Space Flight Mechanics

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Lecture - 08
Body Problem

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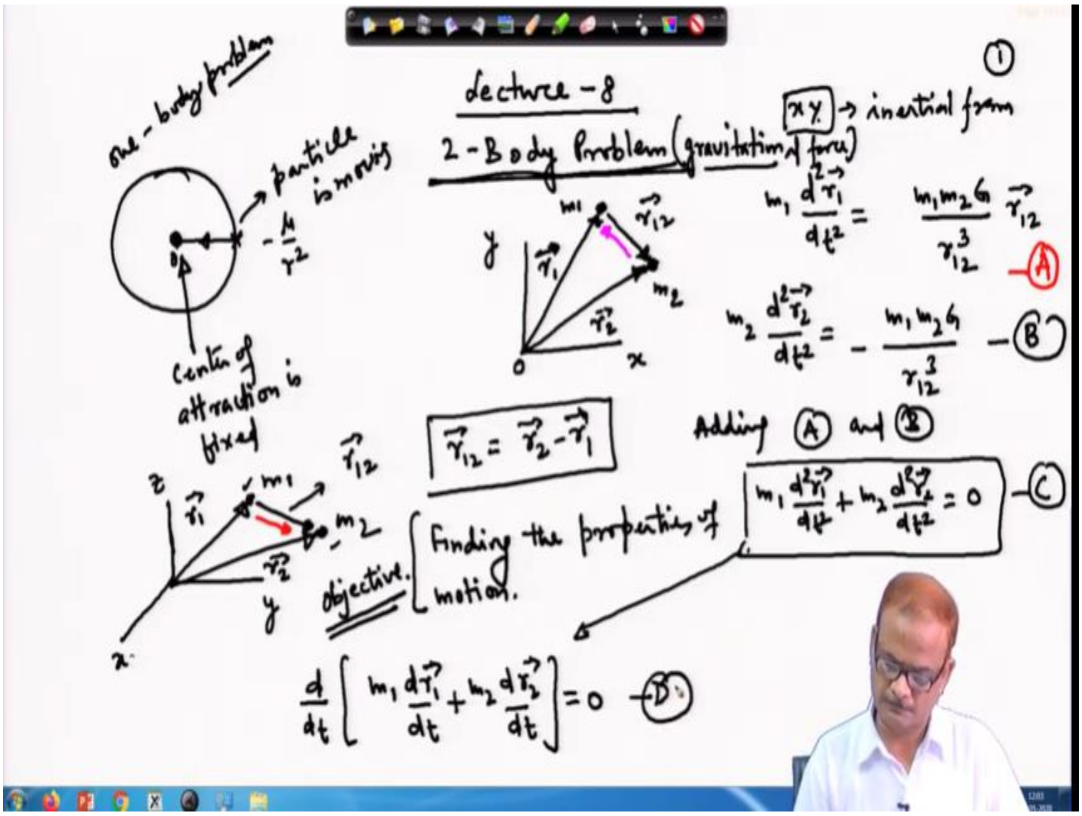

Welcome to lecture number 8. So, till now we have a discussed about the particle which is moving about a centre of force. And thereafter we went into the gravitational force motion which means whether the particle was moving in on a curvilinear path about the centre of a force where the force was directed toward the fixed point O okay and we taking the gravitational force means this force is described by $-\mu / r^2$. So, in this assumption what we have done that we have assumed this to be the this is the centre of attraction to be fixed the centre of attraction is fixed the particle is moving.

So, this will call as the one body problem now the situation is, we have the two-body problem in which I have a resistance say the X, Y and there are two bodies this is the two particles two body problem also this is called as two particle system. Let us say whenever we are discussing we are discussing about a particle remember that Newton's law is applicable to particle and it so happens that once we apply to a system of particles it gets reduced to the same format as the one particle system and therefore we quite often we assume body to be a particle and work with the system.

So, here we assume that this is the radius vector $\boldsymbol{r_1}$ to the particle m_1 and this is the radius vector $\boldsymbol{r_1}$ to the particle m_2 and this is point O. So, X, Y this is the inertial frame okay not necessary that this is only for a two dimensional one this can be X this kind of look like this X Y and Zand we have two particles here which is m_1 and m_2 and this is the $\boldsymbol{r_2}$ vector, $\boldsymbol{r_1}$ vector and another vector is directed.

So, this vector we are writing as $\boldsymbol{r_{12}}$ means it is directed from 1 to 2. So $\boldsymbol{r_{12}}$ this will be written as $\boldsymbol{r_2}$ - $\boldsymbol{r_1}$ and this is a standard notation I will be using quite often. So here if I write that $\boldsymbol{r_{12}}$ means it is going from 1 to 2. Therefore, the vector becomes $\boldsymbol{r_2}$ - $\boldsymbol{r_1}$ this is logical. So here it is a only X, Y plane or it is X, Y, Z does not depend on that you can take X, Y, Z or either this here it looks like this is a planner case and yet it looks like that did it not in this three dimensional plane.

But the motion will be a planner one only it will be confined to the plane as we are going to get the result. So, what we are interested in here finding the properties of motion this is our objective. So, we will start with the Newton's law for working out this problem and Newton's law as you know this is applicable only in inertial reference frame this we cannot apply to a non-inertial reference frame.

So therefore, the starting with the first particle motion we can write as

$$m_1 \frac{d^2 r_1}{dt^2}$$

and the force are acting on this particle will be times $\boldsymbol{r_{12}}$ this vector. So,

$$m_1 \frac{d^2 r_1}{dt^2} = \frac{m_1 m_2 G}{\boldsymbol{r_{12}^3}}$$

So, where the force is directed on this particle m_1 it is directed here along this direction okay it is a, I will show by some other color this is directed along this direction for the particle m_1 okay force is there.

So, under that it is moving so this is our equation number A. Similarly, for the second particle we can write it as

$$m_2 \frac{d^2 r_2}{dt^2} = -\frac{m_1 m_2 G}{\boldsymbol{r}_{21}^3}$$

So, this we are bringing under the mutual gravitational force this description is for mutual gravitational force. So, here $m_1, m_2,$ G r_{12}^3 which is the distance between this and this now in which direction this force is being applied.

So, on m_2 the force will be acting in the opposite direction it is acting here long this so here this will come with a minus sign, so we have to apply here a minus sign this is our equation number B. So, adding A and B just gives us

$$m_1 \frac{d^2 r_1}{dt^2} + m_2 \frac{d^2 r_2}{dt^2} = 0$$

the right-hand side if you add these are the same term so that gets equal to 0. So, this equation we can little rewrite as.

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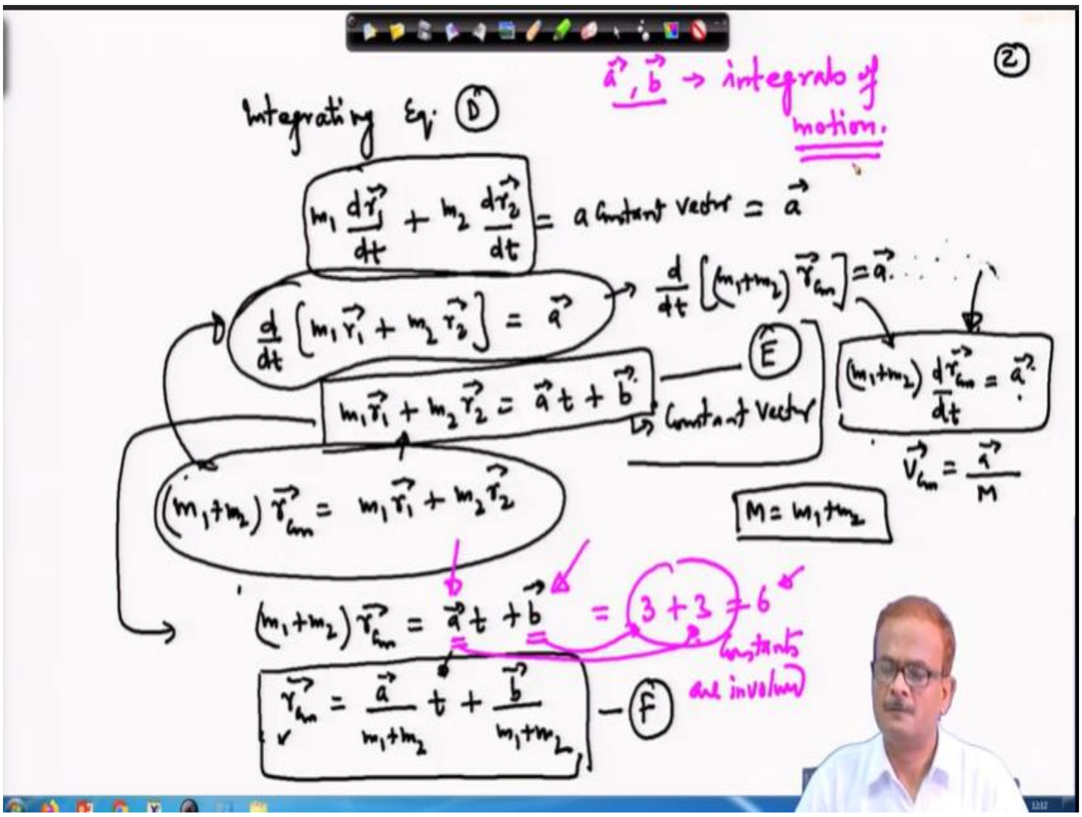

Integrating equation (D) this is a constant a constant vector and let us indicate this by a. Okay next we rewrite it as okay if you differentiate you will get this quantity because the masses are fixed. So, from here we get where b is another constant vector. Now going back here in this place so if you will look here in this place this is the origin and this is the mass m_1 and m_2. So, if you know the definition of the centre of mass. So, the centre of mass definition is m_1 + m_2 . r_{cm} this will be equal to $m_1 r_1$ + m_2 r_2

So, you can see here in this place that this is nothing, but this equation can be written as

$$(m_1 + m_2)\, \overrightarrow{r_{cm}} = \vec{a}t + \vec{b}$$

So, what does this imply this

$$\overrightarrow{r_{cm}} = \vec{a}t \,/\, (m_1 + m_2) + \vec{b} \,/\, (m_1 + m_2)$$

So, this is that the centre of mass changes linearly with time okay or better we can have a look here in this part and this part. So, this part we can rewrite as
d/dt($m_1 r_1$ + m_2 r_2) is $m_1 + m_2$ times r_{cm} = $\vec{a}$ and this implies m_1 + $m_2$if we write it as m.

So, d/dt so this can be written as m_1 + $m_2$we will have to write it in the next line. We can write this because the masses are constant so we can take it outside, so we are using this fact $m_1 r_1$ this expression we are inserting here in this place. Okay so if we insert, we will get it like this

$$m_1 r_1 \frac{dr_{cm}}{dt} = \vec{a}$$

and it is a right-hand side is a constant vector. So, this implies that *v* centre of mass velocity of the centre of mass is equal to a right-hand side is a constant vector.

So, this implies that *v* centre of mass velocity centre of mass = a/M where M= m_1 + m_2 so velocity of the centre of mass is a constant means it is moving in a particular direction so it is moving along that direction it will not change the direction. So, a system of particles free from external force as you know what in the Newton's law, we have learned that the system of particles until unless or either a particle until unless acted upon by external force it will not change its direction or the state of motion.

So, this is what we have learned there so here in this case we have the two-particle system you can consider two particles are there one is constituting one system and the forces are acting which are mutually acting on each other. So, there is no external force on the two particles there is nothing external force okay whatever it is there it will count as the internal force mutual gravitational force.

So, it is free from the external force and therefore the system of particle it is a centre of the mass it is going to move at a constant velocity which is not going to change its direction okay. So, this is the property we are getting here so now coming to the conclusion what we can tell that a system of particles even in the case you can extend it to the three particle system and you will get the same result or in particle system you are going to get the same result.

So, this says that the system of particles if it is free from the external force so the centre of mass moves at a constant velocity okay. So, here if you look back in this equation now we have this constant and this constant so this is a vector this is also a vector so total number of three constants are involved here because this is a vector so 3 + 3 total in this two okay so total 6 constants are involved in the two particle system.

Once we have written these two equations you know that once we integrate it, we will get 3 constants here, 3 constants for total 6 constants. How? $\frac{d^2r_1}{dt^2}$ if it is in three dimensions so this can be written as in matrix notation this can be written as $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$ and $\frac{d^2z}{dt^2}$ So, we have the total of three terms here okay and corresponding right hand term are also there.
So, if you integrate this constitutes one second order differential equation this also constitutes one second order differential equation and this also constitutes one second order differential equation. So, with these two constants are involved these two constants are involved and with this also two constants are involved so total 6 constants are doing well so 6 constants are involved in this 6 constants are involved in this.

So total number of 12 constants are there, so 12 constants are involved okay so out of that we have been able to identify these 6 constants and these 6 constants what we are identifying a and b these are called integrals of motion.

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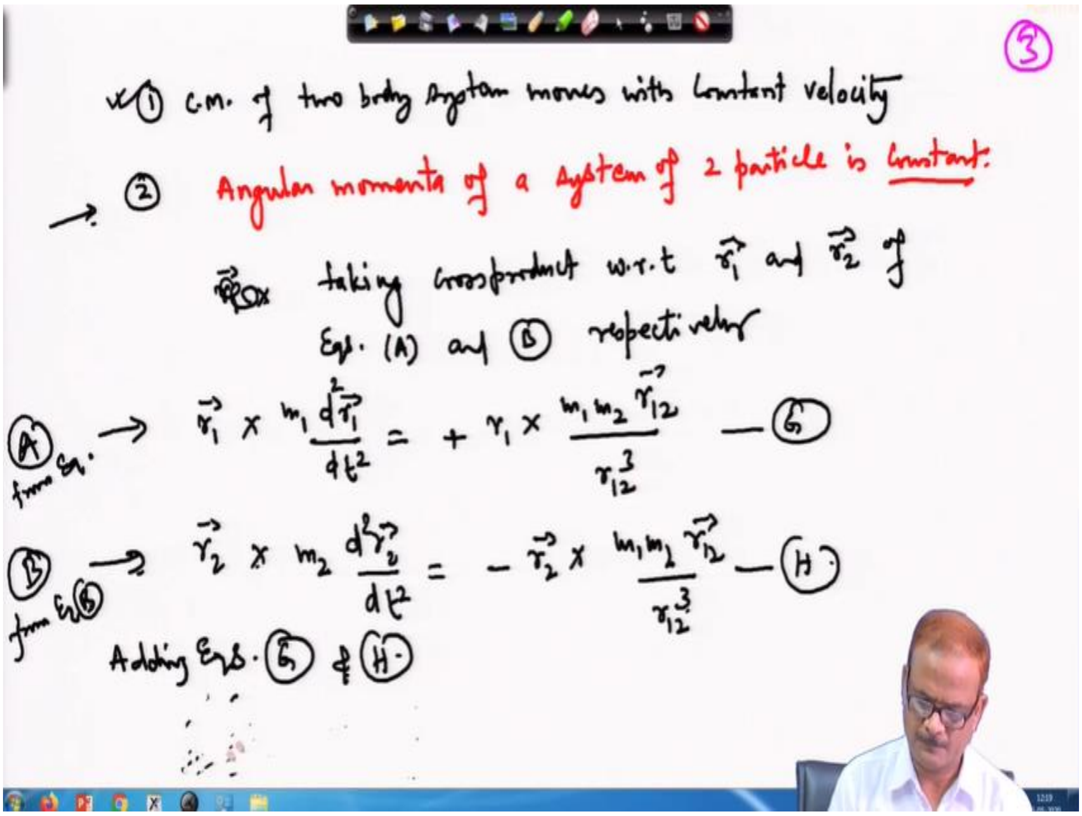

So, 6 we have identified rest remains the 6. So whether we will be able to identify all the 12 constants or not it is a matter of time. So, over a period of time we will come to know so I will not discuss all the things at this stage few of the things I will take it to the three-body problem and then there we will discuss it okay the next one. So, till now what we have got that the centre of mass of two body system moves with constant velocity.

The second one we are going to derive which will be which is the angular momenta okay and one more thing you can get from this place see here in this place I have listed one more thing that this is a total of mass *m* okay so m times V_{cm}cm this is a constant. So, this is the total linear momenta of the system here in this place it is very much visible $m_1 \frac{dr_1}{dt}$, $m_2 \frac{dr_2}{dt}$

So, this is the linear momentum of first particle and this one is the linear momentum of second particle. So, adding this to momentum linear momentum so we call it momenta so momenta of the system this is going to be a constant. So, why this is a constant because these two-particle systems which are free from external force. If any external force is acting on the system, then this will no longer be valid okay.

So, the second one will be the angular momenta of a system of two particles with constant and this we are going to work out okay. So, this is already done and this we are going to do now so multiply equation (A) and (B)/r_1 and r_2 take the cross product. So, taking cross product with respect to $r_1$and r_2 of equations (A) and (B), respectively. So this gives us

$$\overrightarrow{r_1} \times m_1 \frac{d^2\overrightarrow{r_1}}{dt^2} = +r_1 \times m_1 m_2 \frac{\overrightarrow{r_{12}}}{r_{12}^3}$$

We will number the equation this is (F) for this we are obtaining from equation (A) and this we are obtaining from equation (B) from equation (A) from equation (B). So,

$$\overrightarrow{r_2} \times m_2 \frac{d^2\overrightarrow{r_2}}{dt^2} = -r_2 \times m_1 m_2 \frac{\overrightarrow{r_{12}}}{r_{12}^3}$$

this is (H). Now add both of them adding equations (G) and (H) so that will give us and here we will take it outside $\frac{d^2}{dt^2}$ and write it this way $m_1\ r_1$ +$m_2\ r_2$ added for we have missed the term r_1 + r_2 we are missing here. So, we have to add that we will go the next page.

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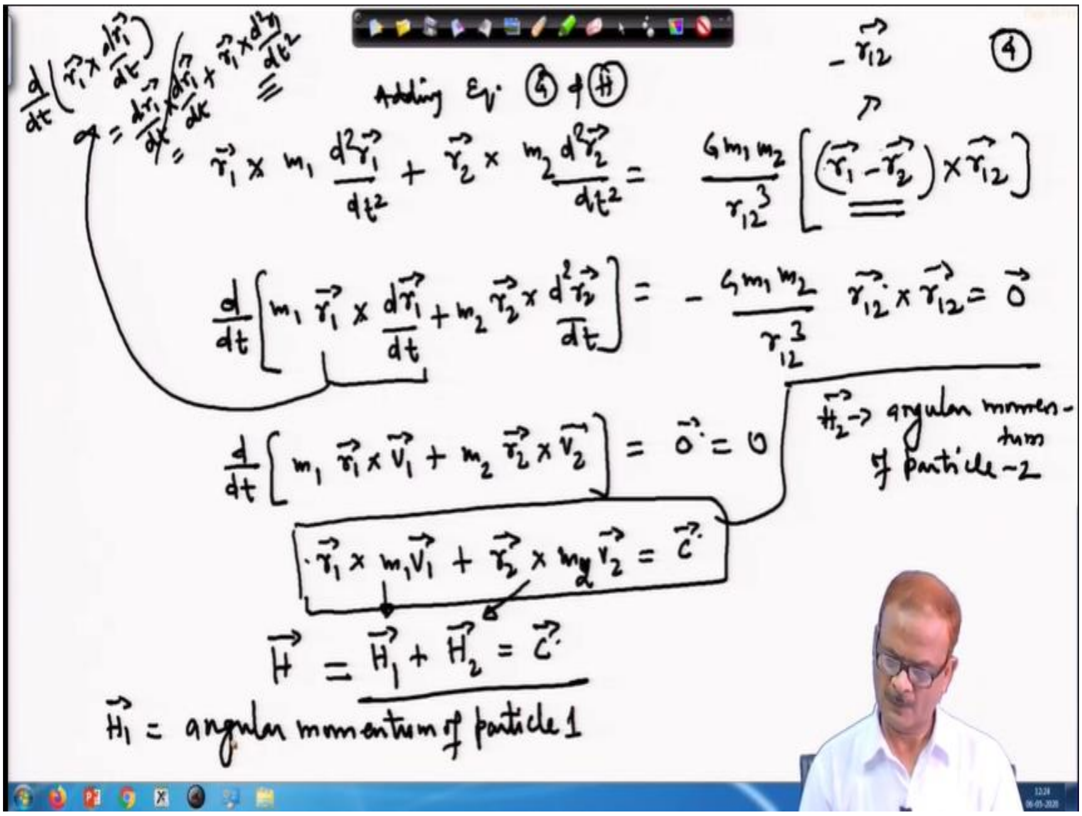

Adding equation (G) and (H) that gives us

$$\vec{r_1} \times m_1 \frac{d^2\vec{r_1}}{dt^2} + \vec{r_2} \times m_2 \frac{d^2\vec{r_2}}{dt^2} = \frac{Gm_1m_2}{r_{12}^3} \left((\vec{r_1} - \vec{r_2}) \times \vec{r_{12}}\right)$$

also this will come out and here this particular thing we are taking this common and just other things will be there. So, this will be $(\vec{r_1} - \vec{r_2})$ $\vec{r_1} \times \vec{r_{12}}$ so $\vec{r_{12}}$ also we will take it out right this is okay. So for $(\vec{r_1} - \vec{r_2}) \times \vec{r_{12}}$ if we write it so this quantity is nothing but -$\vec{r_{12}}$ so this becomes

$$= -\frac{Gm_1m_2}{r_{12}^3} (\vec{r_{12}} \times \vec{r_{12}}) = \vec{0}$$

so this equal to 0 and left hand side we have to process the left hand side if you will process it here.

So, this becomes m_1 times already we have done this kind of manipulation. So, there is nothing great in this so we can write this as d/dt m_1 times r_1 cross dr_1/dt + m_2 times r_2 cross dr_2/dt so this is the left-hand side, and this is the right-hand side. So if you remember we have worked out this particular thing earlier so this d/dt r_1 cross dr_1/dt this simply gets reduced to you can see that this will be dr_1/dt cross dr_1/dt and + r_1 cross $\frac{d^2\vec{r_1}}{dt^2}$.

So, this gets cancelled out and we are left with this, so we recover the original thing. So therefore, we can write this as

d/dt $m_1$times r_1 cross v_1 + m_2 times $r_2$cross v_2 and on the right-hand side this is 0 vector or simply we write this as 0 also. So, once we integrate it, so this is

$$\vec{r_1} \times m_1\vec{v_1} + \vec{r_2} \times m_2\vec{v_2} = \vec{c}$$

this is a constant and let us say this constant we write as $\vec{c}$.

So, what this quantity is this quantity we write it as $\vec{H_1}$ this quantity we will write it as $\vec{H_2}$ this as $\vec{H_1}$ this as $\vec{H_2}$this equal to $\vec{c}$. So, your $\vec{H_1}$is angular momentum of particle 1 and $\vec{H_2}$ is angular momentum of particle 2. So, this we can summarize here

$$\vec{H_1} + \vec{H_2} = \vec{H} = \vec{C}$$

that means the angular momenta of the system gets a constant. Angular momenta of the two-particle system it is a constant. Okay so this was the second property we have written angular momenta of the system of two particle is constant.

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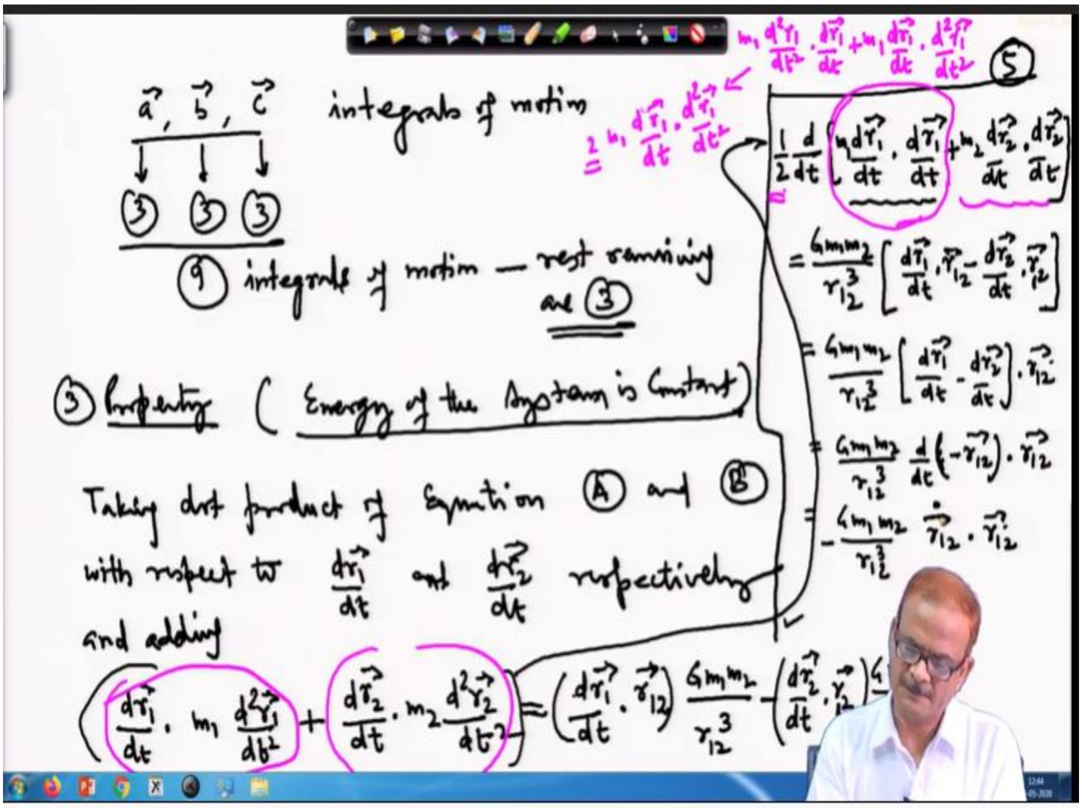

So similarly, we can look into the one more thing we have to discuss. So here how many constants are involved this is in the X, Y, Z inertial frame so XYZ inertial frame. So this we have 3 components, and this is this $\vec{C}$ this overhead arrow this is another constant which is the constant of integral of a constant of integration of the motion or called the integral of the motion okay earlier term we have used here integrals of motion okay.

So, added to this we have one more term which is *C* so what all of you have *a*, *b*, and *c* these are the integrals of motion and each one is of dimension 3 this is of dimension 3 this is also of dimension 3 so that total makes 9 integrals. So, we have been able to identify 9 integrals of motion rest remaining are 3, so 3 more we have to determine what visit we can do that or not we are not still sure.

Now we go to the third property, so we have done one and now the second one is done so the third property energy of the system is constant so already we have observed for the central force motion that this energy of the system is conserved. So, here in two particles system also the energy remains conserved what is the reason that there is no external force acting on the system?

And here we are not assuming that inside there is heat dispersion or there is vibration because of vibration some energy is getting formed of the heat or any kind of things we are just assuming to be absent and the system is free from the external force and therefore the system energy is bound to be conserved which is a principle of conservation of energy so again we start with the equation (A) and (B).

But this time we take the dot product with r_1 and $r_2$okay so taking equation (A) and (B) with respect to dr1dt and dr_2/dt, respectively. If we do this the equation and adding. So, from equation (1) we will have dr_1/dt dot m1 times d^2r/dt^2 this is r_1 and adding this with $2/dt^2$ and on the right hand side you will have

$$\frac{d\overrightarrow{r_1}}{dt}.m_1\frac{d^2\overrightarrow{r_1}}{dt^2}+\frac{d\overrightarrow{r_2}}{dt}.m_2\frac{d^2\overrightarrow{r_2}}{dt^2}=\left(\frac{d\overrightarrow{r_1}}{dt}.\overrightarrow{r_{12}}\right)\frac{Gm_1m_2}{r_{12}^3}-\left(\frac{d\overrightarrow{r_2}}{dt}.\overrightarrow{r_{12}}\right)\frac{Gm_1m_2}{r_{12}^3}$$

okay one more term this comes with minus sign dr_2/dt see here the dot product is commutative in nature.

So, we can write it all on either side that does not matter times $\frac{Gm_1m_2}{r_{12}^3}$. So, we need to go to the next page or either you can work here a little bit and then go to the next page so working out this particular part see what we have done this particular part from here to here okay just writing like this. If you differentiate this let us differentiate only this part and 1/2, we will introduce here so if I differentiate only this part, I will mark this differentiating only this part.

So what it will appear as

$$\frac{1}{2}\frac{d}{dt}\left(m_1\frac{d\overrightarrow{r_1}}{dt}.\frac{d\overrightarrow{r_1}}{dt}+m_2\frac{d\overrightarrow{r_2}}{dt}.\frac{d\overrightarrow{r_2}}{dt}\right)=\frac{Gm_1m_2}{r_{12}^3}\left(\frac{d\overrightarrow{r_1}}{dt}.\overrightarrow{r_{12}}-\frac{d\overrightarrow{r_2}}{dt}.\overrightarrow{r_{12}}\right)$$

So you can see that this is nothing but 2 m_1 times because the dot product is commutative in nature so you can write it like this okay and if we divide by 1/2. So, this two will cancel out and we get the original term which is present here okay. So similarly, this term is written here in this place divided by 1/2 okay.

This right-hand side, we have to work out. So, on the right hand side we can take out $\frac{Gm_1m_2}{r_{12}^3}$ and here we are left with $\frac{d\overrightarrow{r_1}}{dt}.\overrightarrow{r_{12}} - \frac{d\overrightarrow{r_2}}{dt}.\overrightarrow{r_{12}}$okay thereafter some manipulation is there and if you do this we will be able to solve this problem okay so here we have worked out. So, on the right-hand side we have this equation on the left-hand side of this equation.

But here we have missed out the term this$\overrightarrow{r_{12}}$ on the right-hand side this $\overrightarrow{r_{12}}$ is there. In the original equation if you go back and look into this here $\overrightarrow{r_{12}}$ is present and in this place $\overrightarrow{r_{12}}$ we have missed out this $\overrightarrow{r_{12}}$ is also here in this place which we have missed out okay so $\overrightarrow{r_{12}}$ is here. Okay therefore once we work it out, so we need to write there $\overrightarrow{r_{12}}$ this is $\overrightarrow{r_{12}}$and $\overrightarrow{r_{12}}$, so we have missed out wherever we have to insert all of those places.

Now we can reorganize it so if we re organize it the right-hand side we are re organizing and $\frac{m_1m_2}{r_{12}^3}$ and this becomes $\frac{d\overrightarrow{r_1}}{dt} - \frac{d\overrightarrow{r_2}}{dt}.\overrightarrow{r_{12}}$. So right hand side gets reduced to $\frac{m_1m_2}{r_{12}^3}$ and this quantity is nothing but d/dt ($r_2 - r_1$), r_1- r_2 is minus. So, we will write this as –. $\overrightarrow{r_{12}}$. So, finally we summarize here into the right-hand side $r\ _{12}$whole cube this will come with a minus sign here and this is $\overrightarrow{r_{12}}$. $\overrightarrow{r_{12}}$we have d/dt I have removed and placed a dot here in this place. okay now if you can go to the next page.

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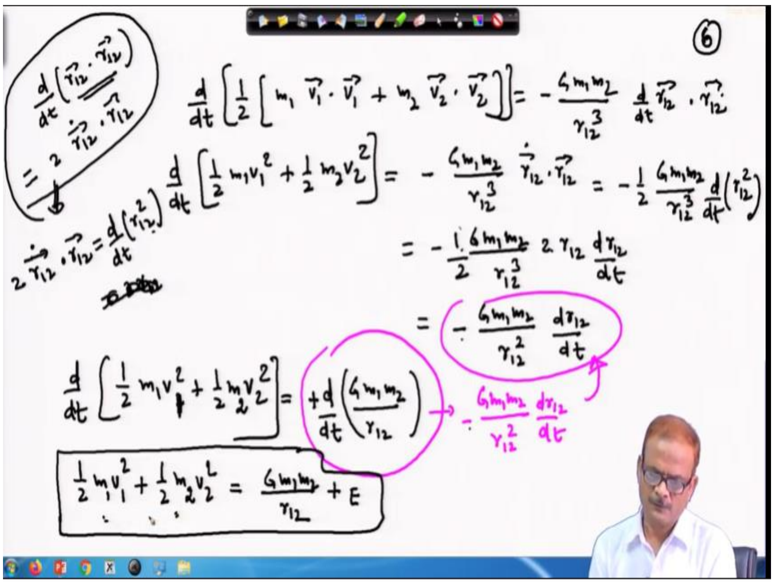

And the left hand side we can rewrite so the left hand side is 1/2 times as written here in this place m_1 times $v_1.\ v_1$+ m_2 times v_2 . v_2 and on the right hand side then we have $-\frac{Gm_1m_2}{r_{12}^3}$times d/d(t$\overrightarrow{r_{12}}$. $\overrightarrow{r_{12}}$). this part we have copied here and on the left hand side we have this part. So, the left-hand side then becomes 1/2 m_1 v_1^2+ 1/2 m_2 v_2^2 on the right-hand side we have to work out r_{12}^3 r_{12} dot times r_{12}.

So, d/dt r_{12} dot r_{12} this will be nothing but two times r_{12} dot r_{12}. So, we utilize this property here in this place and rewrite this as $G\frac{m_1 m_2}{r_{12}^3}$ and then this part this part we can rewrite as this is 2 times r_{12} dot this equal to on the right hand side we will have d/dt(r_{12}^2) .this is nothing but *2d* okay this is okay because we can put it here in this format we are taking the dot product of this.

This becomes r_{12}^2 whole square so if you can replace this by 1/2 d/dt r_{12}^2. And thereafter we simplify it so this is $G\frac{m_1 m_2}{r_{12}^3}$and this is 2 r_{12} times d$/dt($ $r_{12}$$)$ and 1/2 this is multiplied by 1/2. y so minus 2 , 2 cancels out can we get here – $G\frac{m_1 m_2}{r_{12}^2}$,dr_{12}/dt and we explore if there is any possibility of further rewriting the right hand side suppose we have to write it like this and then we check this term by expanding.

So, we will see that this is $-G\frac{m_1 m_2}{r_{12}^2}$ d r_{12} /dt. So, this is nothing but the term here was not present so therefore and on the left-hand side we have made out another part this is 1/2 d/dt okay. So, we need to write here d/dt also d/dt and d/dt so this d/dt which was present here this we were missing so I have added that so with that our equation looks like this.

$$\frac{1}{2}\, m_1\, v_1^2 + \frac{1}{2}\, m_2\, v_2^2$$

Now we can integrate it and it be integrate to we get it in this format 1/2 m okay, so this is minus, and this gets plus sign here this minus is eliminated once we differentiate here this minus sign which is given here. So, what we get here

$$\frac{1}{2}\, m_1\, v_1^2 + \frac{1}{2}\, m_2\, v_2^2 = G\frac{m_1 m_2}{r_{12}} + E$$

E is the constant of integration. So, the left-hand side is the kinetic energy of the two particles.

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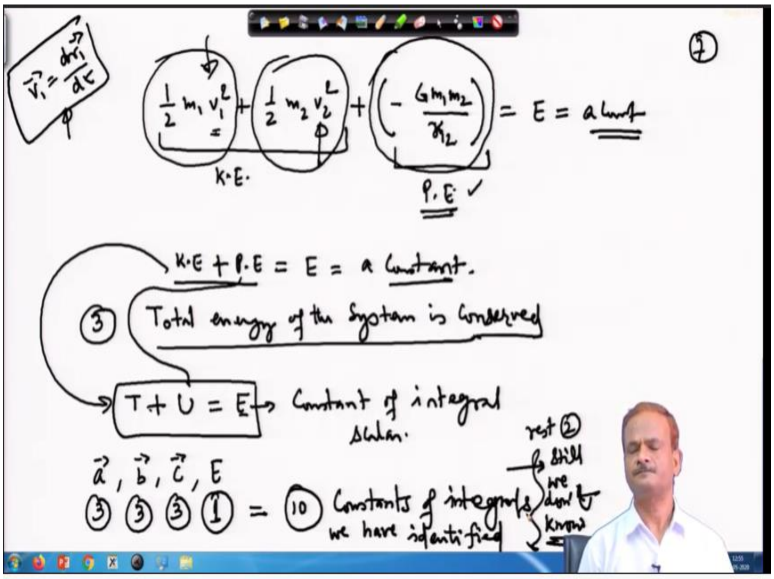

So, going to the next page so

$$\frac{1}{2}\, m_1\, v_1^2 + \frac{1}{2}\, m_2\, v_2^2 - G\,\frac{m_1 m_2}{r_{12}} = E$$

So, this is kinetic energy of the two particles, and this is potential energy mutual potential energy or these two particles. Okay and this we are writing as an E which is it constant so what it says that kinetic energy plus potential of this two-particle system this equal to E is a constant that means total energy of the system is conserved.

This is the mutual potential energy, but this is your this is written in terms of v_1 is nothing but d r_1/dt, so this is an inertial frame okay so what is the inertial frame what is the linear velocity? and in the inertial frame this is another linear velocity of the second particle. So, this constitutes your absolute kinetic energy and this potential energy is due to the mutual proximity how close they are so it depends on that okay so altogether these three terms added together it will remain as a constant.

So, this is the third part we were looking for that the total energy of the system remains conserved. So, we stop at this stage and we will continue in the next lecture. So finally let us conclude it the kinetic energy is written as T and potential energy is written as U so T + U this is equal to E and later on we will find the find description for this also we will discuss about this.

So, this is another constant of integral what this is scalar quantity because this is energy term. So, how many terms we have got *a*, *b*, *c* these are the vectors and *E*, so this is 3 this is 3 this is

3 and this is 1 so total of 10 constants of integral we have identified the rest 2 is still we do not know. So, we will explore whether this can be solved or not solved so these are the two missing integral system.

So, what we have done we are looking for closed form solution of the equation of motion of these two particles. Two particles system we had, and we were looking for the closed form solution obviously if you are given the differential equation for this two particles and the initial conditions are given so you can propagate the system is check okay if the initial conditions are known there is no problem in that.

But that we are not doing what we are doing here we are trying to look for closed form solution okay we are not trying to do it numerically okay. So, in that effort we are just able to identify it until now 10 constants rest 2 are still missing. So once I take up the case for the 3 particle systems so at that time, we will discuss it again. Okay so thank you very much for listening and I while discussing some terms always say if sometimes, we are dropping out like this d/dt was dropped out from this place initially it was missing here.

So, this kind of error attrition while discussing because the attention gets diverted and copying from one place to another place okay. So, but to of course once it comes to the final conclusion, I am correcting all the time as it is necessary not only necessary but automatically if those terms are missing so we will not get the result. So hopefully while you read even in the future this kind of error may creep in and so always look for the later on.

I keep correcting whenever the errors are creeping in so I keep correcting them one by one so thank you very much for listening and next lecture we will take up the new topic and also in the future we will solve some new problems so that not only the theory but you will also acquainted with how to solve the different space mechanics related problem thank you very much.