Space Flight Mechanics
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Lecture-59
General Orbit Perturbation Theory (Contd.)

Ok, welcome to lecture 59, we have been discussing about the general orbit perturbation theory. And in that context we worked out with the Lagrange bracket and evaluated it and then found the Lagrange planetary equation of motion. And then thereafter we wanted to describe the Institute of perturbation potential, we wanted to use the perturbation force. Because many times the perturbation potential is not available but because the force may not be conservative in that case you do not have the ability of the potential, so we have to work with the force.

So, that we were trying to convert the potential model into the force model, so we will continue with that.

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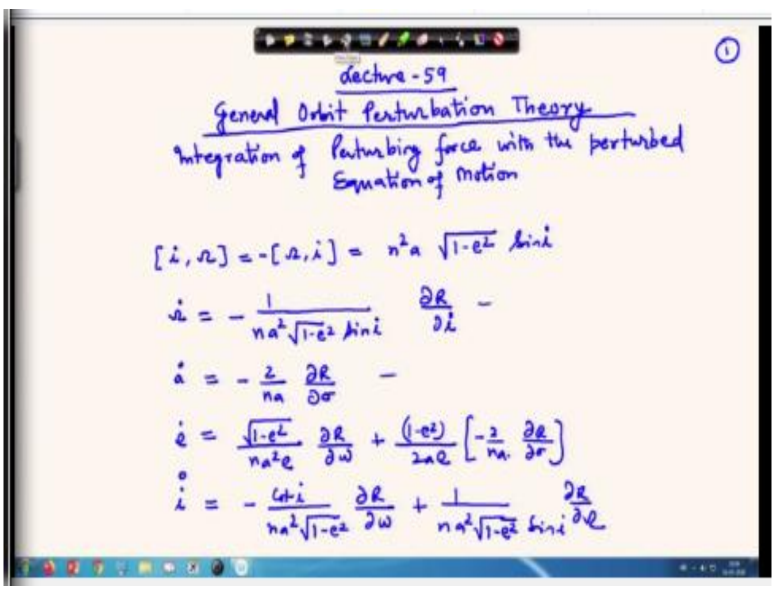

But before that one equation was left out last time, so we will complete that. So, if there was some sign problem with this bracket perhaps there was a $-$ sign written there. So, the correct sign is as indicated here, ok and the corresponding equation for the Ω where the problem might have occurred, ok. So, this is given by $sini$ times ok. So, once again I will write all those equations, so that no confusion remains.

$$\dot{\sigma} = \frac{(1-e^2)}{na^2e}\frac{\partial R}{\partial e} + \frac{2}{na}\frac{\partial R}{\partial a}$$

$$\dot{\omega} = \frac{\cot i}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial i} - \frac{\sqrt{1-e^2}}{na^2e}\frac{\partial R}{\partial e}$$

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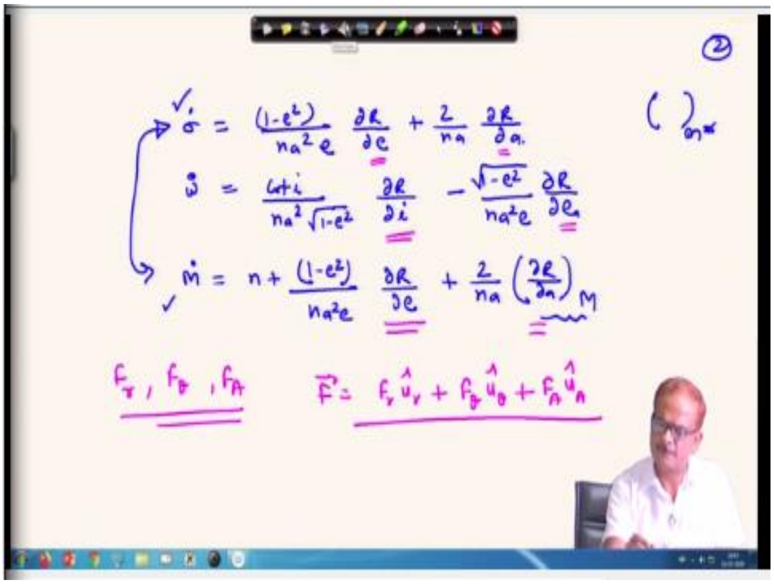

And sigma dot this equal to $1 - e^2$, so we have 4 equations here, a dot e dot ok small omega dot is remaining, so $\dot{\omega}$, this is $coti$. This $\dot{\sigma}$ also it can be expressed in terms of $\dot{m}$. And it can be written as $\dot{m}$ equal to where $\dot{m}$ is the rate of change of the mean anomaly, n is the mean angular velocity or the mean angular rate. This implies that you are keeping M constant whenever we put in bracket, some partial differential and put like this, so that means you are keeping M constant and then you are finding this.

$$\dot{m} = n + \frac{1-e^2}{na^2e}\frac{\partial R}{\partial e} + \frac{2}{na}\left(\frac{\partial R}{\partial a}\right)_M$$

$$\overrightarrow{F} = [F_r\hat{u}_r + F_\theta\hat{u}_\theta + F_A\hat{u}_A]$$

So, either this or this, either of them can be used. Now here in this part what we are trying last time that all these terms appearing here this need to be replaced in terms of we have replaced in terms of we were trying to replace in terms of F_r, F_θ and F_A where F we have described as $F_r\hat{u}_r$, then $F_\theta\hat{u}_\theta$ and $F_A\hat{u}_A$. So, the development we were following, so I will go along with those development but before that I will summarize these equations, so that it is a ready for your reference.

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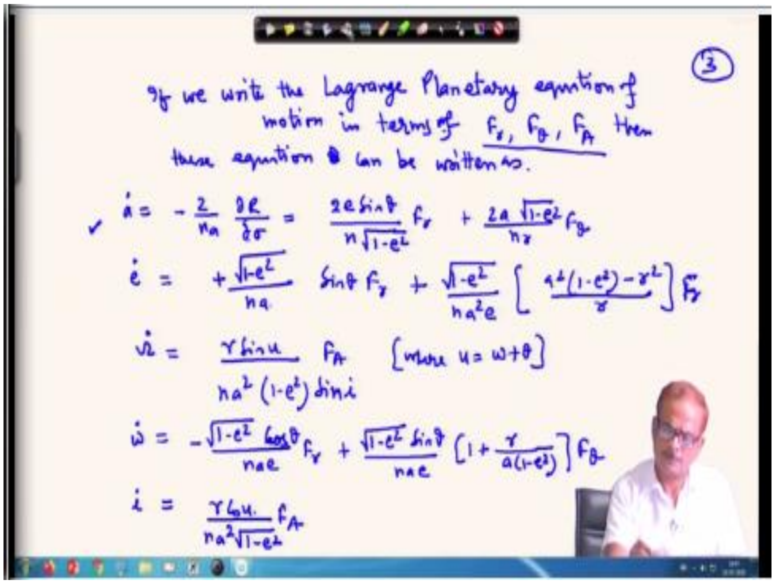

In terms of and we written as this was our Lagrange planetary equation of motion ok. So, $\frac{\partial R}{\partial \sigma}$ if we replace in terms of F_r, F_θ and F_A, so this will get reduce to the format 2esin θ F_r + 1 - e^2 F_r then a dot already we have worked out for the $\frac{\partial R}{\partial \sigma}$ actually what we have done there the last time we have worked out what will the equation for the F_r ok, I will do it again.

$$\dot{a} = -\frac{2}{na}\frac{\partial R}{\partial \sigma} = \frac{2e\,sin\theta}{n\sqrt{1-e^2}}F_r + \frac{2a\,\sqrt{1-e^2}}{n_r}F_\theta$$

$$\dot{e} = \frac{\sqrt{1-e^2}}{na}\,sin\theta F_r + \frac{\sqrt{1-e^2}}{na^2e}\left[\frac{a^2(1-e^2-r^2)}{r}\right]F_r$$

$$\dot{\Omega} = \frac{rsinu}{na^2(1-e^2)sini}F_A$$

$$\dot{\omega} = -\frac{\sqrt{1-e^2}}{nae}cos\theta F_r + \frac{\sqrt{1-e^2}}{nae}\,sin\theta\,\left[1+\frac{r}{a(1-e^2)}\right]$$

$$\dot{\sigma} = \left[\frac{(1-e^2)cos\theta}{nae} - \frac{2r}{na}\right]F_r - \frac{(1-e^2)sin\theta}{nae}\left[1+\frac{r}{a(1-e^2)}\right]F_\theta$$

First we finish this sinu n$a^2$1 – e^2sini F_A where u =ω + θ, 1 - e^2 F_θ and these are the 4 equations and last 2 were.

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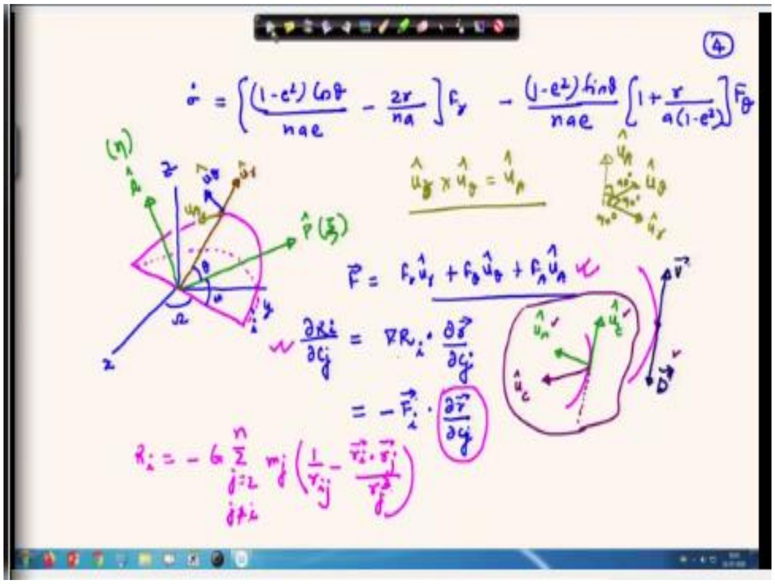

So, these are the 6 equations which are once converted in terms of F_r, F_θ and F_A. So, this is how they appear ok. So, now we have start with what we have been doing. So, what we did that we represented the reference frame first x, y, z and in that then we had the orbit and the projection of the orbit represented like this. Then we had in this direction P cap unit vector we have taken in the direction of ξ.

And then in the direction of $\hat{Q}$ we have taken the η, so this is the reference frame in the plane of the orbit. And then we indicated the satellite position somewhere, so along this direction we had the vector u_r perpendicular to this we have u_θ where the angle theta is indicated like this. And this angle we have represented as a ω, this angle as Ω when angle of inclination this angle is i, inclination with the x, y plane of the orbit, inclination of the orbit with the x, y plane ok.

So, and then we had with other color we can show it and perpendicular to this then we have u A. So, where $\hat{u}_\theta$, $\hat{u}_r$ cross $\hat{u}_\theta$ equal to $\hat{u}_A$, so these are perpendicular. Means, if you have u_r here, this is u_θ, so $\hat{u}_A$ is up, this angle is 90º, this angle also 90º, this angle is also 90º. And then what we were trying to do is that we wrote F as a $F_r\hat{u}_r$, $F_\theta\hat{u}_\theta$ and $F_A\hat{u}_A$.

$$\overrightarrow{F} = -[F_r\hat{u}_r + F_\theta\hat{u}_\theta + F_A\hat{u}_A]$$

And $\frac{\partial R}{\partial c_j}$ we wrote it this way and this quantity is nothing but - F. So, force of perturbation on the RTS particle the RTS planetary body, assuming it to be a point mass. And then of course we have

$\frac{\partial R}{\partial c_j}$ and what we were trying to do is find out this quantity. Because F is already described here in this format, so we have to find out $\frac{\partial R}{\partial c_j}$. And once we do this, so we can take the dot product and this quantity will be available to us where

$$R_i = -G \sum_{\substack{j=2 \\ j \neq i}}^{n} m_j \left(\frac{1}{r_{ij}} - \frac{\vec{r}_i \cdot \vec{r}_j}{r_j^2} \right)$$

So, different models are required in different places say if we are looking for aerodynamic drag. So, aerodynamic drag always by convention say if this is the orbit and then at point the velocity will be tangent to this, velocity tangent we can show it like this, so this is the velocity direction. And therefore drag will be opposite to this, this is by convention ok. And therefore we need to find out the force along the tangent direction while here this is not along the tangent direction.

So, in this case it may be helpful to work with the other representation, like here we can have in the tangent direction we can have u_t as the unit vector. And perpendicular to this then we will have u_n as the unit vector and perpendicular to u_t and u_n we will have another vector which we can name anything like say I name it u_c or whatever. So, here in this case the drag will be model if properly if we represent this in terms of the tangent vector, unit vector and normal to this, this called the principal normal.

And the what is appearing here the as you see this is called a binormal ok. So, along the tangent direction, along the principal normal direction and in the normal direction we require then the all the forces. This model also we will look into shortly once we finish this part. So, let us go ahead and finish this part first because each every model it is a useful in it is own place, it say depends on the situation which will be more convenient to represent ok.

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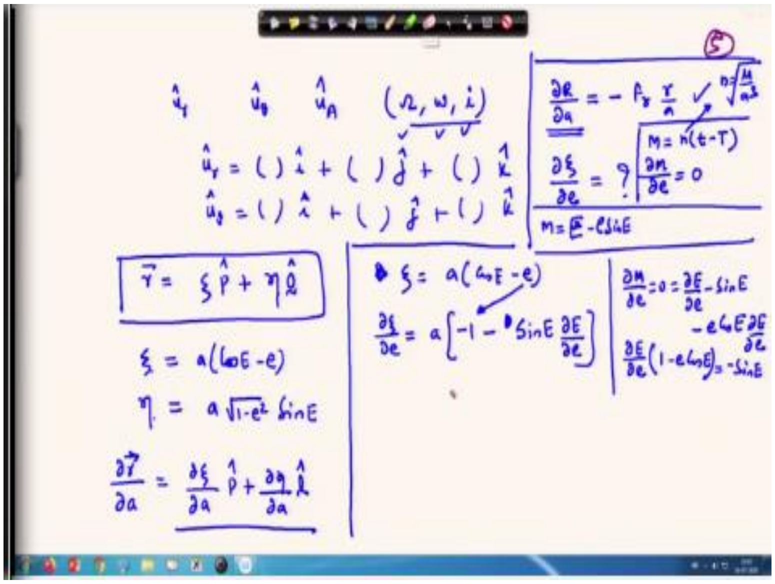

So, thereafter we represented u_r vector and u_θ vector and u_A vector, this we represented in terms of Ω, ω and i. So, these were function of Ω, ω and i, I will not repeat those equations again. So, $\hat{u}_r$ say if this is something times it was $\hat{\imath}$, then $\hat{\jmath}$ and then $\hat{k}$. Similarly $\hat{u}_\theta$ was described and the same way the $\hat{u}_A$ also it was written ok.

So, thereafter we wrote,

$$\vec{r} = \xi\,\hat{P} + \eta\hat{Q}$$

and this model we have used earlier while evaluating the Lagrange bracket. And here we have the ξ as a times cosE - e and η is a times $(1 - e^2)sinE$. And what we require, we require this quantity dou R by dou cj. So, therefore the first example we took off it was $\frac{\partial R}{\partial a}$ and from this place then we wrote it this way ok.

And thereafter we evaluated this term, so in that context we got $\frac{\partial R}{\partial a}$as the quantity - F_r /a ok. So, till this extent we have done last time and where this will go, this will go into the equation we have written earlier $\frac{\partial R}{\partial a}$ you see where it is appearing, so in those places we need to insert it. Here in this place this is not there, here it is not there, here also it is not there, here also not.

$$\frac{\partial \vec{r}}{\partial a} = \frac{\partial \xi}{\partial a}\hat{P} + \frac{\partial \eta}{\partial a}\hat{Q}$$

So, if $\frac{\partial R}{\partial a}$ is present here in this place. So, we need to insert it here in this place, ok. Now once we have done this the other part then one more I will do, which is $\frac{\partial \xi}{\partial a}$. Because each of the derivation it is a quite cumbersome and it is a long, so it is not possible to carry out all the things here in the class. So, this will come as the supplementary material later on ok. So, this quantity is what this we have to derive.

So, again we start writing $\xi = a(\cos E - e)$ and therefore we can see from this place that $\frac{\partial \xi}{\partial a}$ how much this quantity will be. So, this is a times here e will be this is - 1, once we differentiate and this will be cosE once we differentiate this will be sinE $\frac{\partial \xi}{\partial a}$. And why we are differentiating this because we have M equal to E - e sin E ok, and M is the quantity which is written as n_t - T where T is a constant.

And therefore once we differentiate and M is also ok, so first we write this $\frac{\partial M}{\partial e}$, this quantity will be. Now here if we look into n is a function of n equal to $\sqrt{\frac{\mu}{a^3}}$ So, n is not a function of e ok, only a is appearing and therefore this quantity is 0. So, once we differentiate this quantity, so $\frac{\partial M}{\partial e}$ equal to 0 this will be equal to 1 minus and then we are differentiating with respect to sorry, this is yes we are differentiating with respect to small e.

And therefore this will be

$$\frac{\partial M}{\partial e} = \frac{\partial E}{\partial e} - \sin E - e\cos E \frac{\partial E}{\partial e}$$

we combine the terms together and write it. So, this ok, so from here we have $\frac{\partial E}{\partial e}$ times 1 - e cos E this equal to sinE with minus sign ok. Let me write it here, this place is getting sort.

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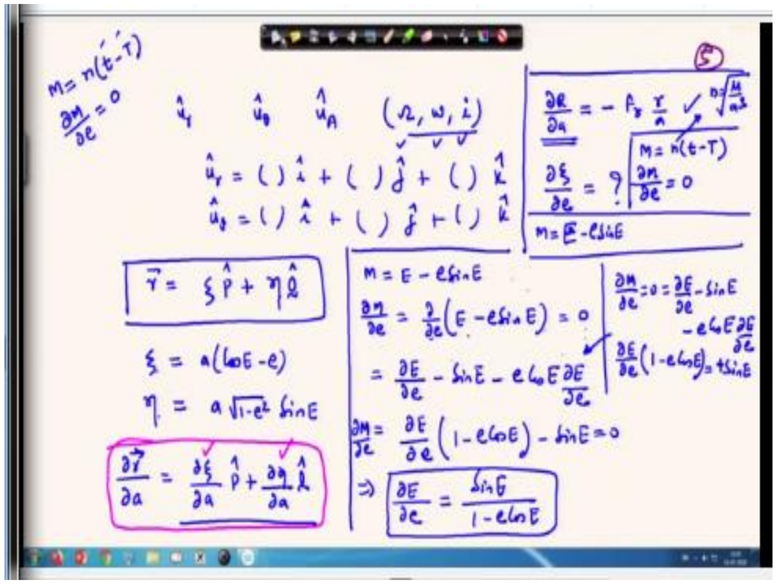

So, M we have written as M esin E and because M is a function of mean angular velocity - T and mean angular velocity and then we are differentiating this with respect to E. So, this quantity will be 0 because here these are not function of e. So, the left hand side then we have set it to 0 and then work with the rest of the things. So, therefore

$$\frac{\partial M}{\partial e} = \frac{\partial}{\partial e}\ (E - esinE)$$

$$\frac{\partial M}{\partial e} = \ \frac{\partial E}{\partial e} - sinE - e\ cosE\ \frac{\partial E}{\partial e}$$

And then once we expand it, so this will be $\frac{\partial E}{\partial e}$, $\frac{\partial E}{\partial e}$ e - sinE and then - e cosE and this we combine and write it as here we have (1 – sinE) this will be plus. So, the same thing I am writing here in a little better way

$$\frac{\partial M}{\partial e} = \ \frac{\partial E}{\partial e}\ (1 - ecosE) - sinE = 0$$

$$\frac{\partial E}{\partial e} = \frac{sinE}{1 - ecosE}$$

and this is the equation we are going to utilize.

Now we have to work with this equation, we have to find out what will we do those ξ by ∂a and $\frac{\partial \eta}{\partial a}$.

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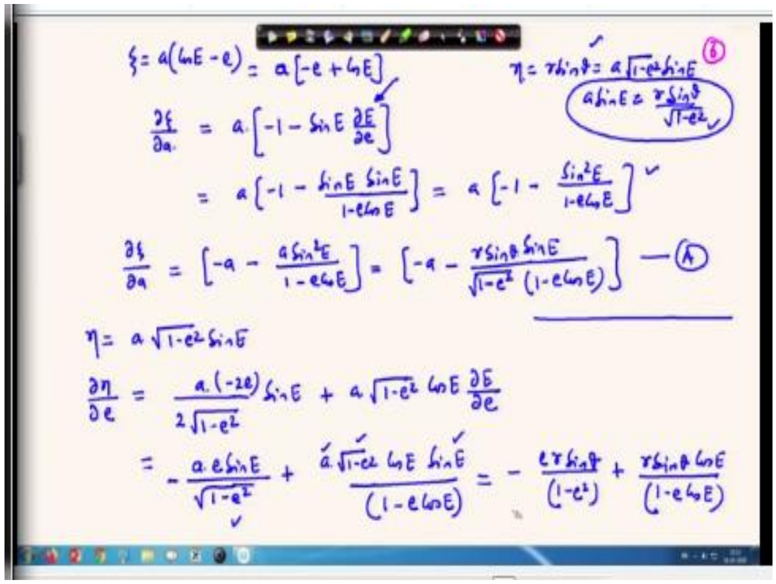

So, we look for $\frac{\partial \xi}{\partial a}$, this quantity because psi is a times cos E - e and therefore once you differentiate this we can write it as a times. So, this is a quantity already we have worked out on the last page. So, we need to insert that value sinE and this is $\frac{\sin E}{1-\cos E}$ and this is $\frac{\sin E}{1-\cos E}$. So, one more point this is a e is missing, so here we are the small e also present, in this place also small e is here.

And we recast this equation into little different form. So, that solution to the problem becomes little easier ok. If we take it a inside, now a sinE if you remember η we have written as equal to r sin theta is equal to a times 1 - e^2, sinE. So, a sin E this quantity is r sin θ divided by $\sqrt{(1-e^2)}$. So, a sin E we will replace by r sin θ divided by 1 - e under root then sin E and rest of the term we have to copy here.

$$\frac{\partial \xi}{\partial a} = \left[-a - \frac{\mathrm{a}\sin^2 E}{1-ecosE}\right] = \left[-a - \frac{rsin\theta sinE}{\sqrt{1-e^2(1-ecosE)}}\right]$$

So, let us say this is our equation A, the same way we have to work out the eta term. So, η is a times $1-e^2$ sin E and therefore $\frac{\partial \eta}{\partial e}$ this quantity will be a times $1-e^2$ - 2 e, 2, 2 cancel out this is esin E with minus sign here. And then the other term this is + a times $1-e^2$ $\frac{\partial E}{\partial e}$, so we need to replace this quantity $\frac{\partial E}{\partial e}$ already we have written on the previous page, $\frac{\sin E}{1-e\cos E}$ ok. Again some of the terms we will change and write it in a proper format.

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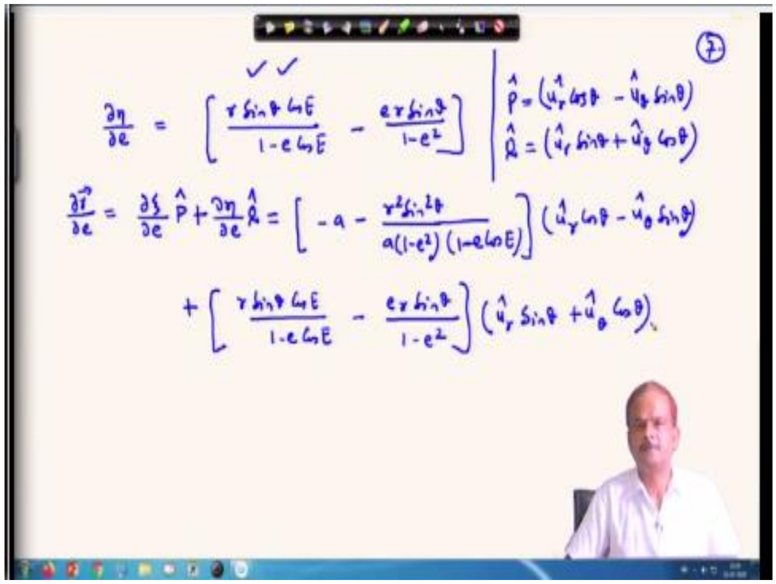

Therefore we have $\frac{\partial \eta}{\partial e}$ this equal to the first term here if you see a $1 - e^2$ sin E is appearing, so this quantity is nothing but rsin θ ok. So, the second term will be replaced by r sin θ ok, first we take into account this part. So, the asin E, asin E already we have written here, asin E we will replace with r sin theta divided by $1 - e^2$. So, for the first term then gets reduced to see here itself let me do it.

So, it will convenient for me to work out, - asin E is r sin θ divided by 1 - e^2. So, r sin θ a we have taken, this is e r and then 1 - $a^2$is already there and this makes it 1 - a^2here. And plus this part, a times 1 – e^2sin E from this place this is r sine theta, so this becomes r sin theta times cos E divided by 1 - e cos E, so r sin θ cos E 1 - e cos E 1 - e^2.

$$\frac{\partial \eta}{\partial e} = -\frac{ersin\theta}{1-e^2} + \frac{rsin\theta cosE}{1-ecosE}$$

So, let us summarize it here, so this becomes $\frac{rsin\theta cosE}{1-ecosE}$. Now we can utilize these terms in finding out $\frac{\partial \vec{r}}{\partial e}$ which we have written as $\frac{\partial \xi}{\partial e}\hat{P} + \frac{\partial \eta}{\partial e}\hat{Q}$. So, inserting the values $\hat{P}$ and $\hat{Q}$, it is available to us, we have $\hat{P}$ equal to u_r cosθ- $\hat{u}_\theta$sinθ already we have worked out this things in the previous lecture.

$$\frac{\partial \vec{r}}{\partial e} = \frac{\partial \xi}{\partial e}\hat{P} + \frac{\partial \eta}{\partial e}\hat{Q}$$

So, we will utilize this information here in this place, this time we are picking up $\frac{\partial \xi}{\partial a}$, a^2sin E 1 - e cos E and we have replaced asin E is $1 - r^2$ ok. One more replacement it is a possible here in this place.

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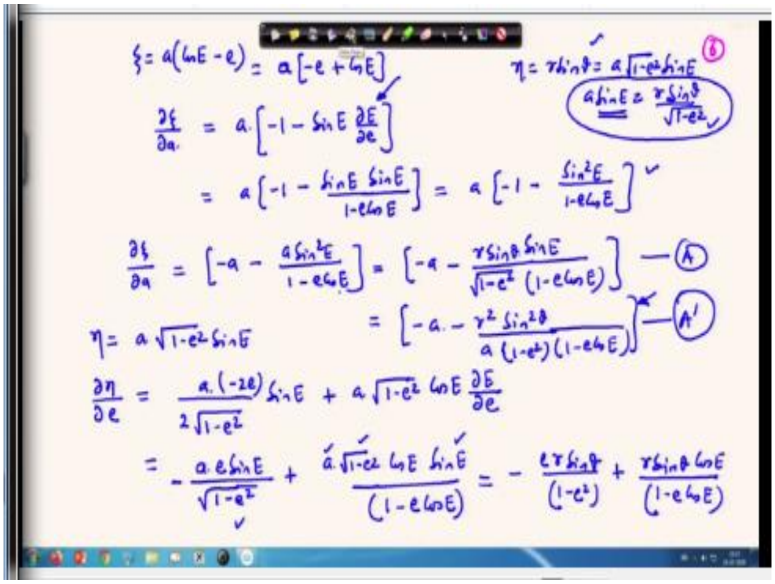

And we can because later on again we will have to replace the term, so better we do it here in this place itself. This sin E also from this place if you see sin E we can write it as r a. So, here this because then the $\frac{r^2\sin^2\theta}{a(1-e^2)(1-ecosE)}$, so and this equation let us write this as a'. So, this equation will be required either we replace write in the beginning or later on but it is required ok.

$$\hat{P} = \hat{u}_r \cos\theta - \hat{u}_\theta \sin\theta$$

$$\hat{Q} = \hat{u}_r \sin\theta + \hat{u}_\theta \cos\theta$$

$$\frac{\partial \vec{r}}{\partial e} = \left[-a - \frac{r^2\sin^2\theta}{a(1-e^2)(1-ecosE)}\right](\hat{u}_r \cos\theta - \hat{u}_\theta \sin\theta) + \left[-\frac{ersin\theta}{1-e^2} + \frac{rsin\theta cosE}{1-ecosE}\right](\hat{u}_r \sin\theta + \hat{u}_\theta \cos\theta)$$

So, $\frac{r^2\sin^2\theta}{a(1-e^2)(1-ecosE)}$ times $\hat{P}$ which is nothing but ($\hat{u}_r$ cos θ - $\hat{u}_\theta$ sin θ). And plus $\frac{\partial\eta}{\partial e}$, so the term which is here, so $\frac{ersin\theta}{1-e^2} + \frac{rsin\theta cosE}{1-ecosE}$ and this then multiplied by ($\hat{u}_r$sin θ + $\hat{u}_\theta$cos θ) ok. So, we will continue with the same equation in the next lecture, we stop it here and thank you very much.