Space Flight Mechanics
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Lecture No - 65
General Perturbation Theory (Contd.)

Welcome to lecture 65, so we have been working on the potential due to an oblate spheroid and that we took the partial differential of that and form the perturbation acceleration.

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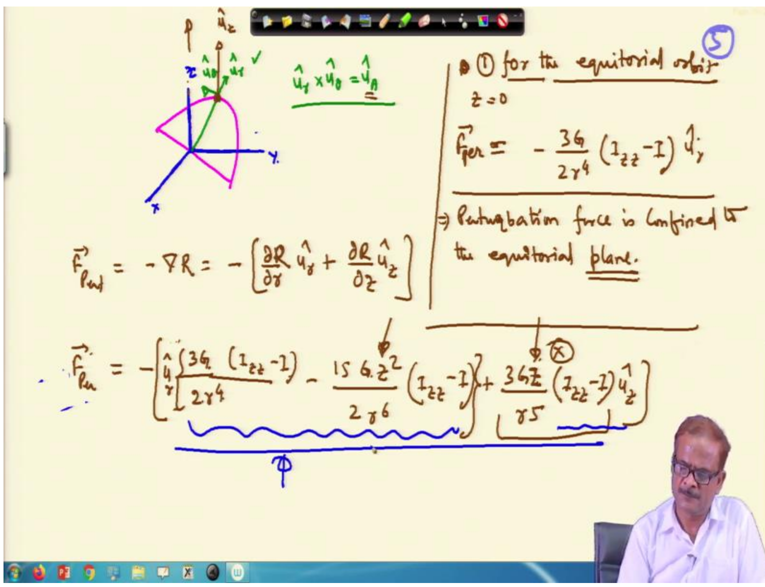

The equation we have written earlier this was our equation. This is the perturbation acceleration or perturbation force equation. We are of course; we have not multiplied by m so this is the perturbation force per unit mass which we multiply by small m so this becomes the perturbation force on the satellite which of course I am not fighting here and F is nothing but it is the force per unit mass which is acceleration.

So, these 2 terms appearing here this two terms I will combine and write in a single term and this term will be the separate term if this is from the last lecture. So, if we combine it, it can be written here in this way.

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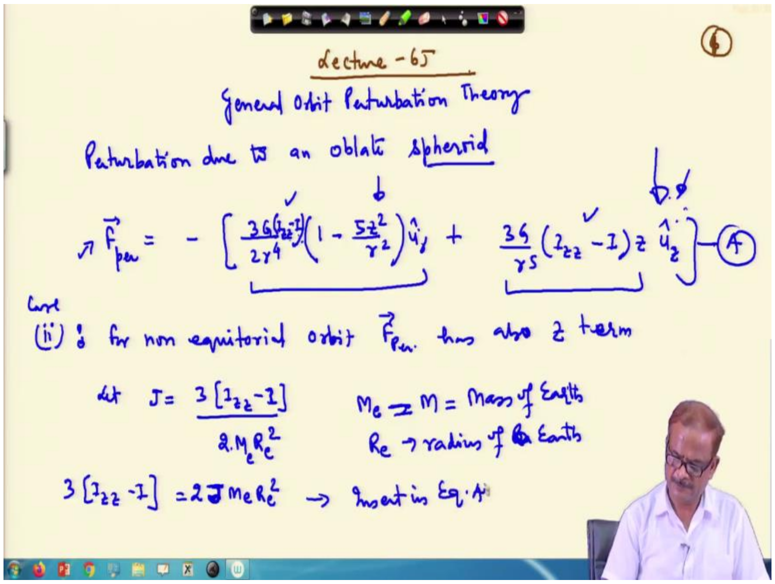

So, that the combine term looks like this and the next term; we just copy likes this. So now if the point is; so the case 2 for non equatorial orbit F perturbation has also z term as shown here. The perturbation acceleration is along z direction. Here the z term is appearing, in this plane z is appearing this is another issue but the force along the z direction this is shown here in this place. So now we do one more simplification ok here we are missing one particular part.

$$3[I_{zz} - I] = 2JM_eR_e^2$$

$$J = \frac{3[I_{zz} - I]}{2M_eR_e^2}$$

This is 3G (I_{zz} – I) which is also appearing here it is also here in this place this particular part. Now let us write I_{zz} - I divided by $2M_eR_e^2$ where M_e is the mass of the earth and where R_e is the radius of the earth. So, M_e is actually already M was appearing which we have left out.

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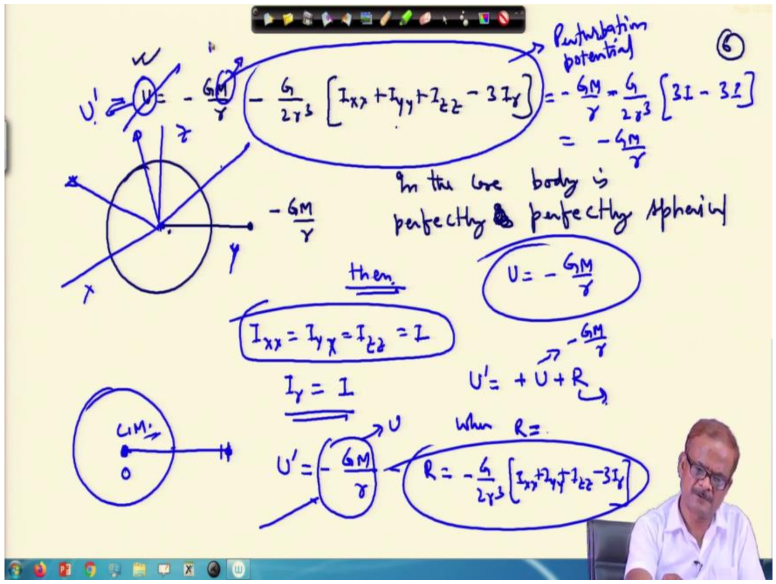

If you look here this M nothing, but the mass of the earth this is M_e that is your main body and under its gravitational acceleration the satellite is moving. So M_e is the quantity M which is the mass of earth and R_e is the radius of Earth or the mean radius of earth. From this place immediately we can see that we can write 3 times I_{zz} - I equal to $2M_eR_e^2$ 2J times M_e R_e^2. And we will insert here in this equation let us name this equation has A insert in equation.

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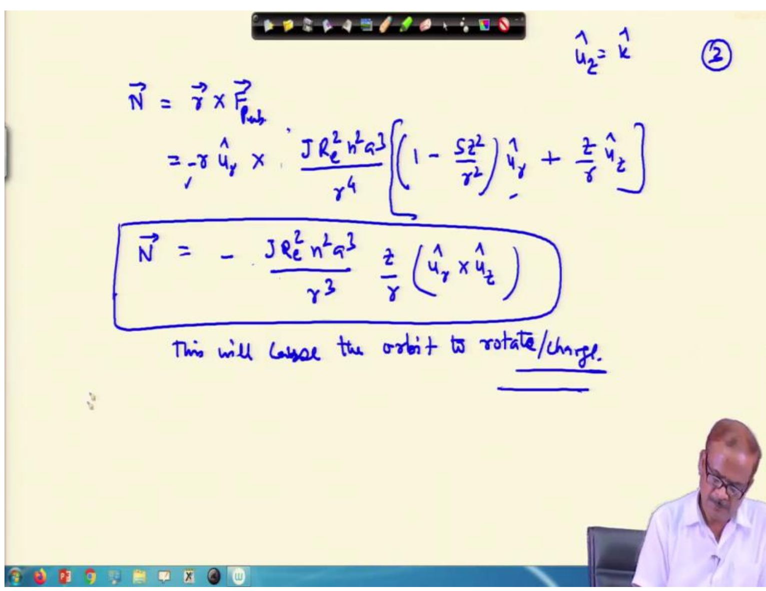

So F perturbation gets reduced - CG ok for G also we are removing G will, G is present here let us keep it separately ok because G times M_e is equal to μ_e which is the gravitational constant for the earth it is not the universal gravitational constant. Universal gravitational constant is G, so if you do this our F perturbation then gets reduced we will have to write it here 2J M_e R_e^2 from this place times G divided by $2r^4$ times 1 - $5z^2$ by r^2 $\hat{u}_r$ M variable also we replace 3G times 2J M_e R_e^2 divided by r^5 and then multiplied by z $\hat{u}_z$.

$$\vec{F}_{Per} = -[\frac{2JM_eR_e^2G}{2r^4}\left(1-\frac{5z^2}{r^2}\right)\hat{u}_r + \frac{3G2JM_eR_e^2}{r^5}z\hat{u}_z]$$

Whatever the simplification we can do here we will do that. These 2 and 2 cancels out. So, this gets reduced to now G times M_e noting but your μ_e and you remember that n^2 is μ by a^3 where n is the mean angular velocity and μ here in this case nothing, but your quantity here maybe; so we can utilize this quantity and μ_e we can replace in μ^2. If we do that we will do it in two steps. So, this is J times R_e^2 μ_e divided by r^4 r^2 and same way this place 6 times μ_e JR_e^2 divided by r^5 times z $\hat{u}_z$ n^2 a^3.

$$\vec{F}_{Per} = -\left[\frac{JR_e^2\mu_e a^3}{r^4}\left(1-\frac{5z^2}{r^2}\right)\hat{u}_r + \frac{6JR_e^2n^2a^3}{r^5}z\hat{u}_z\right]$$

Where a is the semi major axis of the orbit so this is the perturbation force. So, 3 factors repeated here so 3 is not here in this place. In this we have done a 3 factor we are replacing, 3 will not feature in here in this place. This is ok till this place this is ok and thereafter 3 will not feature in here. So this 3 will go because 3 is also present here in this place and also here in this place which we have already replaced by this quantity. So this 3 will go.

So here we inserted 6 and we have 2 present here ok so this constitutes our basic equation which we can little simplify and this way taking out J $R_e^2n^2$ a^3 and r^4. So if you take it outside this gets reduced to 1 – 5 z^2 divided by r^2 $\hat{u}_r$ + z by r $\hat{u}_z$ F perturbation was enforced. So it looks now in a simple format which we can work with.

$$\vec{F}_{Per} = -\frac{JR_e^2\mu_e a^3}{r^4}\left[\left(1-\frac{5z^2}{r^2}\right)\hat{u}_r + \frac{z}{r}\hat{u}_z\right]$$

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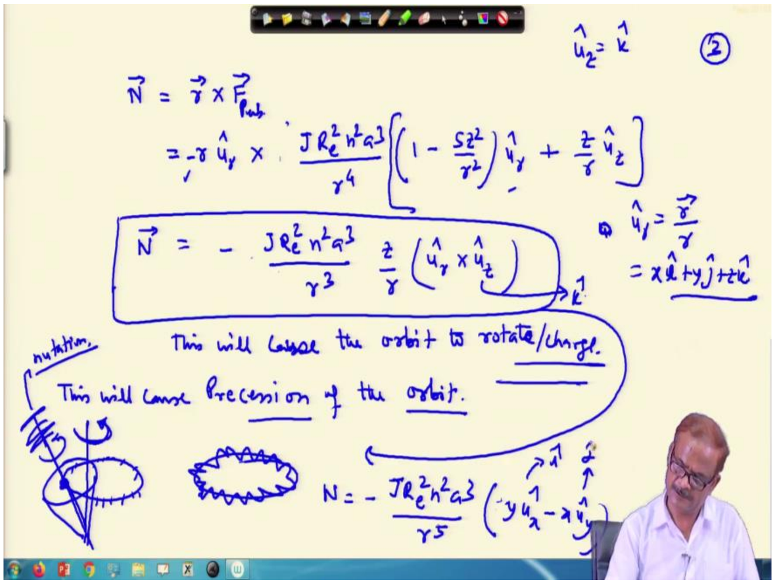

In the next step we will check whether this force F has any twisting moment. So if we write this r times $\hat{u}_r$ and copy it from the previous page F perturbation which is –J R_e^2 m_e^2 a^3 J minus sign already be accounted here J R_e^2 n^2 a^3 divided by r^4 times 1 - $5z^2$ by r^2 $\hat{u}_r$ + z by r$\hat{u}_z$ z by r$\hat{u}_z$. So the cross product with r will vanish and it will leave only the term r times n bracket is outside and bracket is here in this side.

$$\vec{N} = \vec{r} \times \vec{F}_{per}$$

$$\vec{N} = -\frac{JR_e^2 n^2 a^3}{r^3} \frac{z}{r} (\hat{u}_r \times \hat{u}_z)$$

So this gives us J R_e^2 n^2 a^3 divided by r^3 q times z by r $\hat{u}_r$ times $\hat{u}_z$, $\hat{u}_z$ is nothing but $\hat{k}$. Immediately we can see that there is a twisting moment present on the orbit that means orbit will rotate. So, this will cause the orbit to rotate/ change and this is what we are looking for. I am going to explore this. So this will cause precession of the orbit. Precision is a motion where you can see in the case of the top a moving top rotating top it is a spinning on its axis and then also about the vertical axis.

Because of movement about the vertical axis is called the procession. So your spin vector which is in direction. And this top is like this, this is going here in this. This point will be moving over this trajectory. So, this is a Precision motion and if in addition this spin axis if it is going here like this so that means it is motion, will then appear as; the spin axis will appear to move like this. This is the addition. This is called the nutation motion.

Again this is discussed in quite details for the rigid body dynamics in the satellite attitude dynamics control course. Ok so these twisting forces are going to cause the orbit to precise and this is what we are looking for and will work it out.

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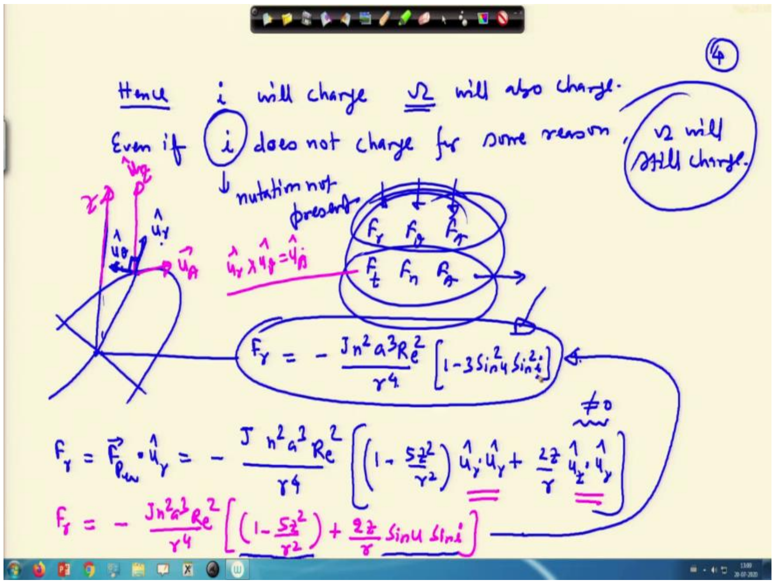

And hence i is the inclination of the orbit will change, i will change and also Ω. This is the location of the nodal line Ω will also change. So even if i does not change for some reason Ω will still change. So here i is not changing this is equivalent to nutation not present. While the precession will still be applicable and here

$$\hat{u}_r = \frac{\vec{r}}{r}$$

u_r is the quantity r by r so this quantity we can write as

$$\hat{u}_r = x\,\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

and accordingly we can work out and these quantities $\hat{k}$ u_z is.

So this can be worked out and I can write here itself this quantity will turn out to be N equal to – JR_e^2 n^2 a^3 by r^3. So this is r^5 combine the terms and then this will result in

$$\vec{N} = -\frac{JR_e^2 n^2 a^3}{r^3}(y\hat{u}_x - x\hat{u}_y)$$

So this is your $\hat{\imath}$ and this is your $\hat{\jmath}$. So, for getting back to the our actually issue which is the direction r, so we are looking for perturbation force in the r direction as we have already described ok Lagrange planet equation from there we have described in terms of F_r, F_θ and F_A.

And also we have described in terms of F_t, F_n and F_A. So these are 2 models we have worked out. This is $\hat{u}_\theta$ here in this direction. So in these directions we have to find the perturbation force F_r, so F_r is required F_θ is required and F_A is required if we use this model. This model used in the aerodynamics, but we would not be able to go into the aerodynamics forces and the sun radiation forces perturbation due to all of them.

So, it is not possible so it is a massive course. Already I have exceeded the 60 lectures while the total 60 lectures we have for this course. So F_r can be written as F perturbation dot $\hat{u}_r$ so component of the perturbation force in the u_r direction so on the previous page already, we have written that expression so we write it here n^2 a^3 times R_e^2 divided by r^4 this particular expression where you perturbation force 1 - $5z^2$ by r^2 and $\hat{u}_r$ dot $\hat{u}_r$ and +2z by r and then what is the quantity there which is $\hat{u}_z$.

So $\hat{u}_z.\hat{u}_r$ this quantity is non-zero. This quantity is not equal to zero because $\hat{u}_z$ is a vector along this direction it is parallel to this z direction while $\hat{u}_A$ up is something like this $\hat{u}_r$ + $\hat{u}_\theta$ equal to $\hat{u}_A$. So u_A and u_z they are not the same there is $\hat{u}_z$. This quantity obviously is equal to 1 but this quantity is nonzero. So this gives us $5z^2$ by r^2 + 2z divided by r and u_z times u_r this quantity we have to insert and this quantity is as I will show you this is sin u times sin i. Ok so this is to force along the r direction.

$$F_r = -\frac{Jn^2a^3R_e^2}{r^4}[\left(1-\frac{5z^2}{r^2}\right)\hat{u}_r.\hat{u}_r + \frac{2z}{r}\hat{u}_z.\hat{u}_r]$$

$$F_r = -\frac{Jn^2a^3R_e^2}{r^4}\left[\left(1-\frac{5z^2}{r^2}\right) + \frac{2z}{r} sinu\ sini\right]$$

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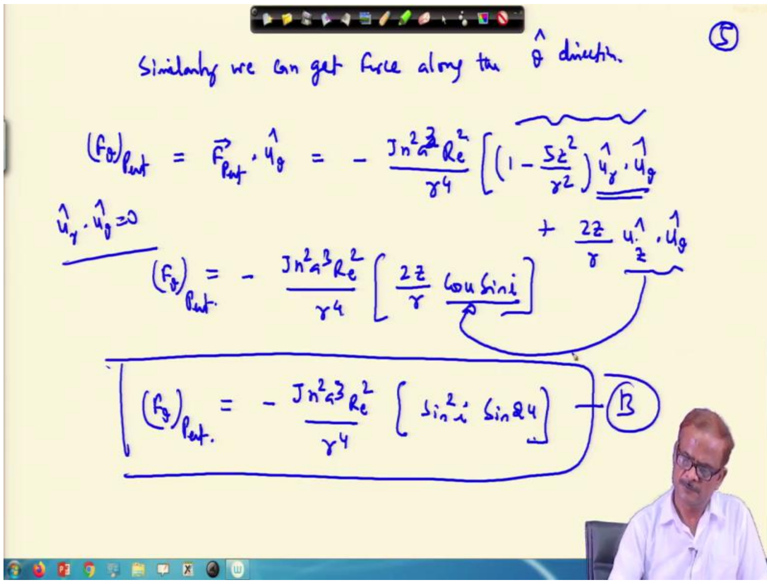

$$(F_\theta)_{Pert} = \vec{F}_{pert}.\hat{u}_\theta$$

Similarly we can get force per unit mass of earth along the perturbation force along the θ direction $\hat{\theta}$ direction. So F_θ perturbation for that case $\hat{u}_\theta$ this is a^3 R_e^2 divided by r^4 and $1 - 5z^2$ divided by r^2 in this is multiplied by $\hat{u}_r$ so this will be $\hat{u}_\theta$ and the other term is to 2z by r this along the z direction $\hat{u}_z$ $\hat{u}_\theta$. Now this quantity $\hat{u}_r$ $\hat{u}_\theta$ this quantity is zero $\hat{u}_r.\hat{u}_\theta$ this is 0 because they are mutually perpendicular to each other as shown here.

$$(F_\theta)_{Pert} = -\frac{Jn^2a^3R_e^2}{r^4}[(1-\frac{5z^2}{r^2})\hat{u}_r\hat{u}_\theta + \frac{2z}{r}\ \hat{u}_z.\hat{u}_\theta]$$

$$= -\frac{Jn^2a^3R_e^2}{r^4}[\frac{2z}{r}cosu\ sini]$$

They are mutually perpendicular to each other and in the same plane in the Plane of the orbit. And therefore this term will dropout this term will dropout and we are left with the second term from – J n^2 a^3 2z divided by r and $\hat{u}_z$ times $\hat{u}_\theta$ this quantity will be equal to cos u sin i. So, quantity is and quantity z and r also related to each other. So here let us say in this expression this I will be using again and reduced to a particular format.

$$F_r = -\frac{Jn^2a^3R_e^2}{r^4}[1 - 3\sin^2 u \sin^2 i]$$

But may be here itself I can write it, F_r then this can be summarised as - J n^2 a^3 R_e^2 divided by r^4 1 - 3 $\sin^2$ u times $\sin^2$ i this is the expression for F_r that means this whole thing which is appearing here this particular term and this term and this term all of them can be reduced here in this format and this we need to do. So, I will take time for that and maybe if this is not possible in this class in the next class also we can do. But first let us summaries the results here.

$$(F_\theta)_{Pert} = -\frac{Jn^2a^3R_e^2}{r^4}[\sin^2 i\; sin2u]$$

This is a F_r and then ok we have kept it blank this is F_θ perturbation and if we include z by r that we replace so this will finally get reduced to –J n^2 a^3 R_e^2 divided by r^4 times $\sin^2$ i times sin 2u F_θ perturbation this is equation B.

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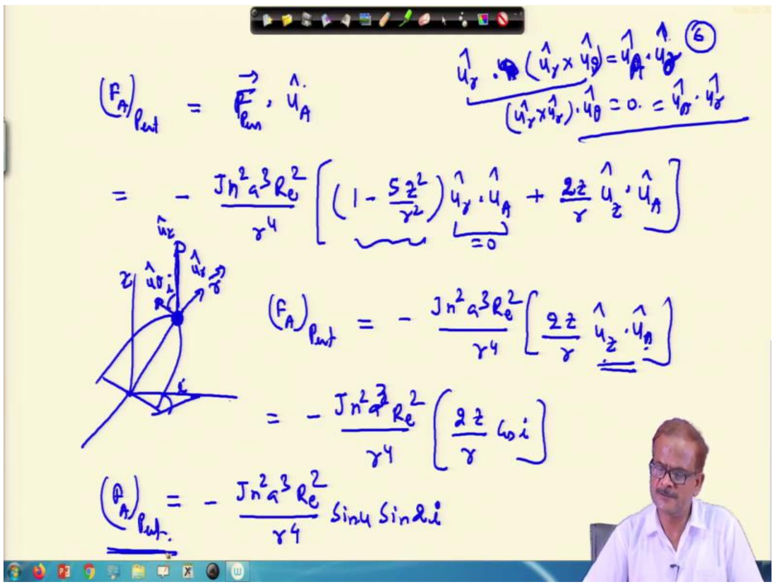

$$(F_A)_{Pert} = \vec{F}_{pert}.\hat{u}_A$$

And this we name as equation A ok the same way the perturbation along the perpendicular to the orbit plane can also be worked out. So, F_A perturbation this is F perturbation dot $\hat{u}_A$ the same format applies everywhere J times n^2 a^3 R_e^2 by r^4 $5z^2$ by r^2 $\hat{u}_r$ and with that we have to take $\hat{u}_A$ and +2z divided by r times $\hat{u}_z.\hat{u}_A$. Now you looking into this part, so in the orbit your orbit is looking like this, r vector is in this direction z vector is here in this direction.

$$(F_A)_{Pert} = -\frac{Jn^2a^3R_e^2}{r^4}[(1-\frac{5z^2}{r^2})\hat{u}_r\hat{u}_A + \frac{2z}{r}\;\hat{u}_z.\hat{u}_A$$

And u_A is coming out of the orbit ok perpendicular to the orbit which I am showing by head of the arrow this blue dot I am creating it is coming out. Therefore it is coming out this is u_θ direction and this is u_r direction. So, u_A is perpendicular to the way we have defined u_r times you $\hat{\theta}$ is equal to $\hat{u}_A$ and therefore the dot product with if you take this dot product which is u_A if you take the dot product on both side u_r times u_A dot.

We have to take dot product with $\hat{u}_r$, $\hat{u}_r$ any way we are defining here if we take this dot products here also, we have to take $\hat{u}_r$ and you know that this will vanish. This $(\hat{u}_r \times \hat{u}_r).\hat{u}_\theta$ so this vanishes. Therefore this quantity is 0 so this term drops out leaving us F_A perturbation this is equal to $-J\ n^2\ a^3\ R_e^2$ divided by r^4 times 2z divided by r $\hat{u}_z.\ \hat{u}_A$. And $\hat{u}_z$ is parallel to this z direction.

$$(F_A)_{Pert} = -\frac{Jn^2a^3R_e^2}{r^4}\left[\frac{2z}{r}\ \hat{u}_z\ \hat{u}_A\right]$$

Here this is $\hat{u}_z$ direction. So this quantity is non-zero and $a^3\ R_e^2$ by r^4 and this quantity as a whole it can be written as 2z by r the angle between the r vectors z vector and A vector. Ok. This is the vertical direction and as the orbitals by angle i. So this angle also this becomes i, so this angle between them is nothing but i angle so this becomes cos i ok and z by r obviously still now we have not written it which I am going to do may be in the next class because it will still maybe 5 minutes to 10 minutes.

$$(F_A)_{Pert} = -\frac{Jn^2a^3R_e^2}{r^4}\left[\frac{2z}{r}\ cosi\right]$$

$$(F_A)_{Pert} = -\frac{Jn^2a^3R_e^2}{r^4}\ [Sinu\ sin2i]$$

This can be written as sin u times sin 2i so this is F_A perturbation along the three directions we have got the perturbative forces. And next this pertrubatship forces are to be inserted into those equation for the planetary motion and once we insert it and look into that we can get the results for the sun synchronous Orbit we will work for the sun synchronous Orbit. How the sun synchronous Orbit can be created. Ok. So then the next lecture I will cover that I will stop with you. Thank you very much.