So, from here we can see that this quantity will be r sin θ by similarly we have ξ equal to A times and this equal to r cos θ.

$$\frac{\partial \vec{r}}{\partial a} = \frac{\partial \xi}{\partial a}\widehat{P} + \frac{\partial \eta}{\partial a}\widehat{Q}$$

Therefore $\frac{\partial \vec{r}}{\partial a}$ which is here. You can write using this 2 information here r cos θ, r sin θ u v r sin θ yet in this place. So, r sin θ divided by A time Q. Now $\widehat{P}$ $\widehat{Q}$ this already we know as earlier we have derived it, okay. So, if we insert that our job will be done.

$$\frac{\partial \vec{r}}{\partial a} = \frac{r\cos\theta}{a}\widehat{P} + \frac{\mathrm{r}\sin\theta}{a}\widehat{Q}$$

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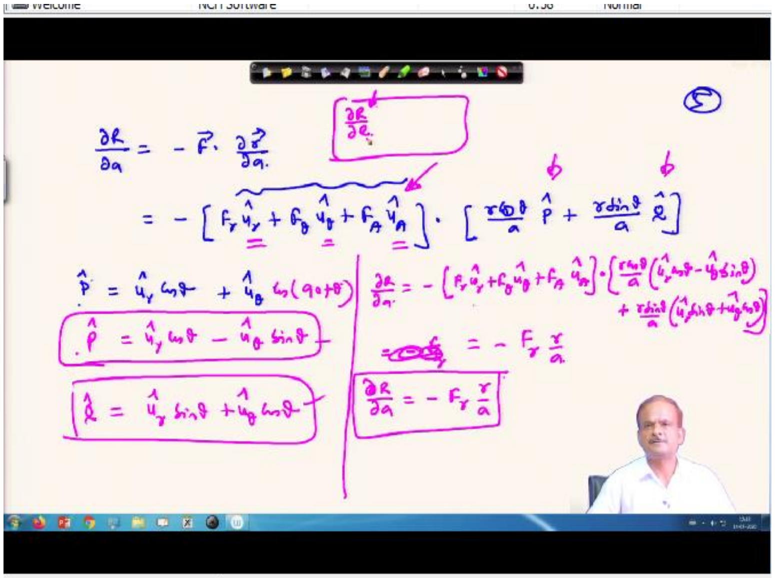

So, therefore $\frac{\partial R}{\partial a}$. So, we want to express this in terms of u_r and u_θ. So, immediately we can see from this place, this kind of exercise numerous times we have done, so again repeating, this is your $\vec{u}_r$. So, along this what the component will be, from here to here angle is θ, you have to $\hat{P}$ the angle between the vector $u_r o \hat{P}$, this is θ and therefore, $\hat{P}$ will become $\hat{u}_r \cos\theta$.

And then the other part we have to take into account which is $\hat{u}_r \cos\theta$ and then we are going to this place u_θ is here. So, this is turning from the $\hat{u}_r$ by 90º, this angle is 90º, as I am showing here, this angle is 90º okay, as shown by this and therefore, this is cos 90 + θ is a component and this gets reduced to $\hat{u}_r \cos\theta$ -. So, this is $\hat{P}$, which is expressed in terms of u_r and u_θ.

These are already expressed in terms of $\hat{u}_r, \hat{u}_\theta$ and this $\hat{u}_A$ obviously because of with a vectors we are taking which the $\hat{P}$ and $\hat{Q}$ they are in the plane of the orbit. So, u_A will not manifest in that $\hat{u}_A$. Similarly, $\hat{Q}$ you can check it, this will be $\hat{u}_r$ sin θ + $\hat{u}_\theta$ cos θ. Now, insert this to here in this expression and thereafter we solve it okay. So, I will write it here itself.

$$\frac{\partial R}{\partial a} = -\vec{F}.\frac{\partial \vec{r}}{\partial a}$$

$$\frac{\partial R}{\partial a} = -[F_r\hat{u}_r + F_\theta\hat{u}_\theta + F_A\hat{u}_A].\left[\frac{r\cos\theta}{a}\hat{P} + \frac{r\sin\theta}{a}\hat{Q}\right]$$

$$\hat{P} = \hat{u}_r\cos\theta - \hat{u}_\theta\sin\theta$$

$$\hat{Q} = \hat{u}_r\sin\theta + \hat{u}_\theta\cos\theta$$

$$\frac{\partial R}{\partial a} = -[F_r\hat{u}_r + F_\theta\hat{u}_\theta + F_A\hat{u}_A].\left[\frac{\mathrm{r}\cos\theta}{a}[\hat{u}_r\cos\theta - \hat{u}_\theta\sin\theta] + \frac{\mathrm{r}\sin\theta}{a}[\hat{u}_r\sin\theta + \hat{u}_\theta\cos\theta]\right]$$

$$\frac{\partial R}{\partial a} = -F_r\frac{r}{a}$$

And this is the dot product and then from here r cos θ divided by a times $\hat{u}_r$ cos θ - $\hat{u}_\theta$ sin θ which becomes $\hat{P}$ and + r sin θ by a and then $\hat{u}_r$ sin θ + $\hat{u}_\theta$ cos θ okay. So, with this once we simplify it taking the dot product obviously, only the dot product will exist with itself with other it will be perpendicular and therefore, they will vanish.

So, this gets reduced to - F_r by F_r this equal to - F_r by a. So, $\frac{\partial R}{\partial a}$, this has the value - F_r by a, using the same technique we can derive for the others. So, in the next class I will do for one more that means, I will work for $\frac{\partial R}{\partial a}$, here once you understand what we are doing with this is a little more complicated finding $\frac{\partial R}{\partial a}$. So, from there this concept will become you will be able to grasp this concept much better what exactly we are doing. So, we stop here and we will continue in the next class. Thank you.