**Space Flight Mechanics**
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**Lecture - 07**
**Gravitational Central Force Motion (Contd)**

Okay welcome to lecture number 7. So, we have been discussing about the gravitational central force motion. So, we will continue with that. So, last time if you remember we have derived the conic section equation from the first principle using Newton's law okay. So, in the Kepler's law as you know there are three of them So, first of them was about the planet moving in elliptical orbit around the sun with the sun at one of its focus So, now rest two are remaining so we will work out the rest two.

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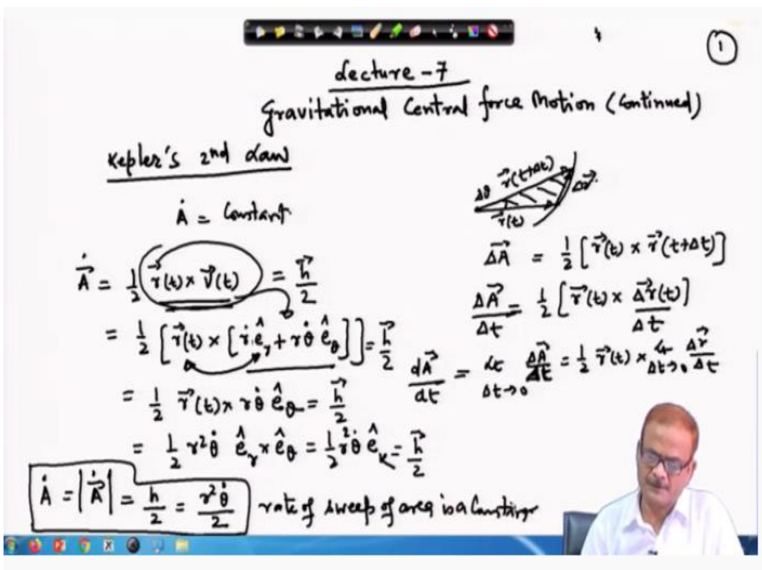


Kepler's second law okay what we have stated that $\dot{\boldsymbol{A}}$ this is a constant and already we have worked out earlier if you remember that if this is the center of attraction or the center of force and this is $r_t$ and this is $r_t$ + Δt. So, this vector will be Δr and so the area of this is $\Delta\theta$ area of this we have written as ½ times $r_t$ and this is vector t + Δt. This is ΔA this is basically this has to be here and from there we have observed that this can be written as $r_t \times \Delta r_t$.

Recalling from the last lecture and therefore ΔA/Δt we have written it this way and in the limit Δt tends to 0. So, we have reduced this to the form

$$\frac{\Delta A}{\Delta t} = \frac{1}{2}\, r_t \times \lim_{\Delta t \to 0} \left(\frac{\Delta \vec{r}}{\Delta t}\right)$$

So this quantity can be written as dA/dt and it is nothing but $\dot{\boldsymbol{A}}$. So,

$$\dot{\boldsymbol{A}} = \frac{1}{2}\; r_t \times \vec{v}(t)$$

and then we redo this to the form writing it as

$$= \frac{1}{2}\; r_t \times \left(\dot{r}\widehat{e_r} + \mathrm{r}\,\dot{\theta}\,\widehat{e_\theta}\right)$$

So, we have replaced this *v* by this as we have derived earlier. And this and this is a cross product will be 0 and therefore we get here $\frac{1}{2}\; r_t \times \mathrm{r}\,\dot{\theta}\,\widehat{e_\theta}$ and of course this gets reduced to

$$= \frac{1}{2}\; r^2\dot{\theta}\;\widehat{e_r} \times \widehat{e_\theta}$$

and which is

$$= \frac{1}{2}\; r^2\dot{\theta}\;\hat{e}$$

And if we look magnitude wise so we have the magnitude of this which we write as $\dot{A}$ here in this case.

So, this becomes $\frac{1}{2}\; r^2\dot{\theta}$ and this quantity as we know this quantity is nothing but your $\vec{r} \times \vec{v}$, we have written this as $h$. So, this becomes magnitude of $h$. So, half of that so this becomes $h/2$ in the right-hand side this is h/2 this is equal to h/2. So, from here we get this quantity as $r^2\dot{\theta}$ this we are writing as $h$. So, $\dot{A}$ it becomes magnitude wise we can write it as h/2 or $r^2\dot{\theta}/ 2$.

$$\boldsymbol{\dot{A} = \frac{h}{2} = \frac{r^2\dot{\theta}}{2}}$$

So, this is your **rate of sweep of area is a constant** this is your **Kepler's second law**. Now remains the third law.

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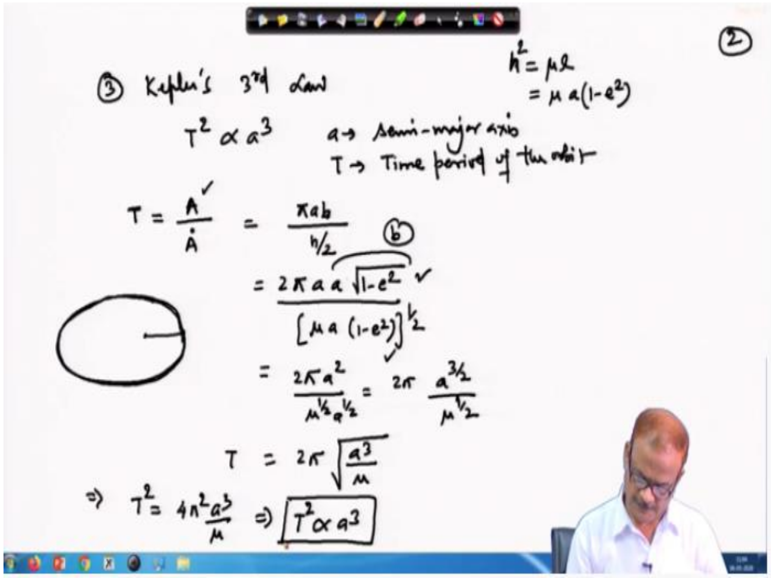


So, third law Kepler's third law and this we have stated as $T^2$ is proportional to $a^3$ where $a$ is the semi major axis and T is the time period of the orbit. So, we know that the time taken for completing the orbit will be nothing but $A/\dot{A}$ . Because $\dot{A}$ is a constant if you remember the rate of shape of area is a constant therefore the total time will be for completing the orbit going once around the orbit is starting from this place and then they turning back over this part to this again.

So, the whole area is covered so in the case of the eclipse we know this area is $\pi ab$ then A= h/2 $\dot{A}$ = h/2. So, this becomes $2\pi a$, $b$ is $a\sqrt{1-\mathrm{e}^2}$ and this is b and h as you remember last time, we have derived this $h^2 = \mu\, l$ and this is nothing but $(1 - e^2)$ . So, insert it here so this becomes $\mu$ times a times $1 - e^2$ and this is under root this two will cancel out $1 - e^2$ and this one will cancel out.

Because both of them are having under root we get here

$$= \frac{2\pi a^2}{\mu^{\frac{1}{2}} a^{\frac{1}{2}}}$$

this is

$$= \frac{2\pi a^{\frac{3}{2}}}{\mu^{\frac{1}{2}}}$$

So, 2 π $a^3$ divided by $\mu$ under root, so this is your T. So therefore,

$$T^2 = 4\frac{\pi^2 a^3}{\mu}$$

and this implies,

$$\boldsymbol{T^2 \propto a^3}$$

and this is your **Kepler's third law**. So, all the three laws we have been able to work out using the first principle okay that is from the; we have started with Newton's law and then we worked out that.

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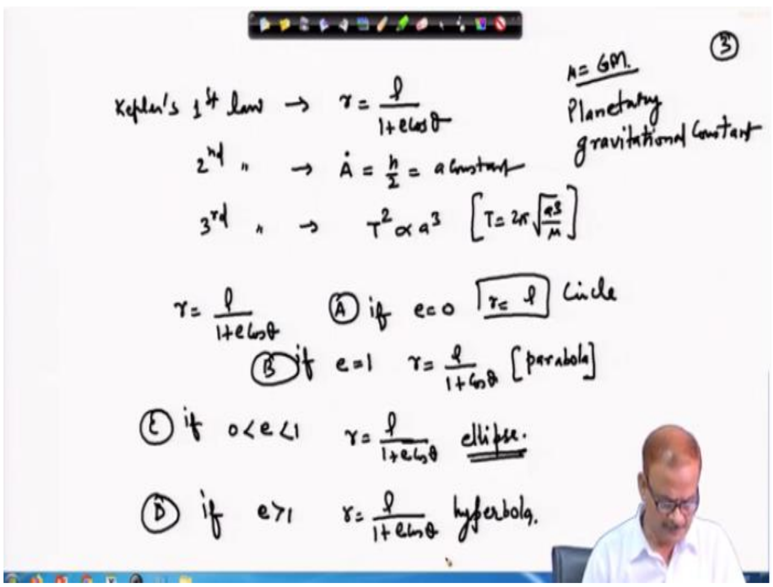


Okay so now we have to summarize the whole thing so Kepler's first law this we are describing as

$$r = \frac{l}{1 + e\cos\theta}$$

Kepler's second law this we are describing as the time period $\dot{A} = \frac{h}{2}$, a constant and third law we have written as $T^2$ proportional to $a^3$ or either we have written in the format as

$$\boldsymbol{T = 2\pi\sqrt{\frac{a^3}{\mu}}}$$

Here $\mu$ = GM. this is called the planetary gravitational constant. Here in this equation. l/1 + e cosθ if e = 0, so you can observe that r= 1. So, this gets reduced to a circle.

Similarly if e = 1 so r becomes $l/1 + cos$ θ we are putting e = 1 so this is 1 + cos θ and for this, this is called a parabola equation if $e$ lies between this is (A), (B) if $e$ lies between 0 and 1. So as it is we write l/1+ e cosθ and here in this case this is the case of an eclipse and (D) if $e$ greater than 1 is in that case again we indicated by the same equation but in this case this is called hyperbola.

Okay so we have finished the Kepler's law and also, we have finished the central force motion. So, before we go into the two-body problem I would like to work out one problem a simple problem.

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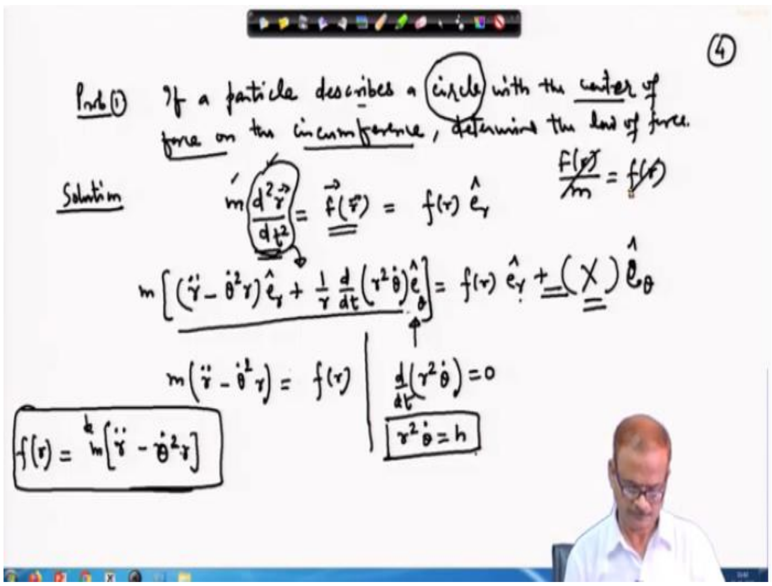


Using the principles still now we have developed. So, now we will discuss a problem which is as stated as so the principles, we have developed so based on that we will discuss. If a particle describes a circle with the center of force on the circumference determine the law of force. So, we apply the Newton's basic principle because it is given that a particle describes a circle with the center of force on the circumference means it is directed toward the point which is lying on the circumference of a circle.

So, these are the important words so here we can write this as f(r) times $\hat{e_r}$ because this is directed toward the center of the circle and m is the mass of the particle. So, if you remember this term we have expanded and got in the last lecture as

$$m\left(\left(\ddot{r}-r\dot{\theta}^2\right)\widehat{e_r}+\frac{1}{r}\frac{d\left(r^2\dot{\theta}\right)}{dt}\widehat{e_\theta}\right)=f(r)\widehat{e_r}+(X)\widehat{e_\theta}$$

this term once expanded this results in this particular term which is written here in the bracket. So, on the right-hand side this will be $f(r)\widehat{e_r}+(X)\widehat{e_\theta}$ is also present. So here $\widehat{e_\theta}$ is also present. So, comparing on both the sides; so we see that $e_\theta$ term $\widehat{e_\theta}$ term is not here on the right-hand side. Therefore, we will write this as $m\left(\ddot{r}-r\dot{\theta}^2\right)=f(r)$ or just comparing these two terms. So, we can write this as f(r) and here $r^2\dot{\theta}$.

Because in right hand side there is no term corresponding to $\widehat{e_\theta}$ ,this term is not present and therefore this quantity d/dt $(r^2\dot{\theta})$ this will be 0 and that gives us as per our usual description $r^2\dot{\theta}$= h we have derived it. So, this is one formulation another formulation is here. So, $\ddot{r}-r\dot{\theta}^2$ this is equal to f(r). So, this is the law of force which we need to work out okay.

I hope that the things are clear we have applied the basic Newton's law mass times acceleration is the force acting on the system and it is possible that we could have written by we could have divided this while writing it divided by *m*, we could have written this as f(r). So, if you do this this will get further simplified but let us carry this this *m*, we will carry it rather than changing it.

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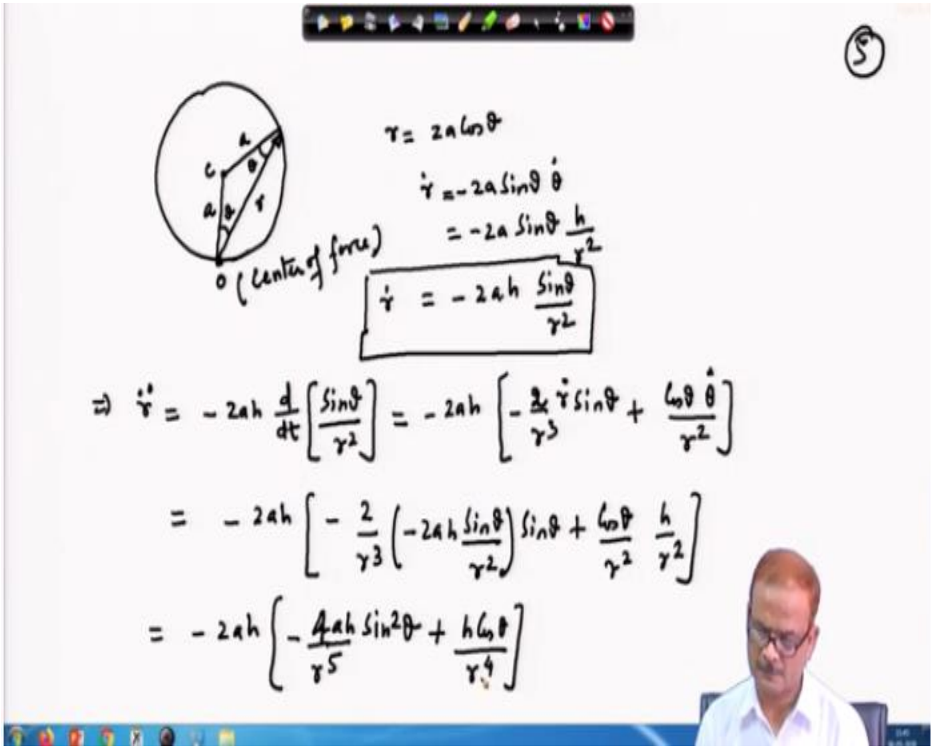


Okay so we have now given that it is the particle is moving on the circumference of the circle this is the center and this is the point the center of force. This angle we will write it as θ this we will write it as *a a,* okay therefore if we write this as *r,* so r becomes 2a cosθ by the property of this circle we are using it these two angles are θonce we differentiate this. So, this is 2a sinθ, $\dot{\theta}$ with minus sign *2 a sin*$\theta$ and then we utilize this relation this particular one this is $h/r^2$.

So, this is -*2a h sin*$\theta$ divided by $r^2$ this is$\dot{r}$ equation. So, therefore $\ddot{r}$ then gets reduced to

$$\ddot{r} = 2ah\ \frac{d}{dt}\left(\frac{sin\theta}{r^2}\right)$$

Here $\dot{r}$ we will insert from this this -2ah sin θ divided by $r^2$ this $\dot{r}$ we have replaced.

$$\dot{r} = -2\text{a h sin}\theta$$

we have this time sinθ + cos θ, $\dot{\theta}$ will replace from the earlier equation $h/r^2$. So, this is

$$\ddot{r} = -2ah\left(\frac{4ah}{r^5}\sin^2\theta + h\frac{cos\theta}{r^4}\right)$$

You need to eliminate this sin θ and cosθ, so we do it on the we carry it on the next page.

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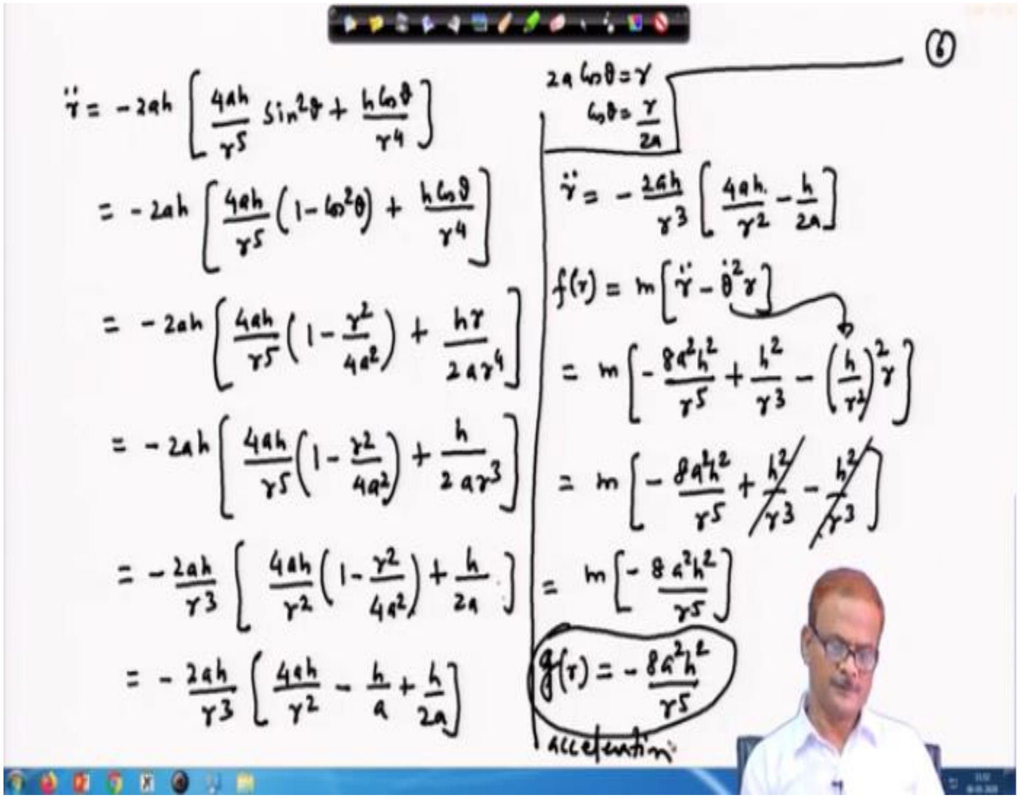


So, we have $\ddot{r}$ and this equal to -2ah and - 4ah minus minus this gets plus, so we have to put it plus here $4ah/r^5$ and then $\sin^2\theta$. So,

$$\ddot{r} = -2ah\left(\frac{4ah}{r^5}\sin^2\theta + h\frac{cos\theta}{r^4}\right)$$

This we need to eliminate here θ. So we do it step by step .

$$\ddot{r} = -2ah\left(\frac{4ah}{r^5}(1-\cos^2\theta) + h\frac{cos\theta}{r^4}\right)$$

and we know that $2a\ cos\ \theta\ =\ r$ so we replace $\cos\theta$, here

$$cos\theta\ = \frac{r}{2a}$$

$$\ddot{r} = -2ah\left(\frac{4ah}{r^5}(1-\frac{r^2}{4a^2}) + h\frac{r}{2ar^4}\right)$$

$$= -2ah\left(\frac{4ah}{r^5}(1-\frac{r^2}{4a^2}) + \frac{h}{2ar^3}\right)$$

So, we will do little simplification at this stage we will pull out here 2ah $r^3$ outside and this is

$$= -\frac{2ah}{r^3}\left(\frac{4ah}{r^2}(1-\frac{r^2}{4a^2}) + \frac{h}{2a}\right)$$

$r^3$ has gone $r^3$ we have already taken outside the bracket. Therefore this is $2ah/r^3$ and then within the bracket $r^2$– 4a 4a cancels out this is $h/a\ +\ h/2a$. So,

$$\ddot{r} = -\frac{2ah}{r^3}\left(\frac{4ah}{r^2} - \frac{h}{2a}\right)$$

and then therefore f(r) this equal to *m* times going back here using this equation okay m times $\ddot{r}$– $\dot{\theta}^2$ r m times $\ddot{r}$– $\omega^2$ r basically this term is.

$$\mathrm{f}f(r) = m(\ddot{r} - \dot{\theta}^2 r)$$

So, we need to insert the terms here m times $\ddot{r}$ this equal to now we will get to the bracket this becomes 4 ×2 this is 8 $a^2$ $h^2/r^5$ and then we get a plus sign here this is 2a 2a get cancels out this becomes $h^2/r^3$ – $\dot{\theta}$ is nothing but $h/r^2$.

So this whole square this particular term and then multiplied by *r* this is m 8 $a^2$ $h^2/$ $r^5$, this becomes $h^2/r^3$.

So, these two terms they cancel out leaving out m times -8 $a^2$ $h^2/r^5$. So, therefore if we eliminate this m and we write here in terms of some other function let us say instead of f we indicate it by some other function. So, I indicate by *g* so *g(r)* then this becomes

$$\boldsymbol{g(r) = -8\frac{a^2h^2}{r^5}}$$

So, this is the **law of force**, so this is basically the acceleration term this is the using the basic principle that we have developed. So, we are able to solve this problem which was to find out that this particle is moving on the circumference of a circle.

So, circle was one condition another condition was that the force itself is lying on the circumference of the circle. Center of the force is lying on the center of force itself is lying on the circumference of the circle. So, this way we have been able to solve the problem. Okay so we will end up this lecture here and continue with the next lecture. Thank you very much for listening.