Space Flight Mechanics

Prof. Manoranjan Sinha
Department of Aerospace Engineering
Indian Institute of Technology – Kharagpur

Module No # 17
Lecture No # 81
Trajectory Transfer (Contd.)

Okay welcome to lecture 81 so last time we have done we have looked into what will be condition for minimum eccentricity. And for going from inner to outer orbit or outer to the inner orbit not in this lecture we start with the generalized trajectory transfer equation derivation. So we have the trajectory transfer we can choose from 2 point of view that we want to have a optimal time or the time is given we have to manage the things going from 1 orbit to another orbit okay.

Or we want to minimize the energy mostly you will see that most of the time because we have to run our satellite for a longer time. So it is a minimum energy requirement most of the time that will be you cannot burn your propellant orbit surely. You should do that then the satellite life is lost so if one satellite mission it cost around 300, 400 crores rupee it has to lost for longer time 10 years, 15 times whatever.

That is the service period of the satellite and if the propellant is burnt earlier then that satellite becomes unusable. So all the control system onboard of maneuvering the satellite orientation that should run for 15 years and also the orbit correction system which is based on the propulsion system that should also run for 15 years. So this is the situation so let us discuss about the generalized co planer transfer.

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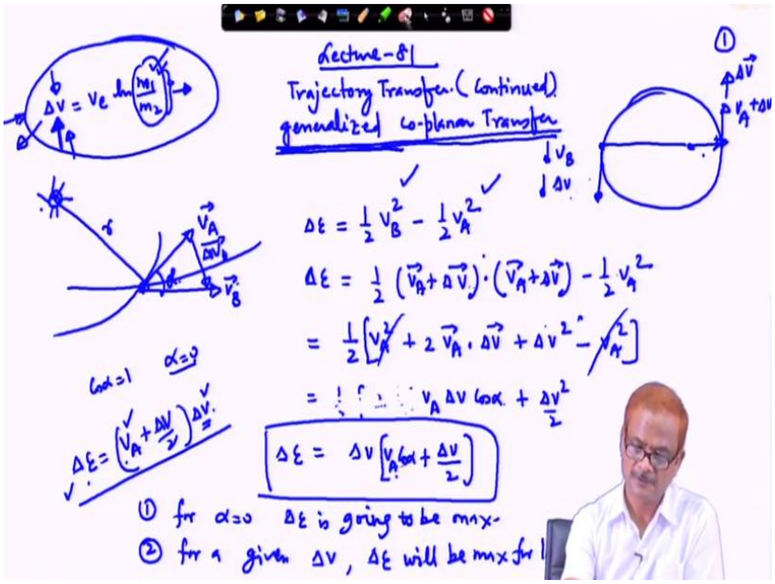

$$\Delta V = V_e \ln \frac{m_1}{m_2}$$

And already we have looked into that this ΔV this is directly related to your $V_e \ln \frac{m_1}{m_2}$ means the propellant mass bond. So minimum energy required means you want to minimize the use of the propellant. I do not want to use a large amount of propellant bond at any time. So when this is going to happen this we can explore. So before going into the actual thing we can have a look of the fact that when your energy is spent will be the minimum.

So let us look at the point A this is the trajectory and V_A is the velocity initially and you want to go into another trajectory. So V_B is the velocity and this is ΔV_B ΔV so the total energy required at that point because the r it is not changing the radius vector this may be the point of attraction okay the center of force. So this you are not changing so the energy this will basically be defined as 1 /2 V_B^2 – 1 / 2 V_A^2 this is the change in energy.

$$\Delta E = \frac{1}{2}V_b^2 - \frac{1}{2}V_A^2$$

$$\Delta E = \frac{1}{2}\left(\overrightarrow{V_A} + \Delta\vec{V}\right).\left(\vec{V}_A + \vec{V}_B\right) - \frac{1}{2}V_A^2$$

So for A given mass bond which is dependent on ΔV okay so what it implies that what A given ΔV once it proper amount of mass is burnt so you get some amount of impulse ΔV okay. And this ΔV is going to manifest here 1 / 2 V_A+ ΔV this^2 means we have to take the dot product V_A+ ΔV. So the question arises when this change in energy will be maximum so if we expand this part.

Δ So we can see that this will be V_A^2 + 2 times $V_A.\Delta V + \Delta V^2 - V_A^2$ this drops out 1 / 2 times 2 this you can write like this V_A times ΔV times Cos this let us say this angle is let us say this angle is α. So we can write V_A times ΔVCos α + Δ V_A^2 divided by 2. ΔV can be taken outside so this is the energy change so for A given mass of propellant bond your ΔV also get fixed and then Cos α is missing here V_A Cos α.

$$\Delta E = \Delta V[\, V_A \cos\alpha + \frac{\Delta V}{2}]$$

So 2 things you can observe from this place if Cos α = 1 means α = 0 so ΔE will be V_A+ ΔV divided by 2 times ΔV. So for A given ΔV and given V_A this ΔE gets maximized missed A change in energy is maximum you are getting the maximum amount of possible energy change in the energy of the orbit total energy of the orbit. Because here; the potential energy is not changing while you are providing impulse at a particular point.

So the orbit the radius is not changing and therefore potential energy remains unaltered only the kinetic energy gets affected. So one observation is that for α equal to 0 ΔE is going to the maximum this is first observation and the second observation what we can have that if for A given ΔVΔ will be maximum when V_A is maximum. So for A given ΔV ΔE will be maximum for largest V_A.

$$\Delta E = \left(V_A + \frac{\Delta V}{2}\right)\Delta V$$

So this implies that in the orbit say if you apply here this is your perigee position and velocity here this is V_A and if you apply impulse ΔV here in this direction in the same direction ΔV. So the change in energy will be maximum okay while you do the opposite part here in this place here this is your V_B point and you provide in ΔV the impulse. Say according to this the V_B is less here in this place and therefore for the same impulse ΔV the change in energy will be less as it is evident from this equation.

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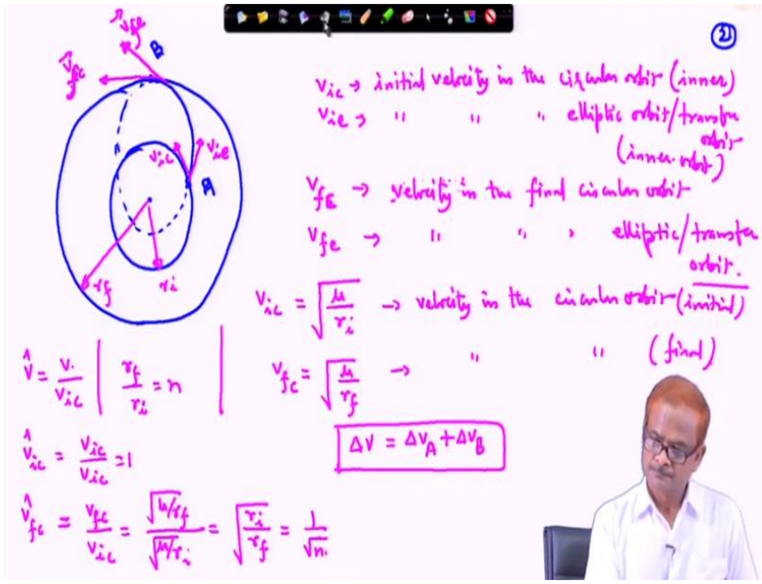

Okay this should be what we have worked out here it should be kept in mind okay when this is going to be maximum. Okay so we explore the think further transfer from first orbit to the inner orbit to the outer orbit is the inner orbit and from point P we have to go into the outer orbit. We can name this as A rather than P this is AB. So this is the orbit and this is generalized transfer not the tangential one.

So at this point we write here as V_{ic} this is r_i and this will indicate as r_f and the velocity here tangent to the transfer orbit which we write it as V_{ie} in the elliptic orbit or the transfer orbit V_{ic} and V_{ie}. Similarly at this point velocity in the circular orbit this will be V_{bc} and tangent to this that transfer orbit this will be V_{bc}. So the initial velocity V_{ic} is the initial velocity in the circular orbit and V_{ie} in the circular inner orbit V_{ie} is the initial velocity in the elliptic order transfer for orbit slash transfer orbit in the inner orbit.

Similarly we are defining V_{fb} we have written V_{bc} okay for the inner we have used the word i so I will change it to the rather than writing at b I will use here f. So accordingly we will define this as V circular final velocity or the velocity in the final circular orbit. This process is applicable to a elliptic orbit also elliptic slash transfer orbit this are some of the nomenclature we are going to use and V_{ic} will be equal to V / r_i under root.

$$V_{ic} = \sqrt{\frac{\mu}{r_i}}$$

$$V_{fc} = \sqrt{\frac{\mu}{r_f}}$$

$$\widehat{V} = \frac{V}{V_{ic}}$$

$$\widehat{V}_{ic} = \frac{V_{ic}}{V_{ic}} = 1$$

$$\widehat{V}_{fc} = \sqrt{\frac{r_i}{r_f}} = \frac{1}{\sqrt{n}}$$

Okay the velocity in the circular orbit, okay with all this nomenclature V_{fc} will be μ/ r_f under root this we have already done. So this is initial this is final more over we define $\widehat{V}$ =V divided by V_{ic} so using this nomenclature we will have V_{ic} =$\widehat{V}_{ic}$ = V_{ic} divided by this equal to 1. And $\widehat{V}_{fc}$ equal to V_{fc} divided by V_{ic} so from here this is μ/ r_f under root divided by μ/ r_i under root and then this becomes r_i / r_f under root and here r_f / r_i we are writing as n.

So with this nomenclature given to us we are ready to work out the energy required. So what we are interested in finding out the impulse ΔV_A and ΔV_B this is will be equal to ΔV.

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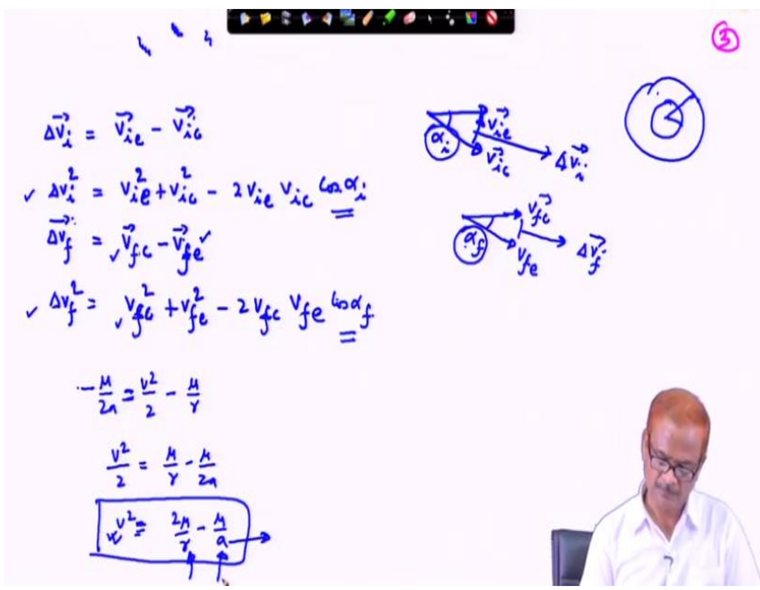

$$\Delta\vec{V_i} = \vec{V_{ie}} - \vec{V_{ic}}$$

$$\Delta V_i^2 = V_{ie}^2 + V_{ic}^2 - 2V_{ic}V_{ie}\mathrm{Cos}\ \alpha_i$$

So at the initial point ΔV_i can be written as $V_{ie} - V_{ic}$ and the vector notation and therefore ΔV_i^2 this can be written as $V_{ie}^2 + V_{ic}^2$ and (()) (15:22) product with the same then you can write it $2V_{ic}V_{ie}$

and the angle between them. This we are defining as Cos α_i this is V_{ie} and this V_{ic} these are the 2 vectors this is α_i this is the change here ΔV.

Similarly ΔV_f^2 this will be V_{ie} V_{ic} – V_{ie} V_{fe} ΔVI will first write in the vector terms and thereafter we convert it ΔV_f^2 V_{ic}^2 + V_{fe}^2 minus we have to distinguish between c and e. We have to little careful in writing this is the velocity at the final point we have to write at which properly this is V_{fc} and here this is V_{fe} both are in the final orbit not in the initial.

And therefore this quantity we have ΔV and this is we are writing as α_f so following that notation here also f all this are fe times Cos α_f. So in this 2 points what are known to us V_{fc} is known to us because you know the radius of the initial and the final orbit these are circular orbit and therefore the velocity is known from that. This α_f and α_i these are not known to us okay these are unknown.

$$-\frac{\mu}{2a} = \frac{V^2}{2} - \frac{\mu}{r}$$

$$V^2 = \frac{2\mu}{r} - \frac{\mu}{a}$$

So this we need to work out and what about the velocity in the elliptic orbit for this we can use this formula minus μ/ 2a equal to V_A^2 divided by 2 – μ / r we utilize this. So V^2 divided by 2 this will be equal to μ/ r –μ/ 2a or V^2 = 2μ/ r –μ by. So at the initial position r is known and if a, is known then our V gets decided. Okay this are the issues that we are going to explore so in which orbit you are going to send it that will decide what will be the value of a.

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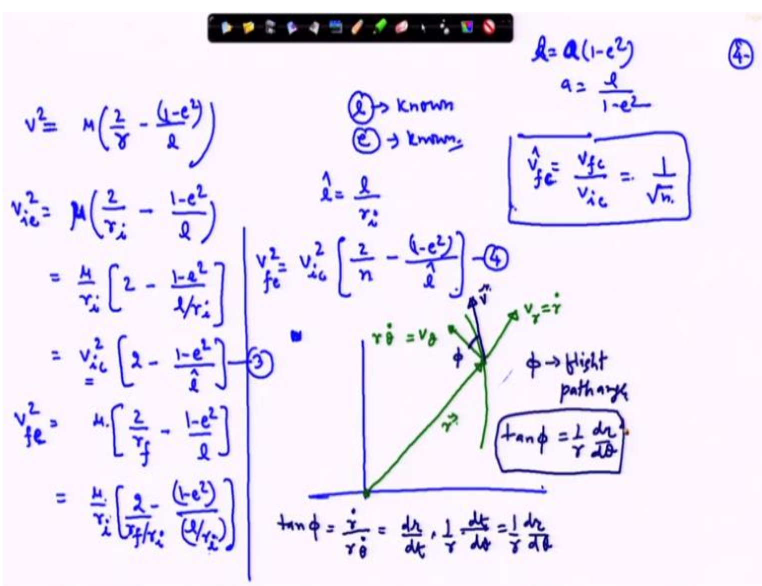

$$V^2 = \mu\left(\frac{2}{r} - \frac{1-e^2}{l}\right)$$

$$V_{ie}^2 = \mu\left(\frac{2}{r_i} - \frac{1-e^2}{l}\right) \text{ where } l = a(1-e^2)$$

$$V_{ie}^2 = V_{ic}^2\left(2 - \frac{1-e^2}{\hat{l}}\right)$$

1 / a so we write it like this l equals a times 1 – e ^{2} so we use this equation l / 1 – e^2. So we will assume here this l is known and e is known you should know in which orbit you are going to send if you know that so everything is then decide it and therefore following this notation V_{ie}^2 this can be written as 2/μ times 2 / r_i can be taken outside. So this is μ/ r_i is 2 minus μ/ r_i c r_i is nothing but v_{ic}^2.

$$\hat{l} = \frac{l}{r_i}$$

Now this is 2 – 1 – e^2 divided by $\hat{l}$ where $\hat{l}$ is been defined as we have used the notation of here l / r_i. Following the same notation we can see that vfe^2 this will be μ times 2 / r_f - 1 – e^2 divided by l. Or it would be better to write it in a way because we are going to express it in terms of V_{ic}. So rather than expressing in terms of r_f taking the r_f here outside let us say that I take here r_i okay.

$$V_{fc}^2 = V_{ic}^2\left[\frac{2}{n} - \frac{1-e^2}{\hat{l}}\right]$$

So this quantity then can be written as r_f / r_i and this quantity here this can be written as l / r_i you will see the benefit of using this. And therefore V_{fe}^2 this gets reduced to V_{ic}^2 times 2 divided by r_f / r_i is n so 1 – e^2 divided by $\hat{l}$. Now this will term as 1 this as the equation 2 and this we will write as equation 3 and this as equation 4. So we can utilize this information to rewrite the equation okay.

$$\hat{V}_{fc} = \frac{V_{fc}}{V_{ic}} = 1/\sqrt{n}$$

So now we have to rearrange the whole thing in a proper way and for that what we require also we have to look into this part that $\hat{V}_{fc}$ we have written as in the final circular orbit this is V_f circular by V_i circular and so got from there we have this as $1/\sqrt{n}$. So keeping in mind all this information we have to rewrite the whole thing. Now α we have to decide what will be the how we are going to work out this angle α which is appearing here.

A is appearing here α_f is appearing here so this we need to work out before we can do any other thing. So rather than writing this α let use this as ϕ instead of reading α_i will use this as ϕ. So if I do this so everywhere I will change it to ϕ the reason for this will be (()) (24:56) as I proceed further I do not want to confuse you. This is ϕ_i ϕ_f so only thing here I have changed here changed and anywhere till now we have not used.

So say this is the trajectory and this is the r vector to the point and this is the center of force it may be focus or whatever. So in this direction you have V_θ and this direction you have ϕ r this you have written as $\dot{r}$ this part is $r\dot{\theta}$. And velocity is tangential here in this direction and this angle we write as ϕ angle. So ϕ is here the flight path angle or maybe we can write in terms of this is okay let us make it ϕ is the flight path angle we have used earlier.

$$\tan\phi = \frac{\dot{r}}{r\dot{\theta}} = \frac{dr}{dt}\cdot\frac{1}{r}\frac{dt}{d\theta} = \frac{1}{r}\frac{dr}{dt}$$

So that this is why I change earlier from α to this flight path angle? So immediately from this place we can see that tan ϕ this will be equal to $\frac{\dot{r}}{r\dot{\theta}}$ this is nothing but dr / dt times 1 / r times dθ/dt we can write like this and this is dr / dθ 1 / r. Tan ϕ is given by 1/ r times dr / dt.

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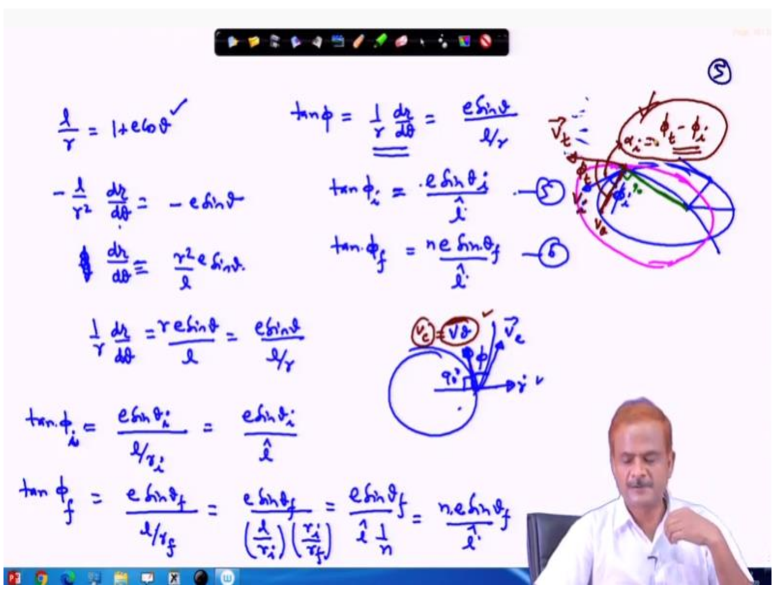

$$\frac{l}{r} = 1 + e\cos\theta$$

$$-\frac{l}{r}\frac{dr}{d\theta} = -e\sin\theta$$

$$\frac{dr}{d\theta} = \frac{r^2}{l}e\sin\theta$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{e\sin\theta}{\frac{l}{r}}$$

Where θ is here the true anomaly $-l / r^2$ dr / dθ that gives us –e sin θ and tan ϕ we have already written on the previous place tan ϕ is equal to 1 / r times dr / dθ. So we need to insert dr / dθ from this place so dr / dθ this equal to r^2 / l e sin θ. Okay and we require here 1 / r so 1/r we keep it here will make it in the next step so we will write it as 1 / r dr / d θ this equal to r times e / l e sin θ divided by l / r.

$$\tan\phi_i = \frac{e\sin\theta_i}{l/r_i} = \frac{e\sin\theta}{\hat{l}}$$

$$\tan\phi_f = \frac{e\sin\theta_f}{(l/r_f)} = \frac{e\sin\theta_f}{\left(\frac{l}{r_i}\right)\left(\frac{r_i}{r_f}\right)} = \frac{ne\sin\theta_f}{\hat{l}}$$

So this quantity then gets reduced to e sin θ divided by l / r therefore tan ϕ_i the flight path angle at the initial point this can be written as e sin θ_i transfer orbit is the same only thing the i tag or the s tag is appearing e sin θ_i divided by $\hat{l}$ similarly tan ϕ_f this will be e sin θ_f divided by l / r_f. According to the notation we are following so this we need to manipulate little bit we can write it as l / r_i, r_i / r_f θ_f divided by ly r_i is $\hat{l}$ and r_i / r_f is 1 / n r_f / r_i we have written as n.

$$\tan\phi = \frac{1}{r}\frac{d\theta}{dt} = \frac{e\sin\theta}{\left(\frac{l}{r}\right)}$$

$$\tan\phi_i = \frac{e\sin\theta_i}{\hat{l}}$$

$$\tan\phi_f = \frac{e\sin\theta_f}{\hat{l}}$$

So this is 1 / n so n times e sin θ_f divided by $\hat{l}$ thus we have tan ϕ_i e sin θ_i divided by $\hat{l}$ and tan ϕ_f is sin θ_f divided by n times $\hat{l}$ this is 5 and this is 6. So now check from this point that from this orbit we have to the velocity vector is directed here. If it is some other orbit elliptical orbit this also at that time else and then you are thrusting it into another orbit. This directly gives you now this tangential to this point and in the (()) (31:34) orbit this is normal this is 90°.

So this constitutes your V_θ direction and this is V in the elliptic orbit and this corresponding $\dot{r}$. In the case of circular orbit $\dot{r}$ 0 what not for the elliptic orbit. So for in the transfer orbit $\dot{r}$ positive initially positive or negative depending where it is lined okay. And you can see this is your flight path angle 5 and this is the reason I have changed it to from α to ϕ. So it is easy if it is an elliptical orbit the situation is then different.

So at that time say here this is the point or either we take another point here this is in the circular orbit elliptic orbit this is V_i and corresponding then I will rather define α_i or say the ϕ_i. And the it has to be sent into some another elliptic orbit say if you have to transfer here in this orbit another orbit we are this is the V_i V in the another elliptic orbit you are doing or V_t we can tag it as V_t. So for that orbit it is easy to okay and what we will do that we will not distort that much will assume that this rotates by certain amount.

You are rotating the orbit by certain amount so from this place it is passing through this to this point and going like this orbit have changed like this and very not good fewer but it is okay. So here in this case your velocity done we can see that this is the radius vector already we have drawn okay this is 90° for the pink orbit we will draw by another part this is V in the transfer orbit it is tangential to the pink orbit.

And radius vectors will along the same direction so perpendicular here this θ direction remains the same. So this is your ϕ in the transfer orbit so you can see that between the V_i and V_f this is the angle this now we can define as α. So α_i equal to $\phi_t - \phi_i$ here in this case in the circular orbit V_θ and V as you can see from this place the velocity itself lies in the V_θ direction.

So here V circular is nothing but equal to V_θ and therefore this is here it as V_c but if the this is elliptic case so you have to take this properly into account okay. So then instead of writing on the previous phase where we have defined in terms of Cosϕ_i will tend to keep it as Cos α_i and Cos α_f. And then α_i and α_f will be defined by α_f is defined like this and similarly α_f will be defined okay so you can see the difference and this we have to keep into account while working. Okay so we will stop here and do best in the next lecture thank you very much.