Space Flight Mechanics
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Lecture No - 66
General Perturbation Theory (Contd.)

Welcome to lecture 66 we have been discussing about the general audit perturbation theory in that context we were working with the oblate and perturbation due to this but before we take that we have already worked on that but perhaps one aspect was left over. So I would like to cover that aspect.

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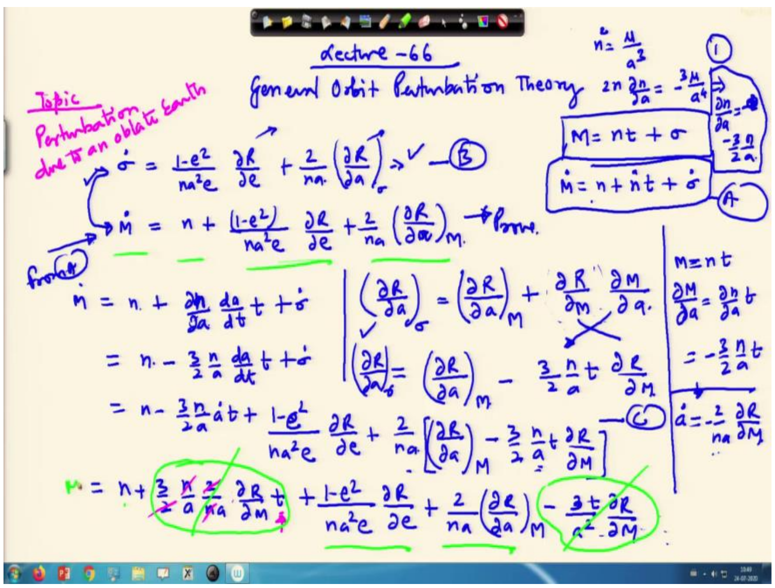


$$\dot{\sigma} = \frac{1-e^2}{na^2e}\frac{\partial R}{\partial e} + \frac{2}{na}\frac{\partial R}{\partial a}$$

$$\dot{M} = n + \frac{(1-e^2)}{na^2e}\frac{\partial R}{\partial e} + \frac{2}{na}\left(\frac{\partial R}{\partial a}\right)_M$$

So $\dot{\sigma}$ is already we know that it can be written in terms of the perturbation potential and this can be reduced to $\dot{M}$ here in this form that means the mean normally at a particular instant of time if it is given to u this can be expanded here in this place where M = nt + σ. So to prove this we just need to write here $\dot{M}$ equal to n differentiate this with respect to t. So this will be $\dot{M} = n + \dot{n}t + \dot{\sigma}$ where n equal to n^2 equal to μ by a^3.

$$2n\frac{\partial n}{\partial a} = -3\frac{\mu}{a^4}$$

And therefore 2n ∂n/∂a this can be written as -μ by a^4, 3 will come here and this implies ∂n/∂a this term will be equal to -3 by 2 and from this place this in a^2 n^2 n by a. So ∂n/∂a is this quantity equal to -3 by 2n and we have to bring it here in this form. In this $\dot{M}$ $\dot{\sigma}$ is already known to us. So we can insert this but we have to replace the other quantities here also.

$$\dot{M} = n + \frac{\partial n}{\partial a}\frac{da}{dt}t + \dot{\sigma}$$

$$= n - \frac{3}{2}\frac{n}{a}\frac{da}{dt}\ t + \dot{\sigma}$$

$\dot{M}$ this becomes n from this equation. From A $\dot{M}$equal to n + n dot is given here. So n dot we can write as dn/da times da by dt times t + $\dot{\sigma}$ and dn/da let us write this as ∂n/∂a. So, ∂n/∂a u already have written here so this is minus -3 by 2 n by a times da by dt t + $\dot{\sigma}$. And $\dot{\sigma}$ is the quantity which is available here so we can insert it later on but we require one more quantity to work with this is ∂R/∂a what appears here in this place this quantity is actually worked for keeping the σ constant.

$$\left(\frac{\partial R}{\partial a}\right)_{\sigma} = \left(\frac{\partial R}{\partial a}\right)_{M} + \left(\frac{\partial R}{\partial M}\right)\frac{\partial M}{\partial a}$$

$$\left(\frac{\partial R}{\partial a}\right)_{\sigma} = \left(\frac{\partial R}{\partial a}\right)_{M} - \frac{3}{2}\frac{n}{a}t\frac{\partial R}{\partial M}$$

$$M = nt$$

$$\frac{\partial M}{\partial a} = \frac{\partial n}{\partial a}t$$

$$= -\frac{3}{2}\frac{n}{a}t$$

Once we have developed this at that time this implied that sigma dot equal to ∂R/∂a to sigma keeping constants. So this quantity is nothing but ∂R by first we differentiate with respect to a but keeping M constant and then ∂R/∂M and ∂M/∂a. And once u insert the values $(\partial R/\partial a)_M$ – ∂R/∂M so this quantity so this quantity discount will be 3 by 2 n a t times; sorry this quantity we have written here ∂M/∂a because this will come in this place ∂R/∂M.

So, m already we are aware of this is nt so ∂M/∂a this quantity will be ∂M/∂a t and ∂M/∂a already we have written 3 by 2 n a t therefore this quantity result here. So, n – 3 by 2 n by a $\dot{a}$ t + $\dot{\sigma}$, so $\dot{\sigma}$we use from let us say this is B and this we have to prove. So, from there then $\dot{\sigma}$ is being insert, this will be 1 - e^2 ∂R/∂a σ. So this we have to insert ∂R/∂a σ this is let us say this is C.

$$\dot{a} = -\frac{2}{na}\left(\frac{\partial R}{\partial M}\right)$$

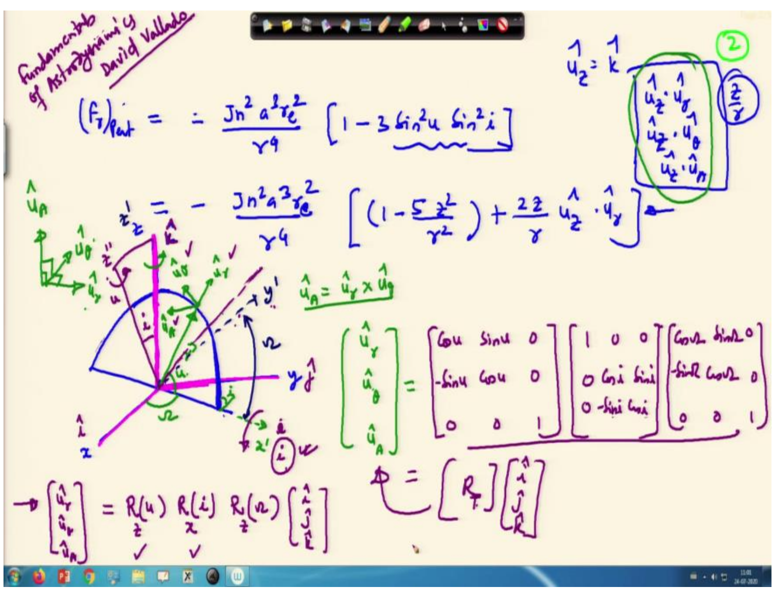
So if we insert here sigma dot we insert this quantity here so 1 - e^2 n a^2_e ∂R/∂r 2 by n a and $\frac{\partial M}{\partial \sigma}$ ∂R/∂a - 3 by 2 n a t ∂R/∂M also, we know that a dot is the quantity already we have data about this quantity is -2 divided by n a ∂R/∂M this we have derived earlier in the perturbation equation while the Langrage planet equation so in that we have derived its quantity.

$$\dot{M} = n - \frac{3}{2}\frac{n}{a}\ \dot{a}t + \frac{1-e^2}{na^2e}\frac{\partial \mathrm{R}}{\partial \mathrm{e}} + \frac{2}{\mathrm{na}}\left[\left(\frac{\partial \mathrm{R}}{\partial \mathrm{a}}\right)_M - \frac{3}{2}\frac{n}{a}t\frac{\partial \mathrm{R}}{\partial \mathrm{M}}\right]$$

So we insert that n - 3 by 2 n a and this is -2 by n a so this gets plus 2 n a ∂R/∂M t + 1 - e^2 – 3 by 2 n a t ∂R/∂M this is n a^2 a is here a is also here and this 2 by n a 3 by 2 so this part we have missed out, let us write it fresh 2, 2 cancels out and this gets 3 n and n will cancel out 3 by a^2 3 by a^2 t ∂R/∂M and if we will look here in this place so here also n and n cancels out 2, 2 cancels out and we get 3 by a and the quantity t is missing here.

$$\dot{M} = n + \frac{3}{2}\frac{n}{a}\frac{z}{na}\ \frac{\partial \mathrm{R}}{\partial \mathrm{M}}t + \frac{1-e^2}{na^2e}\frac{\partial \mathrm{R}}{\partial \mathrm{e}} + \frac{2}{\mathrm{na}}\left(\frac{\partial \mathrm{R}}{\partial \mathrm{a}}\right)_M - \frac{3}{2}\frac{n}{a^2}t\frac{\partial \mathrm{R}}{\partial \mathrm{M}}$$

So we have to insert t also here in this place, t we written already, t is on the back t is already here. So overall this quantity and this quantity along with minus sine they are the same so they cancel out and then what we get n + 1 - e^2 this term and this term and this is nothing but the term which is written here in this place. So therefore $\dot{M}$ is equal to and this is your $\dot{M}$, $\dot{M}$ is nothing, but the quantity here so gets proof. Here was perhaps the left out think so I thought of covering it today.



$$(F_r)_{perturbation} = -\frac{Jn^2a^3r_e^2}{r^4}\ [1 - 3\sin^2 u\ sin^2 i]$$

$$= -\frac{Jn^2 a^3 r_e^2}{r^4}\left[\left(1-\frac{5z^2}{r^2}\right)+\frac{2z}{r}\ \hat{u}_z.\hat{u}_r\right]$$

Now we go to the main topic what we are discussing about. This was the perturbation due to the oblate planet and in that context we have derived a few things. So in that context we have derived F_r perturbation this is equal to $-Jn^2a^3R_e^2$ divided by r^4 1 - 3 $\sin^2$d u and $\sin^2$ i but how this term appeared here this we have not discussed. So, this was part of our equation which was written as - Jn a^2 $a^3R_e^2$ divided by r^4 1 – $5z^2$ by r^2 + 2 z r $\hat{u}_z$ times $\hat{u}_r$.

Where $\hat{u}_z$ is nothing but take $\hat{k}$, $\hat{u}_r$ we have to work out. similarly in the other equations some of the terms which appear the terms like u u_z .$\hat{u}_r$ is here and similarly we get the term as earlier we have worked out in the last class. So $\hat{u}_\theta$ this term is required and one more term was required which was u_z times $\hat{u}_A$. So, these 3 terms required and once we placed them and also z by r this quantity.

So, if replace in this kind of equation so we get the result. So in that context we are discussing that we have the orbit here. Ok, so this is x direction this is y direction and z direction along this we have taken $\hat{k}$. And then we have considered $\hat{u}_r$ in this direction perpendicular to this $\hat{u}_\theta$ and $\hat{u}_A$ in this direction, where $\hat{u}_A$ equal to $\hat{u}_r \times \hat{u}_\theta$. In this u have r vector from this place to this place.

This angle we have taken as u inclination is here i this angle we have taken as Ω. So u can see that for getting the unit vector u_A, u_r and u_θ or u_r, u_A and u_θ actually this looking like this here. From this place it may not be clear so I am writing here $\hat{u}_r$, this is $\hat{u}_\theta$ and this is $\hat{u}_A$. This is 90° and this is also 90°. This is also 90° so $\hat{u}_r \times \hat{u}_\theta = \hat{u}_A$.

May be we can show it here in this direction. So what we are interested in finding out the quantities which are written here for doing that we need to find out each of the unit vectors here. So the unit vectors $\hat{u}_r$, $\hat{u}_\theta$ and $\hat{u}_A$ this can we arrived that by giving certain rotation. So the first rotation we can give about this axis by Ω so your x axis will arrive here in this place.

So, this will be x' and y axis similarly it will go away from this place by certain distance and let us say that here y' comes here with this angle from here to there will be then Ω and z' remains here in this place. The next rotations then we give about this x axis the new x axis yi by doing so your y' will come to the y plane of the orbit. So it will rise from this place and it will come to this Orbit.

So let us edit this I am representing by; so first rotation about the z axis then the next rotation about this axis. And that we are giving by i ok so by doing so u_r I have I vector is right now is the i vector here so that once we move it so corresponding this vector gets transferred here into this place and from there then it will go to the; so the x direction is here y direction μ here z direction remains in the same place rotating about this by i angle.

So y goes to the plane of the orbit and then in the Plane of the orbit again, once we rotate it here. So this will also rotate from this place to this place by i angle so this is your z double' and y double' will come in the plane of the orbit. Remember that by doing so your i remains here in this place. OK so the first rotation what we have done. We have given rotation by Ω about the z axis.

The second rotation then we are get giving by i about the x-axis and the new access and the third rotation then we give about the new z-axis means about this will give the rotation by u. So if u give this rotation by u about the new z axis this will result in this series of rotation and if u operate on this the vector $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ it will result in the vector $\hat{u}_r$, $\hat{u}_\theta$ and $\hat{u}_A$. Either u can refer to my lecture on satellite attitude dynamics and controls for this kind of thing or either u can refer to the Fundamentals of Astrodynamics by David Waller.

This book name is given in the references I have provided u and once u do this u get the $\hat{u}_r$, $\hat{u}_\theta$ and $\hat{u}_A$ in terms of $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$. So, j cap is here along this direction $\hat{k}$ is along this direction this relationship will be available to u and how this are defined? So I will just write here in this place. So this is about the z-axis. So this will appear as 00 ok and I am not explaining how this Matrix appears but u can take it for granted right now.

$$\begin{bmatrix} \hat{u}_r \\ \hat{u}_\theta \\ \hat{u}_A \end{bmatrix} = \begin{bmatrix} Cosu & Sinu & 0 \\ -Sinu & Cosu & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \hat{u}_r \\ \hat{u}_\theta \\ \hat{u}_A \end{bmatrix} = R_z(u) + R_x(i) + R_z(\Omega) \begin{bmatrix} \hat{\imath} \\ \hat{\jmath} \\ \hat{k} \end{bmatrix}$$

Otherwise no way of doing it at this stage until and unless I explain the whole process and that will take time. Ok refer to this to my lecture on Attitude Dynamics and Emission Control course, so this is your R_z so this is rotation here this rotation R_x, we give; we write like this and this is about by the angle i so this is because of cos i sin i minus sin i and the first rotation that we have given is by about the z by Ω this is 1 0 0 0 0 cos Ω cos Ω sin Ω and minus sin Ω.

$$\begin{bmatrix} \hat{u}_r \\ \hat{u}_\theta \\ \hat{u}_A \end{bmatrix} = \begin{bmatrix} \cos u & \sin u & 0 \\ -\sin u & \cos u & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i \end{bmatrix} \begin{bmatrix} \cos \Omega & \sin \Omega & 0 \\ -\sin \Omega & \cos \Omega & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And multiply them and by multiplying let us say this u write this as the rotation matrix R T and this will operate on $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ and this will yield this value this u can check yourself. I have written it here ok. Now what we need to do? Utilize this to solve the problem. So I will write the equation directly here. So this exercise u should do some multiply and check what I am going to write on the next page.

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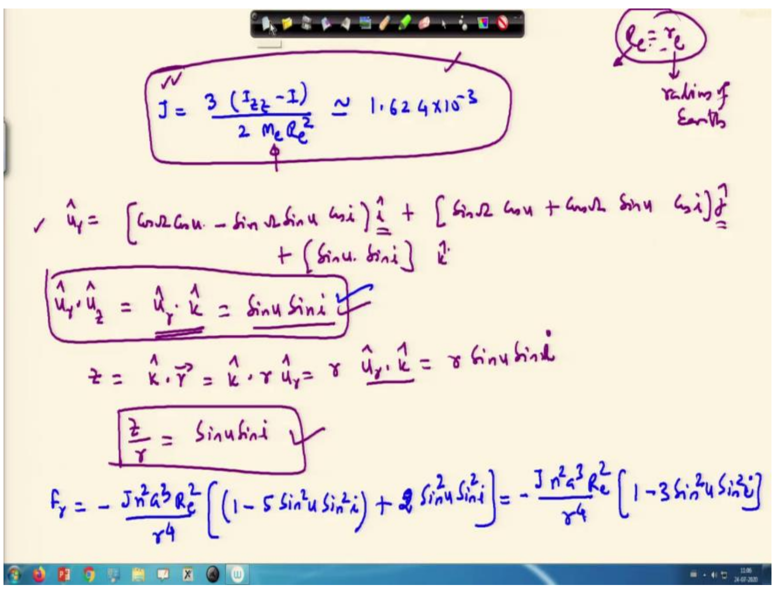

$$J = \frac{3(I_{zz} - I)}{2M_e R_e^2} = 1.624 \times 10^{-3}$$

This term J have appeared in our derivation so this J value is defined like this for the earth where M_e is the mass of the earth and R_e perhaps we have written as a small r_e it is the radius of the earth and we are going to utilize this value. This is the same thing. So $\hat{u}_r$ we can write a cos Ω cos u - sine Ω sine u cos i times $\hat{\imath}$ sine Ω cos u + cos Ω sin u times cos i $\hat{\jmath}$, $\hat{k}$.

$$\hat{u}_r = [\cos\Omega\cos u - \sin\Omega\sin u\cos i]\hat{\imath} + [\sin\Omega\cos u + \cos\Omega\sin u\cos i]\hat{\jmath} + [\sin u\sin i]\hat{k}$$

$$\hat{u}_r.\hat{u}_z = \hat{u}_r.\hat{k} = \sin u\sin i$$

So if you utilize this ok here in this place so to find out what will be the quantity $\hat{u}_r.\hat{u}_z$ this quantity so we just need to take the dot product $\hat{u}_r.\hat{k}$ and immediately we see that this product will vanish because they are perpendicular to each other leaving us with Sin i sin u times sin i besides this we need z by r so the quantity z is nothing but the component of the r vector in the k direction of this is $\hat{k}$ r times $\hat{u}_r$.

Ok, so that means r times $\hat{u}_r.\hat{k}$ and already we have written here u_r. $\hat{k}$. So this becomes sine u sin i and therefore z by r this quantity becomes a sin u sin i. So we are ready to work with these quantities available and this quantities available, insert this here in this equation appear in this part z by r we require so we have to replace this and this quantity also we need to replace and therefore so F_r then this gets reduced to –J n^2 a^3-J n^2 a times R e^2 by r^4 and then 1 -5 z^2 1 - 5 z by r^2.

$$F_r = -\frac{Jn^2a^3R_e^2}{r^4}\left[(1 - 5\sin^2 u\sin^2 i) + 2\sin^2 u\sin^2 i\right]$$

Z by r^2 already we have derived here which is sine u so this $\sin^2$ u $\sin^2$ i this becomes for these quantity if we add this 2 times z by r, these 2. So + 2 times z by r again this is sin u sin i and this multiplied by z times $\hat{u}_z$ times $\hat{u}_\theta$. The quantities here this becomes then $\sin^2$ and once we solve this so this quantity gets reduced to –J n^2 a^3R e^2 divided by r^4 1 - 3 $\sin^2$ u $\sin^2$ i and this is what we have written.

So for the same way the other things can we worked out. So here we have written the u_r and some exercise I will leave it for you.

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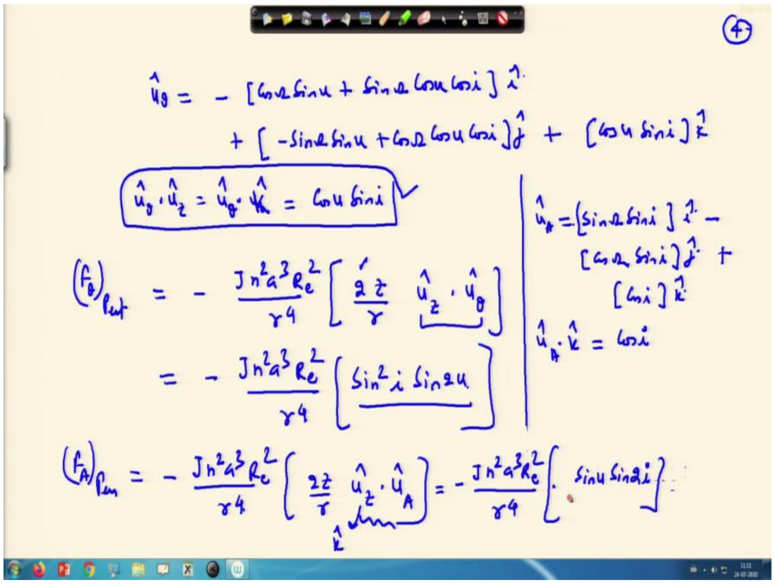

$$\hat{u}_\theta = -[\cos\Omega\sin u + \sin\Omega\cos u\cos i]\hat{\imath} + [-\sin\Omega\sin u + \cos\Omega\cos u\cos i]\hat{\jmath} + [\cos u\sin i]\hat{k}$$

Ok so for the next one is our u_θ, so $\hat{u}_\theta$ similarly from this one the second term if you pick up. We will get the $\hat{u}_\theta$. So we have to multiply the whole thing first and from there the second row you choose so that gives $\hat{u}_\theta$ and this will turn out to be. - cos Ω sine u sine Ω cos u cos i this is you rθ cap. Same way $\hat{u}_\theta$ what we require the quantity that will require $\hat{u}_\theta$ times $\hat{u}_z$u.

$$(F_\theta)_{Pert} = -\frac{Jn^2a^3R_e^2}{r^4}[\frac{2z}{r}\ \hat{u}_z.\hat{u}_\theta]$$

This will be $\hat{u}_\theta\hat{k}$ and immediately we consider that this will result in cos u times sin i. So use this in the equation for F_θ that we have written, earlier this already we have derived all these equations. So F_θ due to the perturbation only this was written as - Jn^2 $a^3$2z by r $\hat{u}_z$ times $\hat{u}_\theta$ and once you have replaced z by r as we have derived here this is z by r, so z by r equal to sine u sine i.

$$(F_\theta)_{Pert} = -\frac{Jn^2a^3R_e^2}{r^4}\ [\sin^2 i\sin 2u\ ]$$

$$(F_A)_{Pert} = -\frac{Jn^2a^3R_e^2}{r^4}[\frac{2z}{r}\ \hat{u}_z.\hat{u}_A]$$

So, if you utilize here in this place z by r and replace by $u_z.u_\theta$ from this place you get the corresponding equation - Jn^2 $a^3R_e^2$ by r^4 times 2, 2 can be observed and this can be written as $\sin^2$ i sine 2u. So sine u times cos u multiplied by this 2 that results in Sin 2u. You can check this part also. And at the last we have F_A perturbation this part was $-Jn^2 a^3R_e^2$ divided by r^4 $\hat{u}_A$. We require this quantity this quantity is again this quantity will be available to you from the last row of this matrix. And once we used to solve that Matrix and write it so this will appear as sine Ω sine i times $\hat{\imath}$ minus cos Ω Sin i $\hat{\jmath}$+ cos i $\hat{k}$.

$$(F_A)_{Pert} = -\frac{Jn^2a^3R_e^2}{r^4}\ [\sin u \sin 2i]$$

And therefore immediately we can see from this place that u_A $\hat{k}$this thread is nothing but $\hat{k}$so this results in cos i so all things are available here and we can immediately write the result for this 2 times z by r sine u sin i and from here u_z .u_A this part we can sort it out. And this write has just sine 2i and sine 2i and these 2 we can remove. Ok so till this extent we have worked out.

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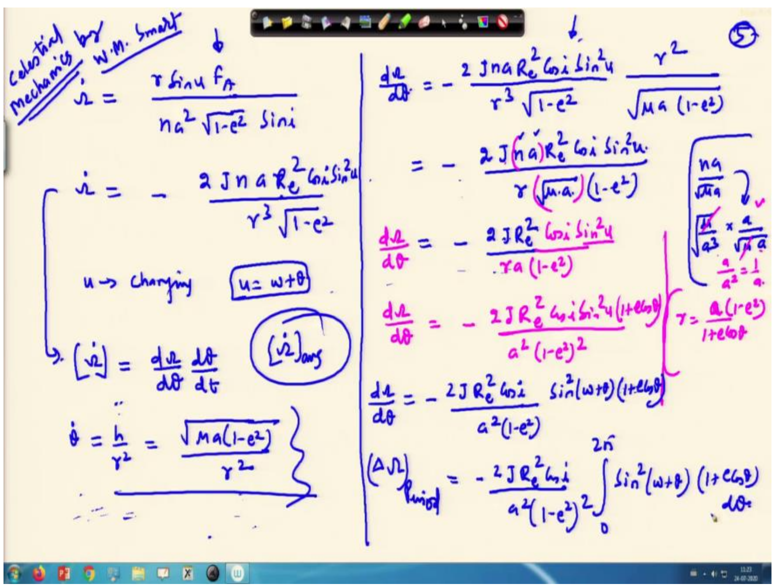

So, if you do little quickly so will be able to work it out. So the $\dot{\Omega}$ this is r sin u F_A this is the Lagrange planet equation we have got in terms of F_r, F_θ and F_A. And F_A we can insert here in this place and once we insert that what we have derived on the last page, so this will get reduced to R_e^2 divided by $r^3 1 - e^2$. So insert the F_A that we have derived on the previous page and reduced it.

$$\dot{\Omega} = \frac{r \sin u\, F_A}{na^2\sqrt{1-e^2}\sin i}$$

$$\dot{\Omega} = -\frac{2Jna\,R_e^2}{r^3\sqrt{1-e^2}}\sin^2 u \cos i$$

So this is what we get. So here u is changing because u we defined as ω plus θ and r also change; r may change. So, these things they give rise to the variation in short period variation whatever it is what we will do that first we are going to average it out to eliminate all the short period variation $\dot{\Omega}$ then this can be written as after average it over a period let us write this first as d ω dθ into dθ by dt whereθ dot is a quantity from our earlier derivation h by r^2 which is μ a times 1 – e^2 divided by r^2.

$$\dot{\Omega} = \frac{d\Omega}{d\theta}\frac{d\theta}{dt}$$

$$\dot{\theta} = \frac{h}{r^2} = \frac{\sqrt{\mu a\,(1-e^2)}}{r^2}$$

So, this gives us dΩ by dθ this will turn out to be; if you insert all these values, so this can be reduced to -2J R_e^2 divided by or let me work it out this particular part so that we do not commit any error here. So, cos i times sine squire 1 - e^2 and then this divided by $\dot{\theta}$ this is your Ω dot this particular part and this we divide byθ dor so we get the desired quantity and $\dot{\theta}$ is here.

$$\frac{d\Omega}{d\theta} = -\frac{2JnaR_e^2\cos i\sin^2 u}{r^3\sqrt{1-e^2}}\frac{r^2}{\sqrt{\mu a(1-e^2)}}$$

If we divide it this will be μ a times 1 - e^2 under root and this multiplied by r^2 now we will reduce this is -2 J n a R_e^2 $\sin^2$ u r times μ under root times 1 – e^2. Ok now we have to do little bit of manipulation here. This is na this quantity and in the denominator μ a under root this can be written as n equal to μ by a^3 under root and this is a and this divided by μ under root times a μ times a under root.

So, what do we get from this place this μ this μ cancels out and is in the numerator and below we get a^2 so this 1 by a. This whole quantity this part and this part we can replace has become r a and this multiplied by 1 - a^2 in the denominator we have written 2J R_e^2 already and this is cos i and this is dΩ by dθ r we can replace in terms of r equal to l by 1 + ecosθ this can be written as a times 1-a^2.

$$\frac{d\Omega}{d\theta} = -\frac{2JR_e^2\cos i\sin^2 u}{ra\,(1-e^2)}$$

$$r = \frac{a\,(1-e^2)}{1+ecos\theta}$$

$$\frac{d\Omega}{d\theta} = -\frac{2JR_e^2\cos i\sin^2 u}{a^2\,(1-e^2)^2}(1+ecos\theta)$$

$$\frac{d\Omega}{d\theta} = -\frac{2JR_e^2\cos i\sin^2(\omega+\theta)}{a^2\,(1-e^2)}(1+ecos\theta)$$

So if you replace this becomes a^2 times $(1 – e^2)^2$ and this part will go in the numerator is$_e$ cosθ. So now if the; and also one thing we should remember let us further write it 2J R_e^2 cos i divided a^2 1 - e^2, u is nothing but ω +θ and then sine this is $\sin^2$ ω +θ and 1 +ecosθ so we integrate it over a every here it over a period of time 2 get the; what we are looking for?

We are looking for $\dot{\Omega}$ average by removing the Short period variation there may be some periodic variation local so all those things will get eliminated. The books the textbook the I have referred to you. So all of them will be not explaining these things that we were given the final equation and working with that. So for look into the details of this you can refer to the book on Celestial Mechanics by W M Smart.

And also the already I have mentioned in the introduction to the fundamentals of astrodynamics by David Valero, that is also a very good book my presentation their presentation will differ a little bit. I have adopted the vector method of working. Valero is also vector method but little bit different. So the way I have presented it is a little bit different. But the basic idea I have given you and based on this basic once the Lagrange planetary position is available only thing remains that find out the perturbation function or the force function there.

And use it and write the differential equation wherever required you want to approximate and whatever the things to be done it follows their after. So those are details which cannot be covered in this course, but of course what has been possible I have done here. So this ΔΩ over a period then this will be given by – 2J R_e^2 cos i as $(1 - e^2)^2$ integrate with 0 to 2π $\sin^2$ 1 + ecosθ dθ.

$$(\Delta\Omega)_{period} = \frac{-2JR_e^2 \cos i}{a^2(1-e^2)} \int_0^{2\pi} \sin^2(\omega+\theta)(1+ecos\theta)d\theta$$

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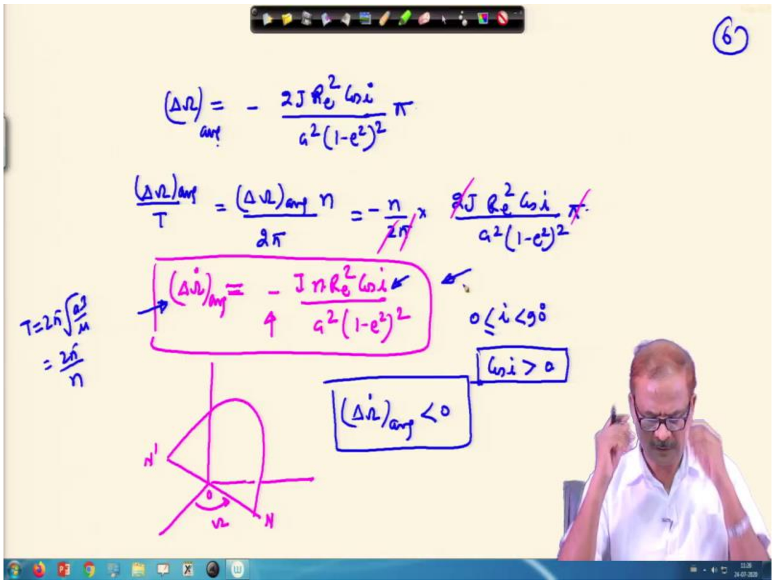

$$(\Delta\Omega)_{avg} = \frac{-2JR_e^2 \cos i}{a^2(1-e^2)^2}\pi$$

$$\frac{(\Delta\Omega)_{avg}}{T} = \frac{(\Delta\Omega)_{avg} n}{2\pi}$$

And once integrated this gives you Δ Ω 2J R_e^2 cos i divided by a^2 $(1 - e^2)^2$ multiplied by π. So mean angular rate what this will be? Δ Ω every this is by Everest divided by T which is the period and this is nothing but Δ Ω is divided by period is 2π a^3 by μ and this is nothing but 2π by μ n so this is n goes in the numerator.

So, n by 2π times 2J with minus sign R_e^2 cos i a^2 $(1 – e^2)^2$. So whatever we can eliminate here. You will eliminate this π this π cancels out 2 2 cancel out. Therefore we get this Ω ω dot this one is the average one equal to –J times n R_e^2 cos i divided by a^2 time 1 - a^2 and this gives you the regression rate of the node this is O and n n' the notation we had been using all along.

$$(\Delta\Omega)_{avg} = \frac{-JnR_e^2 \cos i}{a^2(1-e^2)^2}, \ 0 \le i \le 90°$$

This is Ω the minus sign here it indicate that if i is less than 90° i lies between 0 and 90° at one side equal to 90° we can see that the whole thing vanishes there is no variation in the Ω. So, if you have this situation there i lies between 0 and 90° so we can cos i then this will be greater than 0. And this situation you can see that Δ Ω this will be less than zero.

That means this is a regression. What does this mean? This we have to think of and this is very important for this particular result is very important for our Sun synchronous satellite.

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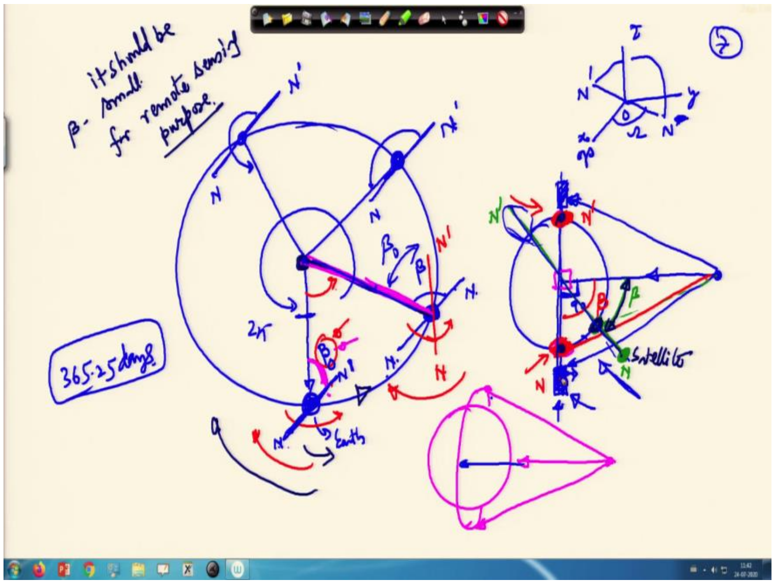

Now look into this picture Sun is here and Earth is here in this place and around the Earth one satellite is going on. OK so the satellite orbit say oriented like this. This is the nodal line N and N' holds somewhere here all the time we have shown here like this. This is your the inertial axis along the vernal Equinox this is x, y and z and then you have the orbit here where N; this is N and this is N' this is O this angle which Ω.

So, this line is your nodal line the vernal equinox I am not showing here. Now consider the case of a satellite which is moving like this period here if you see the sunlight will be directly coming from this place. Now bigger picture I will draw here. So this is the earth and around this the satellite is going on. So it can be shown again like this in another picture. I will show this one is here and sunlight is coming here from this place.

So, if satellite is supposed that in this orbit. It is going like this is perpendicular to the sunlight. This angle is 90° and satellite orbit is like this. If we see from the top so it will appear like this. It is coming up here this place coming up and it is going down here in this place. So, in this kind of orbit I will see that all the time the sunlight will be available. And if u have the satellite antenna here, so satellite antenna will always be illuminated.

And if we are able to maintain this angle of 90° all the time with respect to the sun so here what is the objective? I will write this angle is beta. This is the beta. Ok and can we maintain the angle beta? This is the question? If the node does not rotate so it will remain parallel from this place to this place. Ok, if u go here in this place, this is the Earth here and then the node will be appearing like this. Ok if u go here in this place node again remains parallel remains like this.

So this is the β angle. This is β angle here. This is your N, this is your N and N' we have shown this on here on this side N and N'. This becomes β angle this is N this is N prime. And N’to this line this is your β angle. This is N N'. And N' to this, this constitutes your β angle. So what we can see that it will look here in this picture. So here if the satellite antenna is there so this is getting illuminated.

But suppose I have to do Remote Sensing so in the Remote Sensing work what we require that I have to take picture of the ground here. If it goes here on this side will be taken picture of the ground here similarly on the top it may be required to take the picture here. So on the top of the earth means earth is like this and over this the satellite is going like this and Sun is illuminating from this place, here the sun is located.

So your satellite is getting the illumination but what happens to the ground what will happen at this place, here in this place? Because the sunlight will be just grazing like this here, so it cannot reach to this point. So here this will be dark and if it is dark that means we cannot take photograph of this portion which I have shown by red. Then the whole purpose is defeated. Sun synchronous satellite it is used for maintaining a particular angle with the Sun.

That means this angle we have to maintain this β we have to maintain. So that means this nodal line should rotate. So for keeping this here in this place μ nodal line should rather come to this place. So, sun is moving and along with this nodal line should also move. So, already we are getting here the equation that the nodal line is getting perturbed on an average like this. It is angular orbit is like this.

Ok so can I make it equal to the; here there is a minus sign. So therefore this is also called nodal regression rate because there is not minus sign. If i is positive automatically Ω of Ω will be negative that means instead of moving here in this direction. It will move in the opposite direction. Ok, it will not go here in this direction. If i is positive it will go here in this direction that means β will increase.

But what we are interested in we are interested in maintaining the β angle. So that Sunlight is always available. Now you see that instead of keeping this nodal line here N and N' here in this place. I keep my nodal line instead of here in this place and then this is my β angle and this N and this is N'. So, you can immediately see that is my satellite is located here the sunlight is available from this place.

And ground will be illuminated here in this place as shown by this blue colour deep blue colour. So, the ground is getting l illuminated by the sun and therefore satellite can take picture satellite view is here satellite can take picture. And therefore this region this β angle is maintained small so β small it should be a small. Should be a small for Remote Sensing purpose if it is 90°s this is of no use because you cannot take photographs in these places.

So this is the borderline of the sunlight. What we call as the Terminator line so there the future picture cannot be taken so to maintain this particular β what is required that instead of regression the nodal line should progress. Ok, it should not get regret it should not go here in the back direction, but it should go along this direction because this the sun and earth is moving here in the direction.

So if earth is moving, to maintain with this line to maintain angle with this line which is the β angle I am showing here. To maintain the same angle I have to rotate this nodal line in N' all the time ok so that it maintains a particular β angle and wherever this thing is available we call this as the sun synchronous orbit. And this β can be set to any value and if at the time of launch, whatever you maintain this β so that the same β can be maintained for all the time.

But for that what is required we have to launch it in a particular inclination angle. And that we can see on the next page.

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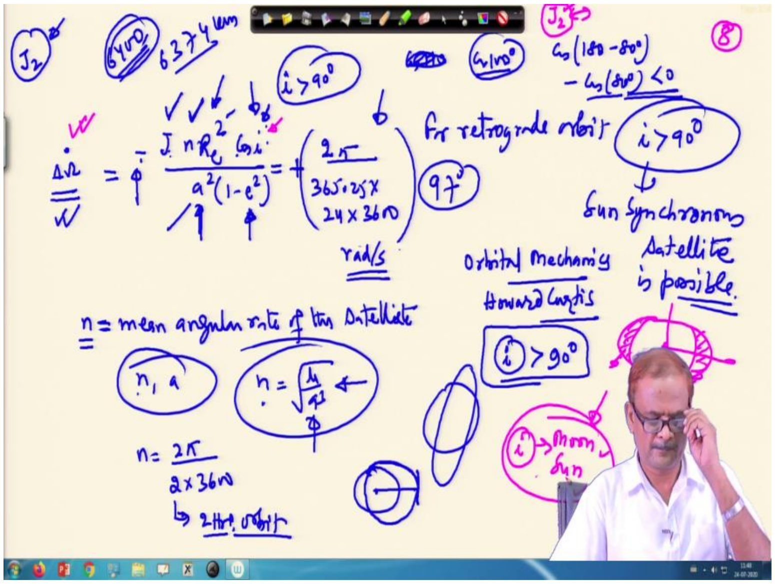

$$(\Delta\Omega)_{avg} = \frac{-JnR_e^2 \cos i}{a^2(1-e^2)^2}$$

So we have -Jn R_e^2 this is the nodal regression rate and we want this to be positive. So, how it can be made positive if I is greater than 90° so then cos i becomes negative so cos say 110° if I put like this or cos 100°, so this quantity is cause 180 - 80° - cos 80° so this quantity is less than 0. So therefore if this is minus this get Ω gets a plus sign and that means the nodal instead of nodal line regressing now, it is progressing.

Ok regressing means going in the backward direction progressing means going in the forward direction. So that means for the retrograde orbit with orbit i is greater than 90° and for that only the sun synchronous satellite is possible. For frigid orbit sun synchronous satellite is not and for that reason we will see that most of the; if you get from somewhere the date. I will see that most of the sun synchronous satellite put the altitude of 700 kilometre surround and will have around 97° of; around this value this angle of inclination will be solved.

So all these are retrograde orbit and to do this what we need to do. This the nodal rate this nodal line should also progress at the same rate as this line is sweeping ok. So this line is sweeping in 2π in 365.25 days. This is the period of this earth around the sun. If it; the earth is moving here so joining the sun and the earth has come here in this place by the time it would move in such a way that this β angle β_0 is also mentioned herein this β_0.

And if this is done you get a sun synchronous satellite means always it is in synchronicity with the sun. Here for this kind of orbit what I have shown this is in this way for this kind of orbit the satellite will be always in the sunlight. Ok. It will be never eclipsed but if you have this β angle available here. Ok so you can see that in this portion in this portion sun is available. But once it goes on the back direction show some gets this is satellite get eclipsed because it comes on the back side of the Earth.

So if it gets eclipse means at during that time the satellite antenna will not get any Sun ray and therefore battery onboard the satellite will not be charged. So, you have to maintain a battery of the capacity that it should have that much of backup that without sunlight the system on the satellite will be running. And for that reason keeping β to a small will keep the satellite for a longer period behind the earth so it gets eclipse by the earth shadow and it is not visible when the sunlight and the battery is not charging.

So 90° angle β equal to 90° is the ideal one, but this limits the thing that no photography can be done and then for the Remote Sensing satellite if there is no photography its meaning is lost. And therefore β is kept small and backup is maintained with the battery that whenever the satellite is on the back side of the Earth then it said that battery is supporting the system onboard.

So if this quantity we keep it as – 2π divided by 365 this plus progressing the line joining the Earth and the Sun is a processing at the state 265 into 24 hours into 36 so this gives you this region converted into Radian per second the right hand side J value already I have written. So if we insert those values n is available to us, n is nothing but the mean angular rate of the Satellite ok this is the mean angular rate of the satellite.

So, depending on the purpose n is already fixed, n and a these are already fixed, n is directly related to a through this equation. And at what altitudes were going to put its all these things are already fixed. So once these things are known say n equal to this is the mean angular rate so this is 2 hour so this will be 2π divided by 2 arguments 2 times 3600. So this is the mean angular rate for the satellite, satellite how much angle is 2π it is going around the earth.

In how much time so this is 2r, 2r orbit if you have larger orbit so the fidget will increase but this thing is then available to us by our design. Ok R_e is the radius of the earth which we can write the 637407 or 76 kilometres sometimes we just approximate as this but for the actual purpose while we are designing the satellite. The real purpose is it has to be launched. So we have to be very accurate in all these estimates.

In the synchronizity of the orbit what kind of orbit we are going to maintain. We cannot keep a very low altitude and high eccentricity because in that case the satellite will go the perigee will be line inside the earth. It will go and hit the Earth so this is not possible. So, all these things are fixed in the beginning and accordingly then this i chosen. So if you; I am leaving this exercise and also you can look into the book by this orbital mechanics.

The book name in the references I have given by Howard Curtis. So there some examples of solved instead of J there J2 is written with the journal harmonic they have used. The same thing in little bit some term are differing ok but the J value I have already stated and you can solve by assuming what is your a the R_e values using properly. What will be your Orbit, Orbit period always you can calculate using n equal to μ by x so once you choose that so n is available to you.

So from there you find out the value of i. So this i once it greater than 90° then you are going to get this quantity as negative this makes it positive and therefore we were getting that in the Ω dot which is matching with the progression of this line. So this line is progressive here in this direction. So at the same rate your nodal line will also progress here in this direction like this.

So, instead of being here it will be shown by the red again. It will come here in this place so that this β is maintained ok here in this place again it should be something like this for this β is maintained and ultimately it will again come back to this place whatever is shown here. So this way your Sun synchronous satellite is maintained. Of course, this because of the perturbation the i also changes and for that u have to maintain it periodically.

And this high perturbation it primarily comes from the moon and sun not from the J_2. J_2 perturbation J_2 terms we have not used but it appears in the spherical harmonics the expansion we do for the earth was; the earth is bulging here. So because of this the perturbation which arises already we have done this.

So instead we have taken another route, but it can also be done in the terms of the harmonic. So in that case we mention it as J_2. So, J_2 is the most powerful term there which affects the satellite motion which will affect the $\dot{\Omega}$ but i it gets primarily affected by the third body which is the moon and the sun ok and for that some small maintenance has to be done from time to time. So we do not have this much of that much of time that we lack time to discuss all these issues there in this course.

But the basics idea is covered so that you can read any book and learn on your own whatever you want.

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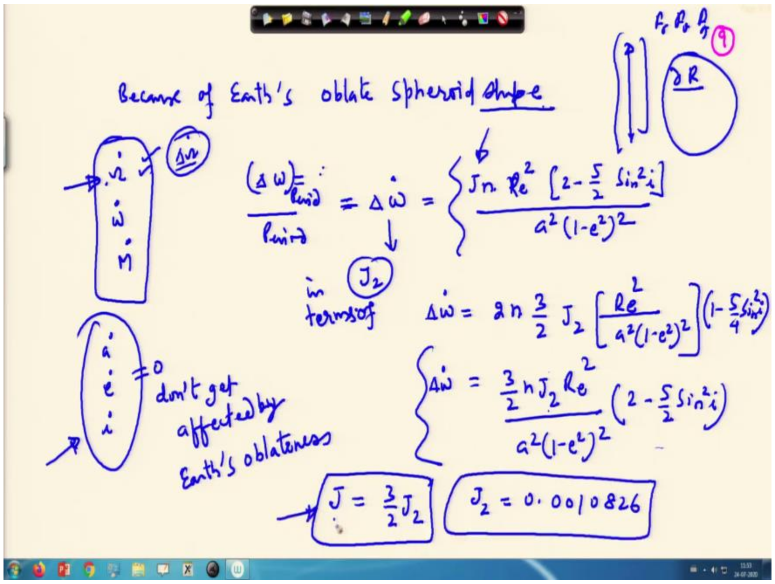

So because of Earth bulging or oblate shape spheroid shape Ω will change ω will change $\dot{m}$ will change. Ok $\dot{a}$, i, $\dot{e}$ they do not get affected by this earth's oblations. Because ultimately everywhere we are to write here in those equations if you look for all the equations and this perturbation equation in terms of F_r, F_θ and F_A perturbation equation we have written ω

The perturbation equation if you look so corresponding to this the perturbation turns will be absent due to the bulge and therefore these quantities will be zero only they exist. So $\dot{\Omega}$ already we have done and $\dot{\Omega}$ is remaining and $\dot{\Omega}$ in the same way $\dot{\Omega}$ equation first we have to write the perturbation equation and that perturbation equation then we have to follow whatever the way we have approached for this.

$$\Delta\dot{\omega} = \frac{Jn\, R_e^2[2-\frac{5}{2}\sin^2 i]}{a^2[1-e^2]^2}$$

So this way just ω per period and divided by period this gives you $\Delta\dot{\omega}$. And this quantity turns out be Jn R_e^2 times 2 - 5 by 2 $\sin^2$ i divided by a^2 times $(1- e^2)^2$ is the denominator is the same as for $\Delta\dot{\omega}$ we have got. The same thing in terms of J_2 so if we write the same thing in terms of J_2 in terms of J_2 it will appear as $\Delta\dot{\omega}$ equals to 2n divided by 3 by 2 J_2 times R_e^2 by a^2 x 1 - e^2 whole^2 times 1 - 5 by 4 $\sin^2$ i it appears like this.

$$\Delta\dot{\omega} = 2n\frac{3}{2}\, J_2[\frac{R_e^2}{a^2(1-e^2)^2}](1-\frac{5}{4}\sin^2 i)$$

$$\Delta\dot{\omega} = \frac{3}{2}\,\frac{nJ_2R_e^2}{a^2(1-e^2)^2}(2-\frac{5}{2}\sin^2 i)$$

So if we take this 2 inside one of them. So this will be 3 by 2 and J_2 times R_e^2 divided by a^2 times $(1 - e^2)^2$ -2 - 5 by 2 $\sin^2$ i. So you can see the difference between this equation and this equation here there is J n is same instead of J there is J_2 R_e^2 rest other terms are same means J and J_2. They are related by 3 by 2 J_2.

And J_2 value this is given to be 0.0010826 and J value already I have stated earlier. This is the average rate of this argument of perigee progression.

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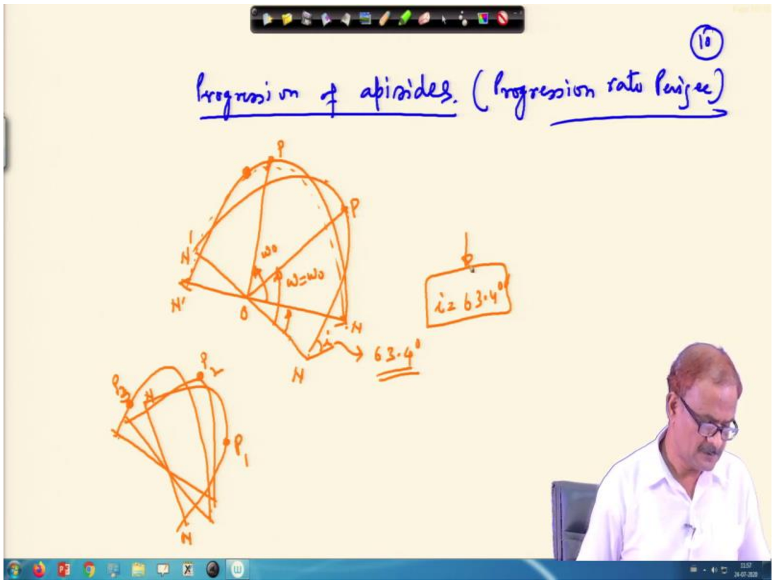

And this is call the progression of apisides progression rate of perigee because it comes with a positive sign here. There is no negative sign before this and therefore this is call progression. So immediately what we can observe from this place that if I shed this quantity $\Delta\dot{\omega}$ is equal to 0 either here in this place or either in this place we set it 0. So what we will get $\sin^2$ i is equal to 4 by 5.

So if this critical angle is applied that your satellite is located inclined at this value which is turns out to be; i will turn out to be 63 point something this value turns out 63.4°. So, if i is equal to 63.4° then you perigee will get fixed mean perigee will not progress. What does this mean that in the orbit say the perigee is located here your this thing is progression the nodal line. This is the noodle line N', so now the N has come here N' has come here.

So your nodal a line is moving here in this direction. So corresponding to this your Orbit will also look like this, orbit has gone like this because of the nodal line rotation your orbit has rotated like this and here i equal to 63.4° if u keep then your perigee location means from the nodal line suppose from this nodal line this perigee location is whatever you have this ω is equal to say ω_0.

So the same perigee location will be maintained all the time. So this will be also ω_0 value. This will not change provided i equal to 63.4° if this is not satisfied that means your perigee will not remain here in the same place, but rather it will move here. This point will move here as it moves from this place to this place. This is moved further in the orbit is rotating so this will keep moving.

That means you have different orbits. You are starting with this place and you have different orbits here.

This is your first orbit here. Perigee is here. The second orbit is here in this place perigee moves from this point it goes to this point. Then this is the third orbit perigee most to this point P_1 P_2 P_3. Perigee will keep progression if you want to stop this then what is required in this one. So these are the things that we can play with.

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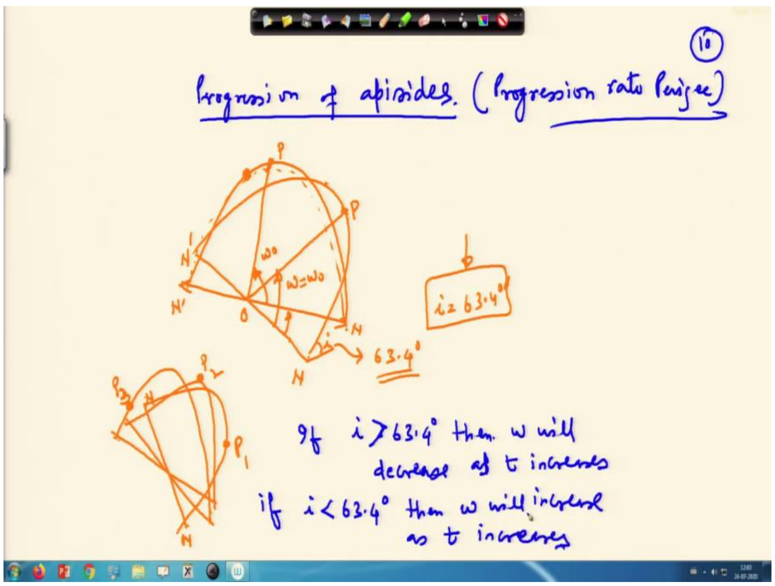

$$\dot{M} = n + n\frac{3}{2}J_2\frac{R_e^2}{a^2(1-e^2)^2}\left[1-\frac{3}{2}\sin^2 i\right]$$

Also one more equation we can write m dot equal to n + n times 3 by 2 J_2 R_e^2 by a^2 times $(1 - e^2)^2$ 1 - 3 by 2 $\sin^2$ i this is call progression of Epoch. Already I have explained that epoch can the perigee they are not the same thing. But the epoch defined by M equal to n t + σ this is your that term n times t - t n t - n T. So, σ is equal to instead of working with this we can convert in terms of M or either we can work in terms of σ also both ways we can work.

So epoch cut a particular instant of time, whatever you have that can also move because of this perturbation and this is given by this equation. So, already we know the relationship here J is related to 3 by 2 J_2. So 3 by 2 J_2 this can be replaced by J and you get the relation in terms of m dot. So all this variation finally I will write here. All these variations are secular in nature. They are varying the time they are not periodic.

Variation in mean orbital elements or secular in nature periodic terms have been removed by averaging over a period. So in one orbital period $\Delta\Omega$ will be $\Delta\dot{\Omega}$ average times T. Shift and orbital node or the nodal line in one period. If you do not want to change this then u have to keep i equal to 90° that is the critical value at which the nodal line will not either regrets for a little progress.

For processing you need i to be positive, i to be more than 90° which is retrograde orbit and for regression you need to be a small but smaller than 90°. So this way this can be managed. So this exercise I am living to you to do something yourself checking by taking the different altitude of the orbit and then working with how much the nodal line will; what will be the; what kind of sun synchronous Orbit you are trying to plan.

This β can be always checked that the time of the launch of the orbit correction the next step we are going to take this the last two weeks will be covering and the time the orbit maneuver. Here also what we can observe that if for this $\Delta\dot{\omega}$ if i is greater than 63.4° so ω will decrease let us write somewhere else. If i is greater than 63.4° you can check it then ω will decrease as T increases.

If i is less than 63.4° then ω will increase as T increases. This is very much visible from this equation if i is a small say i is 0 you can immediately see $\Delta\dot{\omega}$ will be positive. But if i is equal to 90° so in that case this becomes 1, you can see immediately $\Delta\dot{\omega}$ becomes negative. So 90° greater than 63.4 and 0°s less than 63.4 immediately you can observe. So we will wind up this lecture here in this place. Thank u for listening.