Space Flight Mechanics
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Lecture No - 62
General Perturbation Theory (Contd.)

Welcome to lecture 62 we have been discussion about the general audit perturbation theory in that context till now we have looked into how the; due to the extra potential term or the extra perturbation term the equation will look like.

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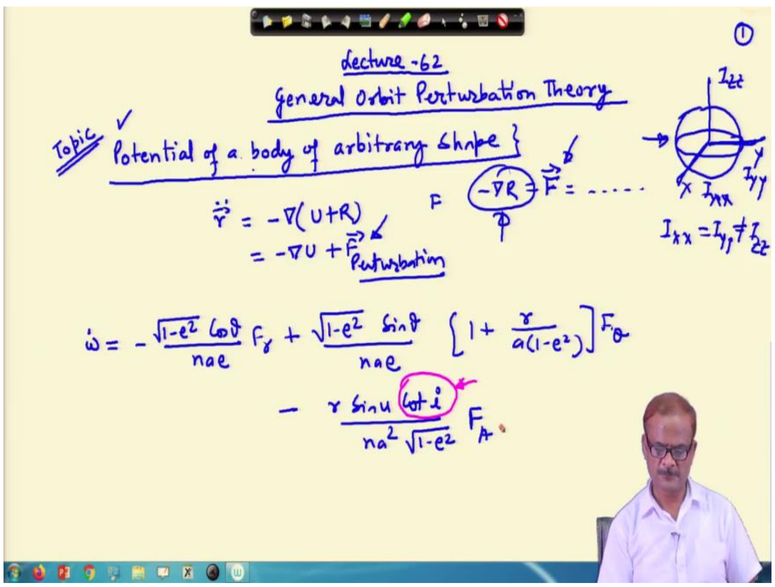

Now actually we have to find out the terms if you remember that we have written

$$\ddot{r} = -\nabla(U + R)$$

$$= -\nabla U + \vec{F}_{Perturbation}$$

and thereafter also in the force from we have written this equation. All these things till now we have worked out. So, today we will discuss about the potential of a body of arbitrary shape. And using this then this will be our first step to know how the perturbation term; already we have looked that mathematically how it is arising in the; but till now we have looked into the F form.

But say if we have this term $-\nabla R$ so this ∇R equal to F ok and we want to know actually how does or what will be the feature of this function F here or the force F here? What are the terms involved in that? So this we need to drive. We will do it for today for oblate sphere means you have a sphere but it is; but here say the the X and Y dimension are the I_{xx} and I_{yy} the moment of inertia about the x-axis and y-axis is same, but I_{zz} is not the same.

So this that is I_{xx} equal to I_{yy} and this is not equal to I_{zz}. This particular part we are going to look through the derivation for the body of an arbitrary shape, first will write the equation and thereafter we will take this particular case and work out how this term will appear but before that one correction from the last time that we have written the $\dot{\omega}$ equation in which we dropped out one term and again, there was a small error in that particular terms.

$$\dot{\omega} = -\frac{\sqrt{1-e^2}\,cos\theta}{nae}F_r + \frac{\sqrt{1-e^2}\,sin\theta}{nae}\left(1+\frac{r}{a(1-e^2)}\right)F_\theta - \frac{r\,sinu\,coti}{na^2\sqrt{1-e^2}}F_A$$

So I will correct that because we are going little fast. So we have less time so it is likely that some of the typing error correction F_θ – r sin u and this is cot i instead of cot i last time it was written as cos i only when the lecture was over I noticed that. This term last time written as cos i which is not correct. So this is the correct term here this multiplied by F_A with this now we start with our main topic for discussion today.

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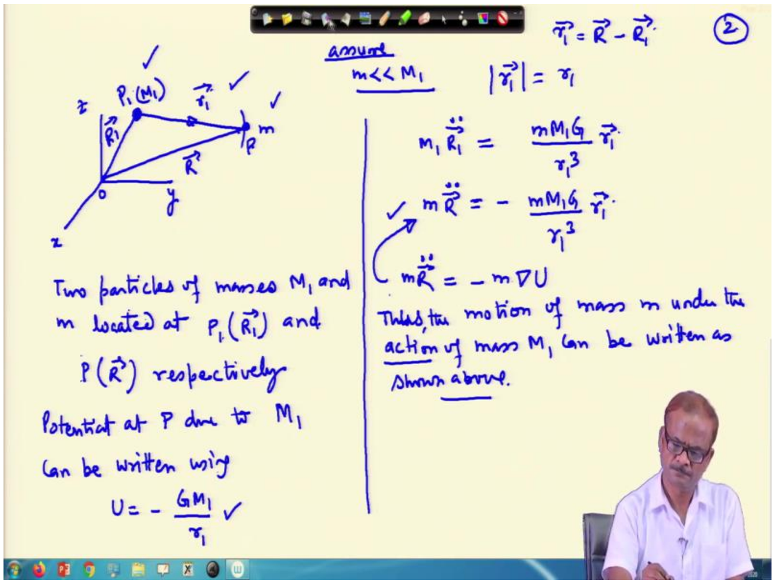

So if you look into this configuration let us first considered that there are two masses m and M_1 so you can put small letter or capital letter it does not matter. So, 2 particles of masses M_1 and m located at P_1 which is described by vector R_1 and P described by vector R respectively. Now consider that this mass m is small ok as compared to; so m is much smaller than in M_1 so under this condition so we can get the potential due to the main body or the particle M_1.

Potential at P due to M_1 can be written using

$$U = -\frac{GM_1}{r_1}$$

where r_1 is; r_1 equal to magnitude of vector r_1 equal to R – R_1 vector. So these are the information available to us and as usual, you know that if you want to write the equation of motion in the reference frame, the inertial reference frame, so we can write $M_1 \times R_1$ double dot is the motion of particle capital which is given by P_1 here this is P_1 so

$$M_1\ddot{\vec{R}} = \frac{mM_1G}{r_1^3}\vec{r}_1$$

sign will be plus because the force is directed along r_1 direction.

Similarly for the small mass we can write this as

$$m\ddot{\vec{R}} = -\frac{mM_1G}{r_1^3}\vec{r}_1$$

$$m\ddot{\vec{R}} = -m\,\nabla U$$

so there is no problem in this we have done it numerous times, so this the same thing. So using this then the equation of motion for the smaller mass can be written as; if we use this, so see here for this is m multiplied by $m\ddot{\vec{R}}$ t and then on the right hand side what are the terms which will appear.

So we know that from our earlier experience or the earlier discussion we have written this as $-m\,\nabla U$. So you will get this quantity this expression from here. So that the motion of mass m under the action of what is the gravitational acceleration of mass M_1 can be written as shown above.

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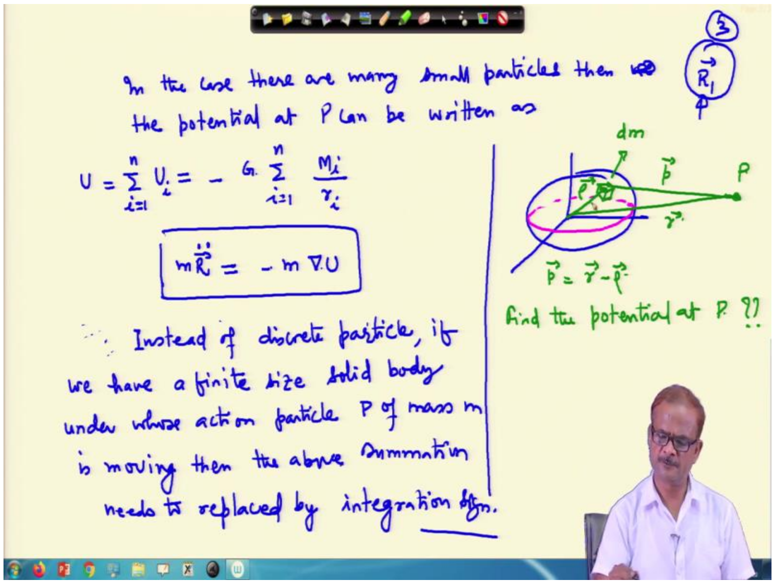

In that case there are number of masses there are many small particles then the potential at P can be written as U equal to r; we can write here just like way U only to represent it better. I will write it as you want. So here I will write as U_i and U will indicate us i equal to 1 to n summation over n number of particles support inertia due to n number of particles of this is -G r I I have used r given for this notation for better representation instead of using R we will use the r.

$$U = \sum_{i=1}^{n} U_i = -G \sum_{i=1}^{n} \frac{m_i}{r_i}$$

So, this is then the potential due to a number of particles and therefore then the equation of motion for this small mass this can be written as; we go back and use this equation. So this is

$$m\ddot{\vec{R}} = -m \nabla U$$

we have this quantity we have written it as 1 where t_1 is defined like this. So, the format of the equation does not change it remains the same. This constitutes our basic principle how the potential at a point P can we work out.

Now if the body or either if instead of a body instead of particles discrete particles, instead of discrete particles if we have finite size solid body Under whose action particle P of mass m is moving. Then the above summation needs to be replaced by integration sign. So now we are talking about; here we have a body of arbitrary shape and there may be elementary mass located in this. Here it is shown very big but its infinitesimally small we can assume the distance from here to here we can show it by $\vec{\rho}$.

This is the vector to this point and then of course, we have the point here. P is the radius vector we can show has r. So this distance we can write as P, so find the potential at P. What will be the potential at that point? So let us assume that this mass is dm elementary mass based on that then we can calculate the potential here and then we integrate over the whole body.

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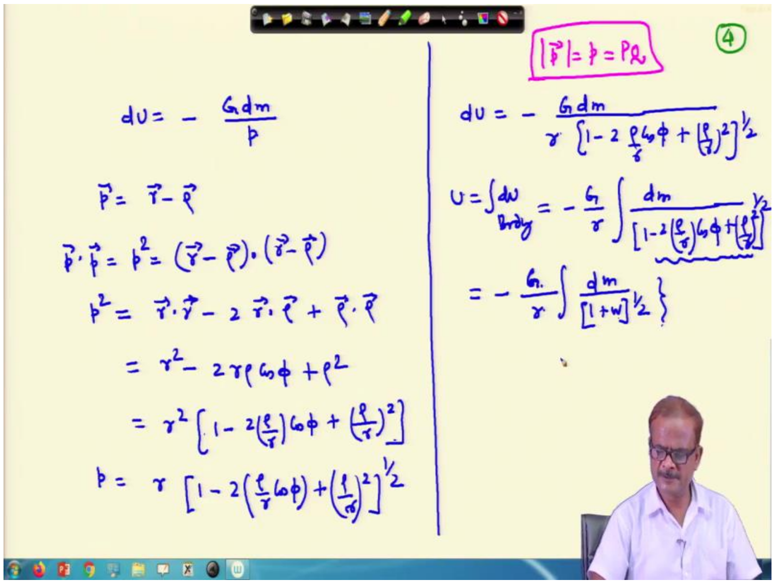

$$dU = -\frac{Gdm}{p}$$

So, the distance p which is this case is nothing but we have written this as Q and right and this point as O and this point we can write as Q. So PQ distances p magnitude equal to p this is nothing but the distance PQ. So this gives such potential and what is the value for the P? For the P already we have written P the vector r - ρ and if we take dot product with the self this become p^2.

$$\vec{p} = \vec{r} - \vec{\rho}$$

So throughout we have taken the vector approach this is not the only way of doing this coloured approach is also there and many of the books may be discussing this matter in some different way, but this is one of the most elegant way of doing the change. This becomes then

$$p^2 = \vec{r}.\vec{r} - 2\vec{r}.\vec{\rho} + \vec{\rho}\vec{\rho}$$

$$= r^2 - 2r\rho cos\phi + \rho^2$$

$$p = r\left[1 - 2\left(\frac{\rho}{r}cos\phi\right) + \left(\frac{\rho}{r}\right)^2\right]^{1/2}$$

this angle is ϕ here.

This is ϕ + ρ^2 when we take r^2 common so this will be 1 - 2 ρ by r cosϕ and therefore potential at P due to the elementary mass this can be written as

$$du = -\frac{Gdm}{r\left[1-2\frac{\rho}{r}cos\phi+\left(\frac{\rho}{r}\right)^2\right]^{\frac{1}{2}}}$$

if we have to find out the potential due to the whole body. So we just need to submit over the whole body and then r is an independent of the integration of the body because it depend the integration is different from the mass where the ρ will factor.

$$U = \int du = -\frac{G}{r}\int\frac{dm}{\left[1-2\frac{\rho}{r}\cos\phi+\left(\frac{\rho}{r}\right)^2\right]^{\frac{1}{2}}}$$

$$= -\frac{G}{r}\int\frac{dm}{[1+w]^{\frac{1}{2}}}$$

Therefore we can write this as 1 - 2 rho by r cos ϕ + ρ by r^2 to the power 1 by 2 we can write it in this way x has nothing to do with that we can use some other notation maybe. Let us say we write here w. Where w is nothing but the quantity from this place to this place.

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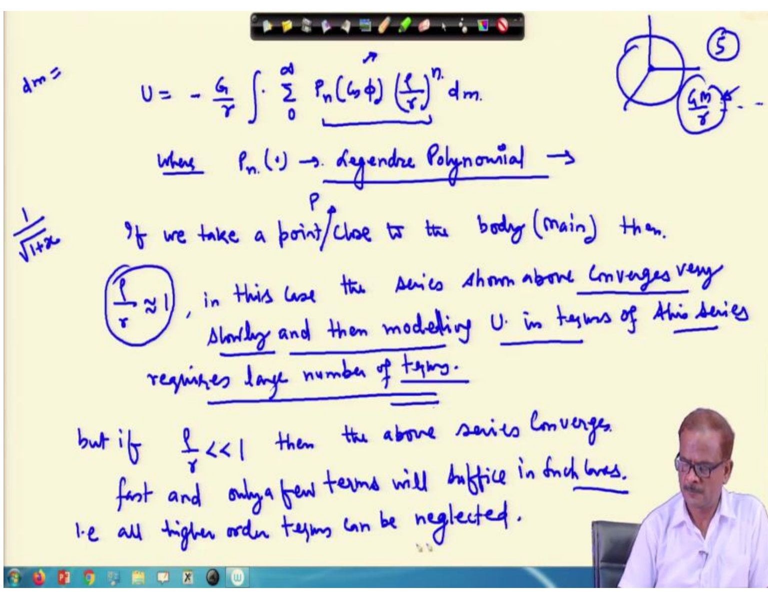

And this expression has got standard representation in mathematics but we will do it by expanding here rather than just going to that first, still let us write this then U becomes –G by r it can be represented as ok and what is dm? dm is also a function of say if the mass is the density is not in uniform. So this will the mass of this is small mass dm real elementary volume will be taking so this will depend on that the density also.

But this is not we are featuring here we go here in a simple way, but of course in the case of if you want to have like your body where the density from one place to another place is a deep thing and if you are ready to model; take the mathematical complexity into account and then you have the observation data available. So it is possible that you can say where the density is more and where the density is less.

This has happened in the case of the Apollo satellites. Apollo satellite and still it is going on those work. Many of the observation data for the lunar Satellite from there they were reduced and then the perturbation acceleration where we worked out and using that it was shown that some of the places where the mass moving may be more dense. So, those kind of analysis it is possible. So you will find it interesting this topic as we progress.

So, 1 by 1 + x under root what we have written, so this whole thing except dm it can be represented in a series form like this is

$$U = -\frac{G}{r} \int \sum_0^\infty P_n cos\phi \left(\frac{\rho}{r}\right)^n dm$$

where P_n the Legendre polynomial. So, this is one of the representation and this representation if we reduce in terms of Legendre function there, the spherical harmonics, they come into picture. This is another way of describing the things.

But the approach right now we are taking it is a little simpler. So, if we take a point close to the point P close to the body or the main body then ρ by r will be of the range around 1 in that case in this case the series shown above converges very slowly and then modelling U in terms of the series require large number of terms. So, in the case of the Moon once the lunar gravity modelling is done. So at that time there are lots of terms from this place it can be expanded.

Rather than expressing here in this format is expressed in terms of Legendre function where the spherical harmonics they come into picture and in that decide the main gravity term which we write as $\frac{Gm}{r}$ many other terms they appear, lots of other terms will appear and using that then the lunar gravity modelling or the potential due to the moon or the perturbation arising under lunar satellite due to the moon. So there are worked out.

So this is because if this reason, if you are in the nearby region where the $\frac{\rho}{\mathrm{r}}$ is nearly equal to 1. So, in this case the series converges very slowly and then the modelling of U in terms of this series requires a large number of terms. So, it may be in terms of lakhs. But if $\frac{\rho}{\mathrm{r}}$ this is much greater than that much less than 1 means r is quite large as compared to ρ much less than 1 then the above series converge fast and only few terms only a few terms will suffice in such cases that is all higher order terms can be neglected.

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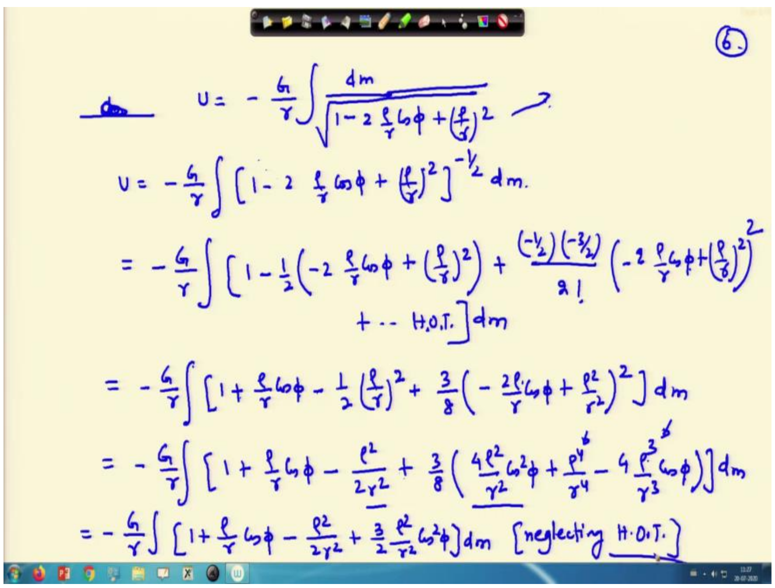

So, this 1 by we required we have written; we must go back here this part already we have written here P equal to the quantity in this place. In this we are going to work out using the Binomial expansion we can work it out.

$$U = \int du = -\frac{G}{r}\int \frac{dm}{\left[1-2\frac{\rho}{r}\cos\phi+\left(\frac{\rho}{r}\right)^2\right]^{\frac{1}{2}}}$$

$$U = -\frac{G}{r}\int \left[1-2\frac{\rho}{r}cos\phi+\left(\frac{\rho}{r}\right)^2\right]^{-\frac{1}{2}} dm$$

This is the equation that we are having and if we expand it to a few terms that is

$$= -\frac{G}{r}\int\left[-\frac{1}{2}\left(-2\frac{\rho}{r}cos\phi+\left(\frac{\rho}{r}\right)^2\right)+\frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{2!}\left(-2\frac{\rho}{r}\cos\phi+\left(\frac{\rho}{r}\right)^2\right)^2+\cdots H.O.T\right]dm$$

this way there will be an infinite number of terms in this expansion.

And this expansion is not very difficult to work out using Taylor series you can do the expansion of any function this kind of organization very easy to do it. And this is a very standard equation available in the books -2 ρ by r cos ϕ plus and so on the higher order terms 1 plus now if we break the bracket here, so this becomes by cos ϕ plus minus sin - 1 by 2 ρ by r^2 plus from this place we have 2 4 8 3 by 8 times -2 ρ by r cos^2 ϕ dm.

$$= -\frac{G}{r}\int[1+\frac{\rho}{r}cos\phi-\frac{1}{2}\left(\frac{\rho}{r}\right)^2+\frac{3}{8}\left(-\frac{2\rho}{r}cos\phi+\frac{\rho^2}{r^2}\right)^2]dm$$

3 by 8 when you square it becomes 4 ρ divided by r^2cos^2 ϕ ρ^4 r - 2 into 2, 4 $\frac{\rho}{r^3}$ cos ϕ. So if we know the higher order terms means the third order and the fourth the third degree and the 4 degree terms will be ignoring it we do that cos ϕ that means only this term and this term we are accounting. So, then this becomes minus ρ^2 by $2r^2$+ 3 by 2 ρ^2 by r^2cos^2 ϕ. Here we are neglecting higher order terms.

$$U = -\frac{G}{r}\int[1+\frac{\rho}{r}cos\phi-\frac{\rho^2}{2r^2}+\frac{3}{2}\frac{\rho^2}{r^2}cos^2\phi]dm$$

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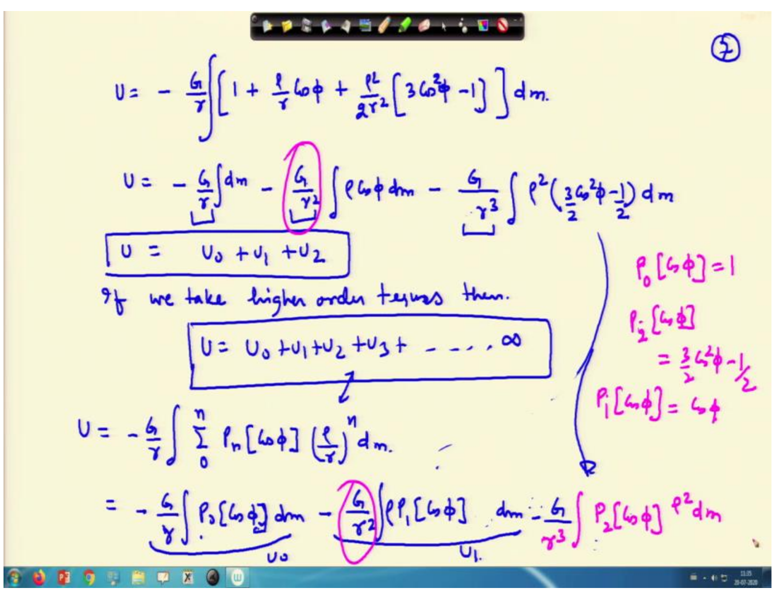

$$U = -\frac{G}{r}\int [1 + \frac{\rho}{r}cos\phi + \frac{\rho^2}{2r^2}[3cos^2\phi - 1]]dm$$

$$U = -\frac{G}{r}\int dm - \frac{G}{r^2}\int \rho cos\phi dm - \frac{G}{2r^3}\int rho^2(3\cos^2\phi - 1)dm$$

this we can represent as

$$U = U_0 + U_1 + U_2$$

So, if we take higher order terms then

$$U = U_0 + U_1 + U_2 + U_3 + \cdots\infty$$

if you remember the term we have written -G by r, U equal to in terms of the Legendre polynomial from 0 to n the summation was rho n cosϕ and then we had ρ by r to the power n dm this is what we have written.

$$U = -\frac{G}{r}\int \sum_0^n P_n cos\phi \left(\frac{\rho}{r}\right)^n dm$$

So, this implies that the first term is equivalent to if we now expand it so this will be P_0 C$os\phi$ and dm – G by r summation here we are breaking the summation so this is P_1 c$os\phi$ ρ by r dm and plus so on that will immediately you can see that this term is corresponding to here U_0 this term is corresponding to U_1 and similarly the next one will correspond to U_2 and so on. This two are the same. Ok. This is the way it should be expanded.

$$U = -\frac{G}{r}\int P_3 cos\phi dm - \frac{G}{r}\int P_1 cos\phi \frac{\rho}{r} dm + \cdots$$

So this is one of the approach but there is another approach in terms of the Legendre function. That is done where we get in terms of the spherical harmonics if time permits I will take that topic or otherwise I will just give you the supplementary material on that particular topic because you already have crossed the number of lectures and we are many other things to cover. But this is the idea involved in working with this problem.

This is the main basic idea. Here instead of writing 2 here in this place we can also write a letter. It is a little more convenient to represent because this is going like this way. So, immediately you can see that what does it mean if we expansion we are expanding so the second term the third term will appear here. So this is corresponding to U_2 so that means r^3ok you here the; let us say that expand it one more time here and write.

So that also you get a idea so the next term will –G by; this ρ by r we can take it outside r we can take outside. So ρ will come here and r we can check out from this place and write here G by r^2 you can see that G by r^2 appearing here. This is also appearing here in this place and just inside this rho P_1 cosϕ and dm multiplied by dm - G similarly here in this place if we expand it, it will be and then P_2 cosϕ ρ^2 dm r^2 we can take it outside and becomes rQ.

Immediately from this place we can refer that P_0 cosϕ from this place immediately you can see this quantity is equal to 1. From here we can guess that this quantity will be P_0 cosϕ that will be equal to 3 by 2 $\cos^2\phi$ - 1 by 2 and similarly this is P_1 not P_0 P_2 cosϕ and P_1 cosϕ from this place we see that this is simply cosϕ. So by this way series can be if you are able to handle that much of mathematics we can see from this place how this works.

But in mathematics this Legendre polynomial with the well researched and well documented any book you can pick up and the books like the Physics on Engineering Mathematics we can refer to those books and you will get these materials. So we stop here and then we will continue in the next lecture. Thank you very much.