Space Flight Mechanics
Prof. Manoranjan Sinha
Department of Aerospace Engineering
Indian Institute of Technology, Kharagpur

Lecture No - 64
General Perturbation Theory (Contd.)

Welcome to lecture number 64 we have been discussing about the potential due to the body of arbitrary shape and later on we reduced it to a simpler format. Now we use that equation and workout for an Oblate spheroid.

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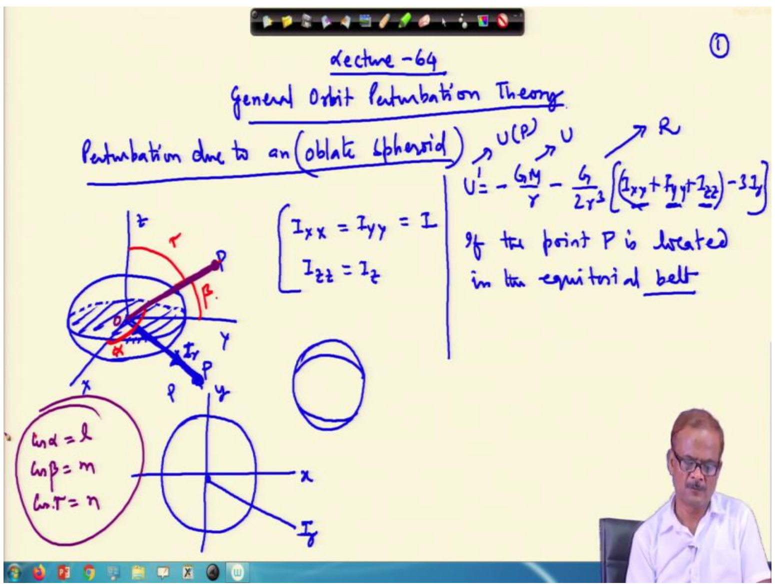

What does this mean? That in the case of the earth do it secured shift but safe I can approximate it in this way. I can show it like this, this way in this place that means here if I write a x, y and z axis. So, I_{xx} I_{yy} this will be equal to I and I_{zz} it will be different, which we will simply writers I_z. What does this mean that this equatorial belt, this is perfectly a circle, this is your x and this is y this is perfectly a circle.

And thereafter on the top, in this, here if it is your sphere and so sphere we are U just face it and make it like this and this becomes an oblate spheroid. It can be described by using this inertia terms and now if U use this then our; the expression for U can be written as - GM by r this does not change but the next term is going to change which is the perturbation term – G by 2 r ^{3} and this was I_{zz} + I_{yy} + I_{xx}– $3I_r$.

$$U = -\frac{GM}{r} - \frac{G}{2r^3}\left[I_{zz} + I_{yy} + I_{xx} - 3I_r\right]$$

$$R = -\frac{G}{2r^3}\left[2I_0 - 3I_r\right]$$

So, if the point, P is located say that P is located somewhere here in the equatorial belt. The point P this is the potential at point P which we are writing here in this case as P' and this particular part we are writing as U and this part we are writing as R, if the point P is located in the equatorial belt. So in this case your equatorial belt it will come here is it not. What will be the I_r in that case? This we have to guess ok and it is easy to do from this place.

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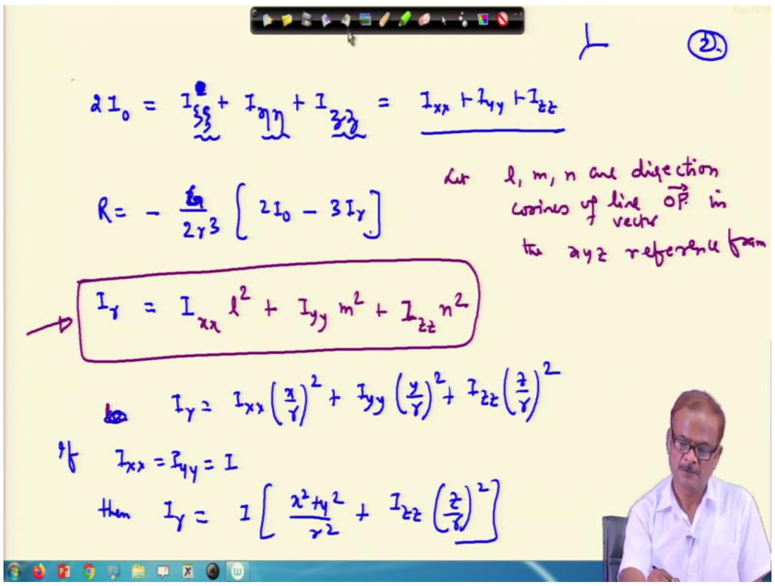

Which will of course we will do once we describe it in a particular format and thereafter will do that? So for the point in; let us write it is general way first and then will come to that. Equatorial belt means you are now located here as shown along this line. Suppose this line is the point P is located. So what will be the moment of inertia about this point? So there this becomes I_r in that case moment of inertia about this line.

If this line is somewhere else then it becomes different. So already we have looked into that 2I 0 we have written has $I_{\xi\xi}$ + $I_{\eta\eta}$+ $I_{\zeta\zeta}$ and this was the co-ordinate notation used for this. So it is a convenient to write this as I_{xx} + I_{yy}+ I_{zz} rather than using this notation and whatever the earlier we have written it because of that only because of ξ, $\eta$and ζ they are there along the x y z direction and therefore with the same thing.

So, the perturbation term R is equal to - 2 by; - G by 2r ^{3} 2I_0 – 3I_r. The moment of inertia if the equatorial plane if we are taking about this line point in the equatorial plane that is in this plane lying in the xy plane. Then it is a simpler case. Ok, but if we take any other point in a general point and letters a the P is located here and then the cosine angles of this P the angle from here to here and the angle from here to here and the angle from here to hear these are called the associated angle and from with this we associate the direction cosines.

And so if we write this is α this angle β, and this angle is gamma because which is the angle made by the OP vector? This is the OP vector this point O and this is P here located. So OP vector is making angle α, β and γ So cos α we will write this as l, cos β as m and cos γ as n direction. So there are direction cosines of line OP. Let l, m and n are direction cosines of line OP or vector OP. In the xyz reference frame so how it is a given this is given as I_{xx} l^2 + I_{yy} m^2 + I_{zz} n^2 where l, n, m as described these are the direction cosines.

And this issue how this result is coming. Again if I try to explain it, it will take longer time. So better again, I am referring you to my lecture on the Latitude Dynamics and Control. So if you are going on look into the moment of inertia part, so there I have worked out all these details and how this expression arises. So, here l is; it is very much visible from this co-ordinate is xyz suppose so the Cos α, then this will become equal to l equal to x by r these distances r from here to here. This is r vector so it is x by r generally this is y by r and this will be z by r.

$$I_r = I_{xx}l^2 + I_{yy}m^2 + \mathrm{I}_{zz}\mathrm{n}^2$$

$$I_r = I_{xx}\left(\frac{x}{r}\right)^2 + I_{yy}\left(\frac{y}{r}\right)^2 + I_{zz}\left(\frac{z}{r}\right)^2$$

$$I_r = I\left[\frac{r^2 - z^2}{r^2}\right] + I_{zz}\left(\frac{z}{r}\right)^2$$

So, therefore we can write here I_r is equal to I_{xx} x by r ^{2} + I_{yy} y by r ^{2} + I_{zz} z by r ^{2} and if I_{xx} is equal to I_{yy} equal to I then the above equation gets reduced to I_r become equal to I times x ^{2} + y ^{2} divided by r ^{2} + I_{zz} z by r ^{2}.

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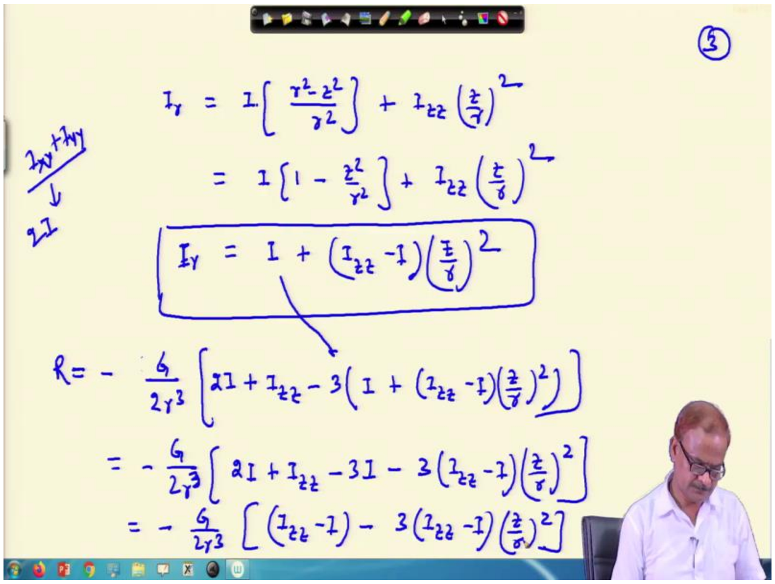

$$I_r = I + (I_{zz} - I)\left(\frac{z}{r}\right)^2$$

$$R = -\frac{G}{2r^3}[2I + I_{zz} - 3(I + (I_{zz} - I)\left(\frac{z}{r}\right)^2)]$$

Here this quantity x ^{2} + y ^{2} becomes we can write as r ^{2} - z ^{2} and then this divided by r ^{2}. So this is the first time here and then the next term which is plus $I_{zz}\left(\frac{z}{r}\right)^2$. So what we get from this place, this is 1 - z ^{2} by r ^{2} + I_{zz} z by r so this is I_y. So for this simpler case we can represent it like this. With this now we can insert this result in the expression for the perturbation potential. So, the perturbation potential R equal to – G by 2 r ^{3} and then the quantities inside the bracket where $2I_0$ which is nothing but this quantity here so this becomes I_{xx} becomes 2I I_{xx} + I_{xx} we are replacing these by 2I.

So this gets 2I and plus I_{zz} and then -3I_r, so this quantity is your I_{zz} + I_{yy} + I_{xx} which is nothing but these quantities and the for the whole bracket then this can be represented as 2I + I_{zz} – 3Ir we are using this expression where 2I + I_{zz} -3I_r 3 times I_r we insert from this place I + I_{zz} - I_z by r ^{2} G by 2 r ^{3} 2I + I_{zz} – 3I -3 z by r ^{2} from immediate we can write here I_{zz} – z by r ^{2}.

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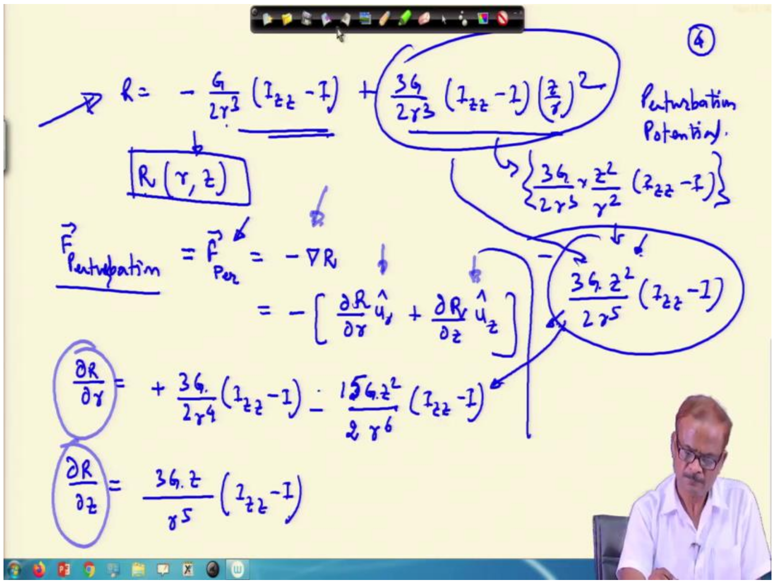

So thus we have R equal to - G by 2 r ^{3} and then multiplied by I_{zz} - I and the other term this is that gets minus and minus sign from this place that will make it plus +3G by 2 r ^{3} and I_{zz} – I z by r ^{2} so this is the perturbation potential. And what it says that R is a function of the capital R which is a perturbation potential which is only a function of r and z that means it depends on the radial distance and z other are not coming into picture.

$$R = -\frac{G}{2r^3}[(I_{zz} - I) + \frac{3G}{2r^3}(I_{zz} - I)\left(\frac{z}{r}\right)^2]$$

And therefore if we will try to find out the perturbation force F perturbation which will simply write as F P perturbation force of; let us say if so this will be equal to – dou R by or – del R as per our earlier notation rewrite it minus del R. And because this is only a function of r and z so we can write simply this

$$\vec{F}_{per} = -\nabla R$$

$$= -[\frac{\partial R}{\partial r}\hat{u}_r + \frac{\partial R}{\partial z}\hat{u}_z]$$

From here we have got the perturbation potential.

Sorry, this is R this is R and here also this is R. Now which of these quantities can be determined so $\frac{\partial R}{\partial r}$ this quantity we differentiate this take the partial differential of this. So this will be minus this becomes + 3G by to 2 r^4 I_{zz} – I, I hope you are aware of partial differential and the next term on this, from we need to simplify little bit. This term is 3G by 2 r^3 times z^2 by r^2 and I_{zz} – I.

This gets reduced to 3G z^2 divided by 2^5 I_{zz} - I and this term then we are replacing with this particular one. So we have to differentiate this quantity take the partial differential of this. So, partial differential of this will then appear as 3G with respect to r this becomes 3 into 5 this becomes 15 and then minus sign will appear in this point 15G z^2 divided by $2r^6$ – I_{zz} – I.

$$\frac{\partial R}{\partial r} = \frac{3G}{2r^4}(I_{zz} - I) - \frac{15Gz^2}{2r^6}(I_{zz} - I)$$

$$\frac{\partial R}{\partial z} = \frac{3Gz}{r^5}(I_{zz} - I)$$

And $\frac{\partial R}{\partial z}$ then here there is no z present. This is present in R but we did not differentiate that way. We just need to differentiate with respect to z here in this place. Ok and I will do that we can write this as 3 into 2 cancel 3G z divided by r^5 I_{zz} – I this comes with the plus sign ok now we can; this two expressions we can utilize and insert in the expression for acceleration and that will give us the perforations force. And that perturbation is along the u_r direction and u_z direction.

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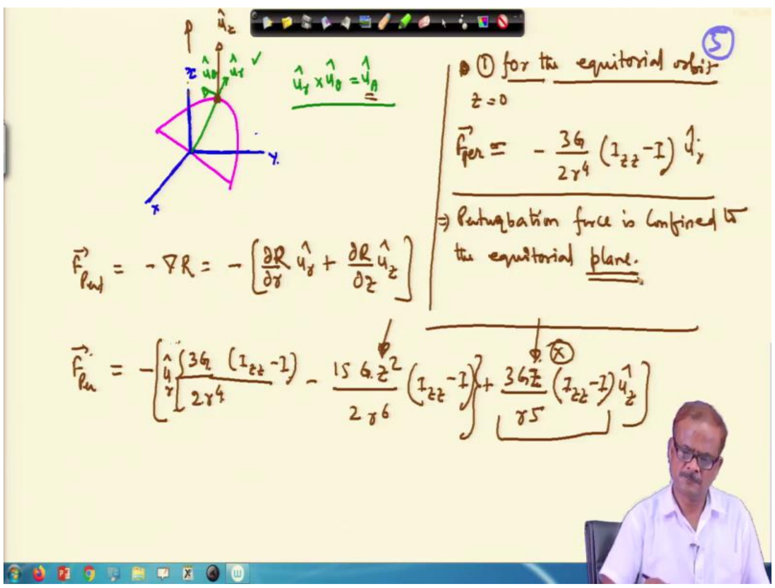

$$\hat{u}_r \times \hat{u}_\theta = \hat{u}_A$$

So what does this mean that if we have this Orbit and already if you remember we have the arbitral located here. So this is the point P u r is located here in this direction. And this is $\hat{u}_\theta$ and $\hat{u}_A$ is perpendicular to $\hat{u}_r \times \hat{u}_\theta = \hat{u}_A$ perpendicular to both of them. So here in this case the $\hat{u}_r$ is appearing and z is appearing but here the z is in a direction; z is lying along this direction $\hat{u}_A$ is perpendicular to its orbit.

But z is along parallel to the direction parallel to the z direction. So here you have u_z curve; when we will need to convert this in proper format, so let us go ahead and do this work. So

$$\vec{F}_{Pert} = -\nabla R = -[\frac{\partial R}{\partial r}\hat{u}_r + \frac{\partial R}{\partial z}\hat{u}_z]$$

Insert this quantities we have worked out earlier. So from here this place 3G by 2 r to the power 4 3G divided by 2 r^4 I_{zz} - I -15G z ^{2} -15 Gz ^{2} divided by $2r^6$ I_{zz} – I.

$$= -\left[\frac{3G\,(I_{zz}-I)}{2r^4} - \frac{15\,Gz^2}{2r^6}(I_{zz}-I) + \frac{3GI}{r^5}\,(I_{zz}-I)\hat{u}_z\right]$$

And the last one this is 3G z by y^5 this comes with the Plus sign $\hat{u}_z$ and this 2 come with $\hat{u}_r$ and this one multiplied by $\hat{u}_z$ so this is $F_{perturbation}$. Now with this we are ready to work out our Lagrange planet equation. We write here first for the equatorial Orbit for the equatorial orbit z equal to 0 if you do that, so this term will vanish so this term will drop out only leaving out ok and z is also present here, so this term will also dropout.

$$\vec{F}_{perturbation} = -\frac{3G}{2r^4}(I_{zz}-I)\,\hat{u}_r$$

Leaving us with only one term let us check it ok fine we have done one error has crept in here so we need to correct that. So the expression is ok I checked it here what we are getting 3G z This part is for the equatorial orbit z equal to 0 because z is appearing here. And also the z is appearing here. So both this terms they will drop out and therefore the F perturbation this gets reduced to -3G by $2r^4$ I_{zz} – I$\hat{u}_r$.

So, this implies that the perturbation force this implies force is confined to the equatorial plane. So, the rest of the things will discuss it in the next lecture. Thank you very much for listening.