Space Flight Mechanics

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Lecture No - 67
General Perturbation Theory (Contd.)

Welcome to lecture 67 in the previous lecture we just finished the satellite movement earth about an oblate spheroid and then we looked into how the sun synchronous orbit can be generated. Now, we look the perturbation on the satellite due to the third body. So here in this case we can do a general treatment for this particular topic, but it is a very long and this lecture is going to be the last lecture on the general orbit perturbation theory.

So in very brief, I will discuss about the movement of moon around the earth and getting perturbed by the sun. So, this is the topic and I will discuss it briefly so that at today we finish this particular topic and then we get into the orbit determination problem.

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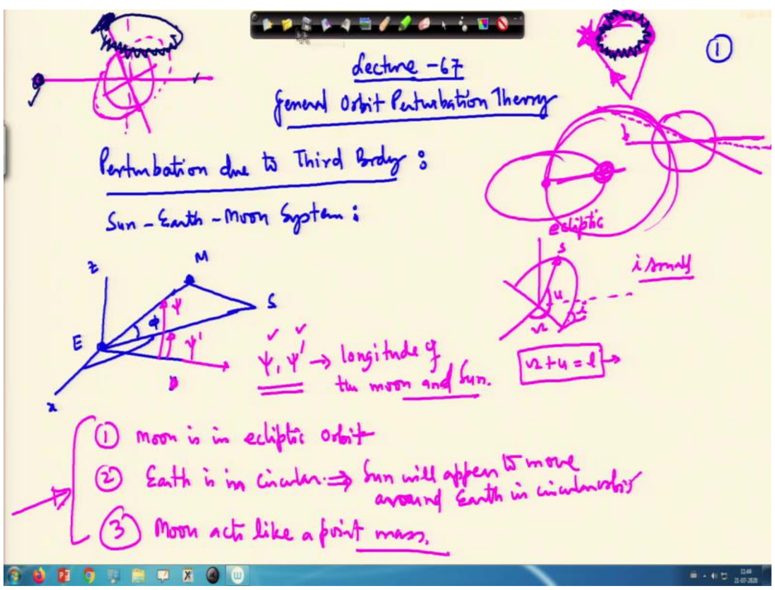

So the earth is located here. So this is earth and moon is here and the sun is located here this angle we write a ϕ. This angle will indicate by Ψ' and this angle will indicate by Ψ. So, this is not necessarily y what I have shown here. This is just the Ψ and Ψ' it is a longitude of the moon and sun. So in what assumption we are trying to work with moon is in epileptic orbit that means; if this is Sun and around the Sun the Earth is moving so this plane we are called as the ecliptic plane.

And earlier I have shown that if this is earth and Earth equator is; and if this is the ecliptic plane so here the earth is there and this is the ecliptic plane which is shown here. So, the equator of the earth will be in clan like this at 23 and half ° and moon's orbit it is say round 5° inclined with the plane of ecliptic. So moon orbit will be somewhere say at 5°. So with the plane of ecliptic moons inclination is not much.

°

We have to simplify the problem. So in that case we can indicate that their longitude $\Psi, \dot{\Psi}$. So in a general problem say in the case of; say if this is a satellite this is Ω and we know this angle is u = ω +θ, Ω plus u this is written as the Longitudinal. But if the angle of inclination is large then it is not valid because Ω is in a different plane and u is in a different plane. It is not valid, but if the angle is a small i is small, so for a small i we can write longitude equal to Ω plus u. So we assume that moon is in the ecliptic orbit.

And then we also assume Earth in circular orbit. So, Earth is in circular orbit so this implies that if you look from the Earth, Sun will appear to move in a circular orbit. So this implies Sun will appear to move around earth in a circular orbit. And the third assumption we made moon acts like a point mass. So here the Sun is far away from the Earth and if this is earth and Sun is at a large distance but still the oblate shape of the earth causes precession motion of the axis of spin.

As we know you are in the right in the beginning of this course, I have discussed this part that the earth axis it moves in a circle. So this is the spin axis of the Earth so it will move in a circle and right now we will say it is a pointing toward the Polaris what we called as a pole star in Hindi we call it the Dhrupthara. Right now the spin axis it pointing toward the pole star. But it will not be so all the time.

Another 5000 years this pole star **(FL: 06:33)** will no longer be the Polaris for the earth. In around 25800 years or nearly 26000 years this earth spin axis goes one end of moves in a circle. So, this motion we call it as precession motion and besides this we have also the nutation motion of the spin axis of the earth this is called the nutation motion it will keep bubbling here. It will keep it will rotate like this and also it will keep bubbling about this is called the nutation motion this happens because of this one and it can be computed also.

So if you look on problem you will find in; if you have done mechanics course, then you will see in the Real Johnston book on the mechanics that there is a problem related to the earth. The inertia of the Earth is given. So this problem is related to the satellite attitude dynamics and then you have to find the period of what will the period of precession given the moment of inertia of the earth and distance of the sun on all these things are available.

So based on that the year in how many years it will complete one precession so that has to be computed period of precession. So that turns out to be around 26000 years 25800 nearly.

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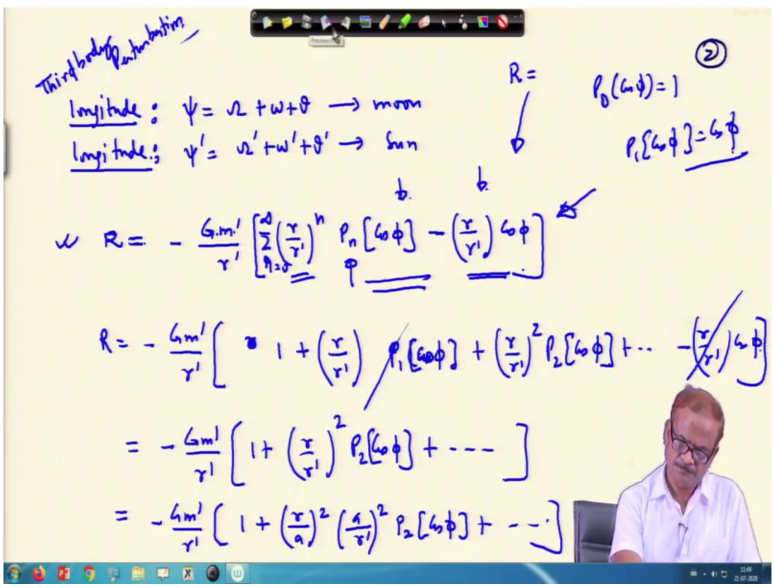

$$\Psi = \Omega + \omega + \theta$$

$$\Psi' = \Omega' + \omega' + \theta'$$

So here we then define Ψ is equal to $\Omega + \omega + \theta$ and Ψ' equal to $\Omega' + \omega' + \theta'$ this is for the moon longitude of Moon and this is for Sun. Already we have looked into that R the perturbation term due to the third body is given by Gm' by r' and this part we have ok this part we have not done. Ok, but still we do not have much time to carry out this work.

$$R = -\frac{Gm'}{r'}\left[\left(\frac{r}{r'}\right)^n P_n cos\phi - \left(\frac{r}{r'}\right)\cos\phi\right]$$

But what we do here that; if we do the whole analysis, it will take little time. Ok I will supply that those materials to you. So basically the perturbation due to the third body it can be written as $(r/r')n$ P_n Cos ϕ - r by r' cos ϕ and from where it is coming so I will supply you the lecture the corresponding lecture for that. See exactly what we have done if we go back and look here in this place.

The moon is moving around the earth and this is getting perturbed by the sun. So what will be the perturbation force acting on the moon for this we have already done in the three body problem we have written that particular equation. So that particular equation can be utilized and the equation can be written rather than discussing all those things because we do not have much time so we just concentrate on this.

Given that this equation is the perturbation potential can be expressed here in this format, which of course I will supply you the respective material. We can expand this and write this as Gm' by r' and here this is your Legendre polynomial as we have worked for the oblate spheroid. And this is summation 0 to infinity n equal to 0 to infinity and here. This is minus. This is a separate term so once we use this and expand it for the first term you can see immediately P_0 cos ϕ this term is equal to 1.

$$R = -\frac{Gm'}{r'}[1 + \left(\frac{r}{r'}\right)P_1[cos\phi] + \left(\frac{r}{r'}\right)^2 P_2 cos\phi + \cdots - \left(\frac{r}{r'}\right)cos\phi]$$

So therefore this gets reduced 1 n equal to 0 this will be this part also be equal to 1 this way we can expand it and we can write this r by r prime and Cos ϕ P_1 cos ϕ $(r/r')^2$ P_2 cos ϕ and + 1 – (r/r') cos ϕ. So, P_1 cos ϕ this is equal to cos ϕ and we utilise this value of this term and this term will drop out and then we are left with Gm' by r' 1 + $(r/r')^2$ P_2 cos ϕ.

$$R = -\frac{Gm'}{r'}[1 + \left(\frac{r}{r'}\right)^2 \left(\frac{a}{r'}\right)^2 P_2 cos\phi$$

See here we are taking them as a point, mass and therefore the perturbation due to the oblate shaped and other things they are not coming into picture. This is just because of the third body perturbation. Again, I am reminding this is the third body perturbation and from where we have got this equation? This I will supply later on while the course runs. Ok and so on all the terms can be written. This we can write as r by a^2 times $\left(\frac{a}{r'}\right)^2$ times P_2 cos ϕ like this.

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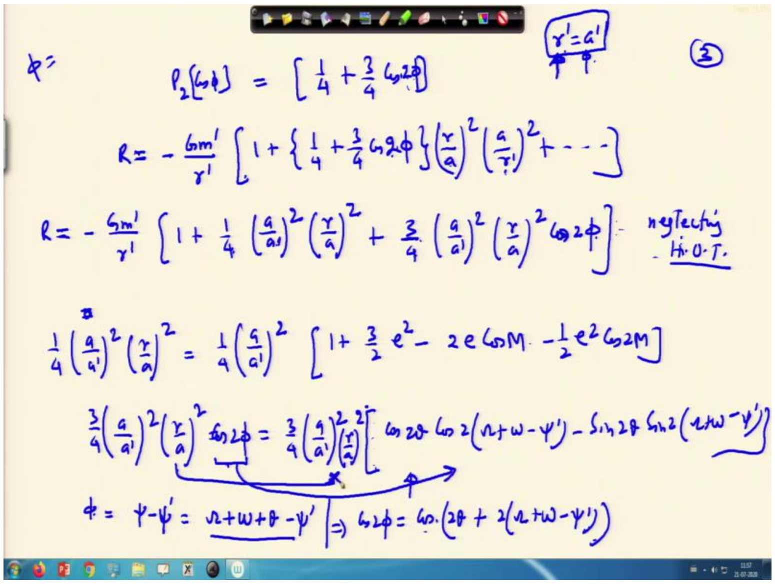

$$P_2 \cos \phi = \left[\frac{1}{4} + \frac{3}{4}\cos 2\phi\right]$$

$$R = -\frac{Gm'}{r'}[1 + \left\{\frac{1}{4} + \frac{3}{4}\cos 2\phi\right\}\left(\frac{r}{r'}\right)^2\left(\frac{a}{r'}\right)^2 + \cdots$$

And P_2 cos ϕ this term in the Legend polynomial this is equal to 1 by 4 + 3 by 4 cos ϕ and therefore R gets reduced to - $\frac{Gm'}{r'}$ and see here once we are writing here in this format. So I am done nothing wrong here in that a we have taken in the numerator denominator and separated out in these two terms cos 2 ϕ this is cos 2 ϕ $\frac{r}{r'}$ we have changed. So r by plus higher order terms a general treatment of this for whatever we are doing it is a part of a general treatment.

Actually I wanted to cover that general treatment, but already exceeded large number of lectures by mistake in that part and all those things I am going to supply separately as supplementary material and this is a small portion I am just dealing with. Now some of the results we are going to use here which will find in those supplementary materials provided. So using those results this can be written as 1 by 1 + 4 a by; now r'will be equal to a'.

$$R = -\frac{Gm'}{r'}\left[1 + \frac{1}{4}\left(\frac{a}{a'}\right)^2\left(\frac{r}{a}\right)^2 + \frac{3}{4}\left(\frac{a}{a'}\right)^2\left(\frac{r}{a}\right)^2\cos 2\phi\right]$$

Here r' this is the this is a standing for Sun and if you are assuming the Sun to be in circular, orbit, OK then this will be equal to a'. So this r' we can replace with in terms of $a'$and we can write as

a'^2 and then the other part remains r by a^2 these are the minute manipulation which are done 3 by 4 a by a'^2 r by a^2 times Cos 2ϕ so we neglect the higher order terms.

$$\frac{1}{4}\left(\frac{a}{a'}\right)^2\left(\frac{r}{a}\right)^2 = \frac{1}{4}\left(\frac{a}{a'}\right)^2\left[1+\frac{3}{2}e^2 - 2e\ cosM - \frac{1}{2}e^2 cos2M\ \right]$$

So we terminate the higher order terms and simply write it as neglecting higher order terms. Now 1 by 4 a by a'^2 r by a can be expanded all these materials you will find their and written here in this format. Cos m actually this can be taken also as the exercise problems 1 + 3 by 2 e^2 – 2e cos m - 1 by 2 e^2 cos 2m a^2 times cos 2 ϕ. This we can write as r by a, that already we expanded so this we delete here and cos 2 ϕ we have to expand.

So, ϕ we have not still defined so ϕ is here; this is the angle between ϕ equal to Ψ minus Ψ′. This angle we have to find sofa using this 5 week and expanded so this file is Ψ - Ψ′ and Ψ is how much this is Ω plus u and Ω u we will expand so this will be Ω +θ – Ψ′. So this way we can write this cos 2 ϕ as cos to ϕ implies cos 2ϕ will be equal to cos 2θ + 2Ω + ω – Ψ′. We write this way and then expand this term here.

$$\Psi - \Psi' = \Omega + \omega + \theta - \ \Psi'$$

$$\frac{3}{4}\left(\frac{a}{a'}\right)^2\left(\frac{r}{a}\right)^2 \cos 2\phi = \frac{3}{4}\left(\frac{a}{a'}\right)^2 [cos2\theta \cos 2\ (\Omega + \omega + \Psi') - sin2\theta\ sin2(\Omega + \omega - \Psi')]$$

So this will be cos2θ times cos 2 (Ω + ω – Ψ′). Mall Ω -3 Prime minus, sin 2θ, sin 2 (Ω +ω – Ψ′). So, this expansion to be written this r by a here because for this extension we do not have a space to write here so let us retain it here r by a^2. So, only the cost 2ϕ term that we expanded here in this place. And this r by a term we have retained here.

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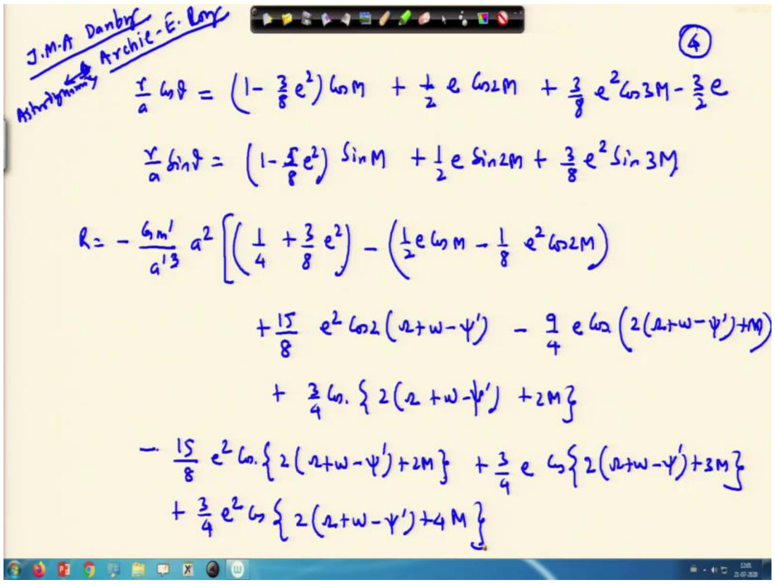

$$\frac{r}{a} cos\theta = \left(1 - \frac{3}{8}e^2\right)\cos M + \frac{1}{2}e\cos 2M + \frac{3}{8}\ e^2\cos 3M - \frac{3}{2}e$$

r by a cosθ so r by a relationship we are aware from our earlier derivation so this quantity will be equal to cos m + 1 by 2 these are some of the expansion part frequently used in celestial mechanics. You can use book by J M A Danby on Celestial Mechanics also by Smart; another book I will give you shortly in the in the next class. And Archie E Roy this is on the fundamentals of Astrodynamics.

$$\frac{r}{a}\sin\theta = \left(1 - \frac{5}{8}e^2\right)\sin M + \frac{1}{2}e\sin 2M + \frac{3}{8}\ e^2\sin 3M$$

All these materials I will provide the soft copy and written soft copy. So these are the expression so this we need to utilise in the this particular parts so we can expand sin 2θ and Cos 2θ here and replace this and where θ is the true anomaly of the Moon. So if we do this, so r can be written as minus Gm'divided by a'^3 a^2 1 by 4 + 3 by 8 e^2. So we need to insert all these values here in this and then expand.

$$R = -\frac{Gm'}{a'^3}\ a^2\ \left[\left(\frac{1}{4} + \frac{3}{8}e^2\right) - \left(\frac{1}{2}\ e\cos M - \frac{1}{8}e^2\cos 2M\right) + \frac{15}{8}e^2\cos 2(\Omega + \omega - \Psi') - \frac{9}{4}e\cos 2\big((\Omega + \omega - \Psi') + M\big) + \frac{3}{4}\cos(2(\Omega + \omega - \Psi') + 2M) - \frac{15}{8}e^2\cos\{2(\Omega + \omega - \Psi') + 2M\} + \frac{3}{4}e\cos\{2(\Omega + \omega - \Psi') + 3M\} + \frac{3}{4}\ e^2\cos\{2(\Omega + \omega - \Psi') + 4M\}\right]$$

So this can be written in this format and reorganized + 15 by 8 cos 2 (Ω+ ω $-\Psi'$) remember this is an approximate analysis +m this way we can expand it and reorganized to write in this format.

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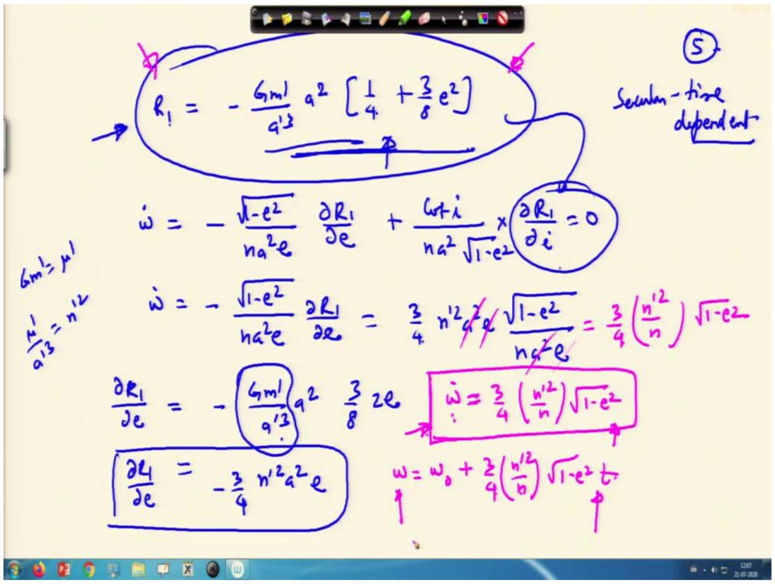

$$R_1 = -\frac{Gm'}{a'^3} a^2 [\frac{1}{4}]$$

So the first time here this is a constant term. So this term gives rise to the secular perturbation this is referring to secular perturbation so let us say that we write R_1 equal to - $\frac{Gm'}{a'^3}$ 1 by 4 + 3 by 8 e^2. And look into the corresponding how the periapsis is getting affected because of this; $\dot{\omega}$ can be written as 1 - e^2 under root this is from our Lagrange planetary equation $\partial R_1/\partial e$ + cot i divided by n a^2 1 - e^2 $\partial R_1/\partial e$ only we look into the effect of this particular term which is arising from this term multiplied by this particular from here.

$$\dot{\omega} = -\frac{\sqrt{1-e^2}}{na^2e}\frac{\partial R_1}{\partial e} + \frac{\cot i}{na^2\sqrt{1-e^2}}\frac{\partial R_1}{\partial i}$$

$$\dot{\omega} = -\frac{\sqrt{1-e^2}}{na^2e}\frac{\partial R_1}{\partial e}$$

Immediately we can see that because of this from this will be equal to 0 this particular part. This multiplied by $\frac{\partial R_1}{\partial i}$ this quantity is 0 because i is nowhere present here. So we are left with just estimating n a^2 e $\partial R_1/\partial e$. So, we have to estimate this quantity. So here we have replaced by R_1 so let us write here this as the R_1. We will write this as the R_1 because we are just looking effect of this term.

$$\frac{\partial R_1}{\partial e} = -\frac{Gm'}{a'^3} a^2 \frac{3}{8} 2e$$

$$\frac{\partial R_1}{\partial e} = -\frac{3}{4} n'^2 a^2 e$$

We are not looking the effect of all the terms we are separately looking the effect of different terms. So this will give rise to the secular perturbation means a time-dependent. Ok secular means time dependent. So, $\frac{\partial R_1}{\partial e}$ this quantity from here we get as - $\frac{Gm'}{a'^3}$ a^2 3 by 8 times 2e and this quantity that is nothing but mu prime by a'^3 which is n'^2.

So, $n'^2 a^2$ and this we can write as 3 by 4 with minus sign e. So, $\frac{\partial R_1}{\partial e}$ is this quantity. Where Gm' we are replacing by μ' so μ' by a'^3 is equal to n'^2. And of course 1 - e^2 then we have to we will evaluate this item the ones we insert this so we get minus minus sign gets plus this gets 3 by 4 n'^2

a^2 e 1 - e^2 under root divided by na^2 e then this term drops out a^2 a^2 then drops out and we are left with 3 by 4 n'^2 divided n times 1 - e^2 under root so $\dot{\omega}$ is 3 by 4 n'^2 divided by n times 1 - e^2. So this gives you the rate at which the periapsis of the moon will move because of this term only ok not all the terms. So you should try to match the result with the moon perigee movement with rising because of this term, you would not be able to do.

Because the moon perigee movement it will involve all the terms effect of all the terms while here this is giving just because of this from this we called as secular term. It is a constant term appearing here. Now if we integrate it, we will see that the t will manifest here in this place. So by direct integration we do not do we; some of the terms were say $\dot{\Psi}$ it is involved. So there we do not do the direct integration.

$$\dot{\omega} = \omega_0 + \frac{3}{4}\left(\frac{n'^2}{n}\right)\sqrt{1-e^2}$$

In this case we can do and write this is ω equal to ω_0 + 3 by 4 n'^2 divided by n 1 - e^2 t. So this is showing that this is for changing with as that t is changing and therefore we call this is the secular perturbation.

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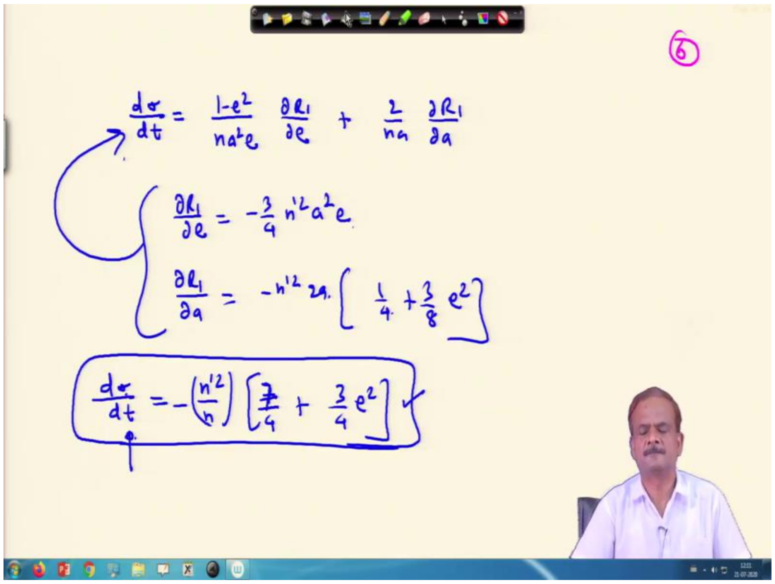

$$\frac{d\sigma}{dt} = \frac{1-e^2}{na^2e}\frac{\partial \mathrm{R}_1}{\partial \mathrm{e}} + \frac{2}{\mathrm{na}}\frac{\partial \mathrm{R}_1}{\partial \mathrm{a}}$$

$$\frac{\partial \mathrm{R}_1}{\partial \mathrm{e}} = -\frac{3}{4}\, n'^2 a^2 e$$

$$\frac{\partial \mathrm{R}_1}{\partial \mathrm{a}} = -n'^2 2a\, \left[\frac{1}{4} + \frac{3}{8}e^2\right]$$

Similarly the perturbation in the σ we can work out 1 - e^2 n a^2 e ∂R_1/∂e + 2 by n a $\frac{\partial \mathrm{R}_1}{\partial \mathrm{e}}$ so this Lagrange Planetary equation and we replace this by 1 and look back into the equation $\frac{\partial \mathrm{R}_1}{\partial \mathrm{e}}$ already we have calculated $\frac{\partial \mathrm{R}_1}{\partial \mathrm{e}}$ -3 by 4 n prime^2 a^2 e and $\frac{\partial \mathrm{R}_1}{\partial \mathrm{a}}$ we have to compute. So R_1 is given here. So this quantity will be Gm', So, for Gm' quality free article writing as this quantity you are writing as n'^2. So n'^2 2a then once we differentiate with respect to a so this will give us 2.

$$\frac{d\sigma}{dt} = -\left(\frac{n'^2}{n}\right)^2 \left[\frac{7}{4} + \frac{3}{4}e^2\right]$$

So, this gives us n'^2 times 2a and then multiplied by rest of the quantity in the bracket. And then insert this 2 results into this equation and that gives you $\frac{d\sigma}{dt}$ if you work it out the whole thing this will result in minus n^2. I am skipping some of the steps 7 by 4 + 3 by 4 e^2. This is also you can see that this is just dependent on t. And this will be a factor of this is also a secular terms.

So, this way these terms are giving rise to the secular perturbation. So particular celestial body movement, it is a combination of the; now you take the various terms all returns are available in that as we go back and look into this. So, we have neglected many of the terms but some of the terms which are visible here. Here, you can look into these particular parts. All these terms are there. Ok if you work for them will get perturbation for various terms.

But out of that; some the periodic some will be the long periodic some short periodic secular terms and so on. So the effect of them all can be combined and looked into the motion of the Moon. So, we stop here and I will supply you all the related materials because we required for the at least some 7, 8 hours to complete this part related to third party perturbation which we do not have. So, we stop here and I will supply the rest of the material in supplementary format. Thank you very much.